

Website: <http://www.physik.uzh.ch/en/teaching/PHY519/>

Exercise 1 [Metric of a static star]

The exterior spacetime of extended spherical objects can be approximated by the Schwarzschild metric. In this exercise we will derive the interior metric modeling the star by a perfect fluid with energy-momentum tensor $T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}$. As a first step we will consider the general static, spherically symmetric metric

$$ds^2 = \exp[2\alpha(r)]dt^2 - \exp[2\beta(r)]dr^2 - r^2d\Omega^2. \quad (1)$$

The corresponding non-vanishing Christoffel symbols are (up to permutations of the two lower indices)

$$\begin{aligned} \Gamma_{tr}^t &= \partial_r \alpha \\ \Gamma_{tt}^r &= \partial_r \alpha \exp[2(\alpha - \beta)] & \Gamma_{rr}^r &= \partial_r \beta & \Gamma_{\theta\theta}^r &= -r \exp[-2\beta] & \Gamma_{\phi\phi}^r &= -r \sin^2(\theta) \exp[-2\beta] \\ \Gamma_{\theta r}^\theta &= r^{-1} & \Gamma_{\phi\phi}^\theta &= -\sin(\theta) \cos(\theta) \\ \Gamma_{\phi r}^\phi &= r^{-1} & \Gamma_{\phi\theta}^\phi &= \cot(\theta), \end{aligned}$$

while the Ricci tensor takes the form

$$R_{tt} = \exp[2(\alpha - \beta)] \left(\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \alpha \right) \quad (2)$$

$$R_{rr} = -\partial_r^2 \alpha - (\partial_r \alpha)^2 + \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \beta \quad (3)$$

$$R_{\theta\theta} = 1 - \exp[-2\beta] + (\partial_r \beta - \partial_r \alpha) r \exp[-2\beta] \quad (4)$$

$$R_{\phi\phi} = \sin^2(\theta) R_{\theta\theta}. \quad (5)$$

- (i) In the fluid rest frame the velocity is pointing in the timelike direction. Using this and the normalization $u_\mu u^\mu = 1$ show that the energy momentum tensor can be written as

$$T_{\mu\nu} = \text{diag} \{ \exp[2\alpha] \rho, \exp[2\beta] p, r^2 p, r^2 \sin^2(\theta) p \}. \quad (6)$$

- (ii) Write down the Einstein equations inside the star.

It will be convenient to define

$$m(r) = \frac{1}{2G} (r - r \exp[-2\beta]) \quad (7)$$

and to rewrite the Einstein equations in terms of $m(r)$.

(iii) Use the energy-momentum conservation $\nabla_\mu T^{\mu\nu}$ to show

$$(\rho + p) \frac{d\alpha}{dr} = -\frac{dp}{dr}. \quad (8)$$

Use the above equation together with the rr -component of the Einstein equations to derive the Tolman-Oppenheimer-Volkoff equation of a star

$$\frac{dp}{dr} = -\frac{(\rho + p) [Gm(r) + 4\pi Gr^3 p]}{r [r - 2Gm(r)]}. \quad (9)$$

This is the relativistic version of the equation of hydrostatic equilibrium for a star.

(iv) Assuming an incompressible fluid with constant density ρ_* out to the surface of the star, solve for the pressure $p(r)$ and write down the metric. What is the maximum mass a star of a given radius R can have?