Applications of General Relativity

Solutions 2

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Exercise 1 [Time delay in gravitational lensing and FERMATs principle]
Considering the weak field metric in transverse gauge

$$\mathrm{d}s^2 = \left(1 + 2\frac{\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - \left(1 - 2\frac{\Phi}{c^2}\right) \mathrm{d}\boldsymbol{x}^2. \tag{1}$$

we are going to calculate light deflection and the SHAPIRO delay. We will consistently perform the calculation at first order in the metric perturbation Φ. In typical configurations of gravitational lensing the distances along the line of sight are much bigger than the distances perpendicular to the line of sight and the deflecting mass is localized such that the photon trajectory can be modeled by two straight lines with a kink in the lens plane as shown in Fig. 1 of the exercise sheet.
Note that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are two-dimensional positions in the source- and lense plane respectively, thus $\boldsymbol{\eta}, \boldsymbol{\xi} \perp \boldsymbol{e}_z$.

(i) Using $\rho(\boldsymbol{x}) = \Sigma(\boldsymbol{\xi}')\delta^{(\mathrm{D})}(z)$ we obtain for the gravitational potential

$$\Phi(\boldsymbol{r}) = -G \int \mathrm{d}^2\xi' \frac{\Sigma(\boldsymbol{\xi}')}{|\boldsymbol{r} - \boldsymbol{\xi}'|}, \tag{2}$$

Associating the line of sight with the z-axis, we write the unperturbed geodesic as $\boldsymbol{\gamma}(z) = z\boldsymbol{e}_z + \boldsymbol{\xi}$. Thus the derivative perpendicular to the path is $\boldsymbol{\nabla}_\perp = \boldsymbol{\nabla}_{\boldsymbol{\xi}}$ and we have for the deflection angle

$$\begin{aligned}
\hat{\boldsymbol{\alpha}} =& \frac{2}{c^2} \int \mathrm{d}z \, \boldsymbol{\nabla}_\perp \Phi = -\frac{2G}{c^2} \boldsymbol{\nabla}_{\boldsymbol{\xi}} \int \mathrm{d}z \int \mathrm{d}^2\xi' \frac{\Sigma(\boldsymbol{\xi}')}{|z\boldsymbol{e}_z + \boldsymbol{\xi} - \boldsymbol{\xi}'|} \\
=& \frac{2G}{c^2} \int \mathrm{d}z \int \mathrm{d}^2\xi' \frac{\Sigma(\boldsymbol{\xi}')(\boldsymbol{\xi} - \boldsymbol{\xi}')}{|z\boldsymbol{e}_z + \boldsymbol{\xi} - \boldsymbol{\xi}'|^3} \\
=& \frac{2G}{c^2} \int \mathrm{d}^2\xi' \frac{\Sigma(\boldsymbol{\xi}')(\boldsymbol{\xi} - \boldsymbol{\xi}')}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|^2} \left[\frac{z}{\sqrt{z^2 + (\boldsymbol{\xi} - \boldsymbol{\xi}')^2}} \right]_{z=D_{\mathrm{ds}}}^{-D_{\mathrm{d}}} \\
=& \frac{4G}{c^2} \int \mathrm{d}^2\xi' \frac{\Sigma(\boldsymbol{\xi}')(\boldsymbol{\xi} - \boldsymbol{\xi}')}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|^2}
\end{aligned} \tag{3}$$

In the second to last line we used that the distances between source, lens and observer are way bigger than the distance of the photon trajectory perpendicular to the deflecting mass distribution $D_{\mathrm{ds}}, D_{\mathrm{d}} \gg |\boldsymbol{\xi} - \boldsymbol{\xi}'|$.

(ii) As we saw above, at first order the perpendicular gradient can be pulled out of the integral

$$\hat{\boldsymbol{\alpha}} = \frac{2}{c^2} \boldsymbol{\nabla}_{\boldsymbol{\xi}} \int \mathrm{d}z \, \Phi = -\frac{2G}{c^2} \boldsymbol{\nabla}_{\boldsymbol{\xi}} \int \mathrm{d}z \int \mathrm{d}^2\xi' \frac{\Sigma(\boldsymbol{\xi}')}{|z\boldsymbol{e}_z + \boldsymbol{\xi} - \boldsymbol{\xi}'|}. \tag{4}$$

From the above we can immedideately identify the lensing potential and obtain

$$\begin{aligned}\psi(\boldsymbol{\xi}) =& \frac{2}{c^2}\int \mathrm{d}z\, \Phi = -\frac{2G}{c^2}\int \mathrm{d}z \int \mathrm{d}^2\xi' \frac{\Sigma(\boldsymbol{\xi}')}{|z\boldsymbol{e}_z + \boldsymbol{\xi} - \boldsymbol{\xi}'|} + \text{const.}\\ =& -\frac{2G}{c^2}\int \mathrm{d}^2\xi' \Sigma(\boldsymbol{\xi}') \left[\ln\left(z + \sqrt{z^2 + (\boldsymbol{\xi} - \boldsymbol{\xi}')^2}\right)\right]_{z=D_{\mathrm{ds}}}^{-D_{\mathrm{d}}} + \text{const.}\\ =& \frac{4G}{c^2}\int \mathrm{d}^2\xi' \Sigma(\boldsymbol{\xi}') \ln\left(\frac{|\boldsymbol{\xi} - \boldsymbol{\xi}'|}{\xi_0}\right) + \text{const.}\end{aligned} \tag{5}$$

where ξ_0 is an arbitrary normalization scale and $\boldsymbol{\xi}$ independent terms have been absorbed into the constant.

(iii) For the Shapiro delay we consider a null path $\mathrm{d}s = 0$ leading to

$$\left(1 + \frac{\Phi}{c^2}\right) c\, \mathrm{d}t = \left(1 - \frac{\Phi}{c^2}\right) \mathrm{d}l. \tag{6}$$

Integrating along the photon trajectory from the source to the observer we obtain

$$c\, t = \int \left(1 - 2\frac{\Phi}{c^2}\right) \mathrm{d}l = l - \frac{2}{c^2}\int \mathrm{d}l\, \Phi. \tag{7}$$

The first term is the geometric time delation because the light ray is deflected in the lens plane and can be calculated using simple geometric considerations

$$\begin{aligned} l =& \sqrt{(\boldsymbol{\xi} - \boldsymbol{\eta})^2 + D_{\mathrm{ds}}^2} + \sqrt{\boldsymbol{\xi}^2 + D_{\mathrm{d}}^2}\\ \approx& D_{\mathrm{ds}} + D_{\mathrm{d}} + \frac{1}{2D_{\mathrm{ds}}}(\boldsymbol{\xi} - \boldsymbol{\eta})^2 + \frac{1}{2D_{\mathrm{d}}}\boldsymbol{\xi}^2 \end{aligned} \tag{8}$$

From this we subtract the zeroth order path length $D_{\mathrm{ds}} + D_{\mathrm{d}}$ and reorder to obtain

$$\frac{1}{2D_{\mathrm{ds}}}(\boldsymbol{\xi} - \boldsymbol{\eta})^2 + \frac{1}{2D_{\mathrm{d}}}\boldsymbol{\xi}^2 = \frac{D_{\mathrm{s}} D_{\mathrm{d}}}{2D_{\mathrm{ds}}}\left(\frac{\boldsymbol{\xi}}{D_{\mathrm{d}}} - \frac{\boldsymbol{\eta}}{D_{\mathrm{s}}}\right)^2 + \frac{1}{2D_{\mathrm{s}}}\boldsymbol{\eta}^2 \tag{9}$$

The last term is constant for all possible trajectories and can thus be absorbed into a constant. The second part of Eq. (7) is the gravitational time dilation and can be calculated integrating along the unperturbed path as above in Eq. (5). Finally, we obtain for the Shapiro delay

$$c\delta t = \frac{D_{\mathrm{s}} D_{\mathrm{d}}}{2D_{\mathrm{ds}}}\left(\frac{\boldsymbol{\xi}}{D_{\mathrm{d}}} - \frac{\boldsymbol{\eta}}{D_{\mathrm{s}}}\right)^2 - \psi(\boldsymbol{\xi}) + \text{const.}, \tag{10}$$

where the constant is the same for all rays from the source at $\boldsymbol{\eta}$ to the observer.

(iv) We can straightforwardly apply Fermats principle to Eq. (10) describing the Shapiro delay. The gradient of the first term is easily calculated and for the second term we can use the defining equation for the lensing potential $\boldsymbol{\nabla}_{\boldsymbol{\xi}}\psi = \hat{\boldsymbol{\alpha}}$.

$$0 = \frac{D_{\mathrm{s}}}{D_{\mathrm{ds}}}\left(\frac{\boldsymbol{\xi}}{D_{\mathrm{d}}} - \frac{\boldsymbol{\eta}}{D_{\mathrm{s}}}\right) - \hat{\boldsymbol{\alpha}} \tag{11}$$

Finally, we have for the lens equation

$$\boldsymbol{\eta} = \frac{D_{\mathrm{s}}}{D_{\mathrm{d}}}\boldsymbol{\xi} - D_{\mathrm{ds}}\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}). \tag{12}$$

Note that this is an implicit equation for the unknown deflection point $\boldsymbol{\xi}$.

Exercise 2 [$f(R)$ modified gravity]

(i) By varying the action

$$S = \frac{1}{16\pi}\int f(R)\sqrt{-g}d^4x$$

with respect to the metric $g^{\mu\nu}$, one can derive the equation of motion. As an initial step we can determine

$$\delta S = \frac{1}{16\pi}\left[\int d^4x f(R)\delta\sqrt{-g} + \int d^4x\sqrt{-g}\frac{\partial f}{\partial R}\delta R\right],$$

which can be rewritten as

$$\delta S = \frac{1}{16\pi}\Big[\underbrace{\int d^4x f(R)\delta\sqrt{-g}}_{❶} + \underbrace{\int d^4x\sqrt{-g}\frac{\partial f}{\partial R}g^{\mu\nu}\delta R_{\mu\nu}}_{❷} + \underbrace{\int d^4x\sqrt{-g}\frac{\partial f}{\partial R}R_{\mu\nu}\delta g^{\mu\nu}}_{❸}\Big]. \tag{13}$$

Let us consider the three terms in eq. (13) separatly.

❶ In the first term we find

$$\partial\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g. \tag{14}$$

The variation of the determinant g can be treated properly if we take into account that it is always possible to change in a coordinate frame where $g_{\mu\nu}$ is diagonal. In that case, we get

$$\begin{aligned}\delta g &= \delta\left(\prod_\nu g_{\nu\nu}\right) = (\delta g_{11})g_{22}g_{33}g_{44} + g_{11}(\delta g_{22})g_{33}g_{44} + \dots \\ &= g\, g^{\mu\nu}\delta g_{\mu\nu}.\end{aligned} \tag{15}$$

Thus eq. (14) becomes

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}g(g^{\mu\nu}\delta g_{\mu\nu}) = \frac{1}{2}\sqrt{-g}\,(g_{\mu\nu}\delta g^{\mu\nu})\,. \tag{16}$$

❷ In the second term $g^{\mu\nu}\delta R_{\mu\nu}$ needs to be written in terms of variations over Christoffel symbols using the definition of the Riemann-Tensor

$$R^\rho{}_{\mu\lambda\nu} = \Gamma^\rho_{\nu\mu,\lambda} - \Gamma^\rho_{\lambda\mu,\nu} + \Gamma^\rho_{\lambda\sigma}\Gamma^\sigma_{\nu\mu} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\lambda\mu}$$

and variations of the connections that lead to a transformation like

$$\Gamma^\rho_{\nu\mu} \to \Gamma^\rho_{\nu\mu} + \delta\Gamma^\rho_{\nu\mu}.$$

Since $\delta\Gamma$ is also a 4-dimensional tensor we can build the covariant derivative

$$\delta\Gamma^\rho_{\nu\mu;\lambda} = \delta\Gamma^\rho_{\nu\mu,\lambda} - \Gamma^\sigma_{\lambda\mu}\delta\Gamma^\rho_{\nu\sigma} - \Gamma^\sigma_{\nu\lambda}\delta\Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\lambda\sigma}\delta\Gamma^\sigma_{\nu\mu}.$$

After contracting the tensor, we get for variation of the Ricci-Tensor

$$\delta R_{\mu\nu} = \delta\Gamma^\lambda_{\nu\mu;\lambda} - \delta\Gamma^\lambda_{\lambda\mu;\nu}.$$

Multiplying by $g^{\mu\nu}$ and changing some dummy indices, this becomes

$$g^{\mu\nu}\delta R_{\mu\nu} = \left(g^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^{\lambda}_{\lambda\mu}\right)_{;\sigma}. \tag{17}$$

Now we are using the definition of the Christoffel symbol and (17) becomes

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= -\frac{1}{2}\Big[g^{\mu\nu}(g_{\lambda\mu}(\delta g^{\lambda\sigma}_{;\nu}) + g_{\lambda\nu}(\delta g^{\lambda\sigma}_{;\mu}) - g_{\mu\alpha}g_{\nu\beta}(\delta g^{\alpha\beta;\sigma})) \\ &\quad -g^{\mu\sigma}(g_{\nu\lambda}(\delta g^{\nu\lambda}_{;\mu}) + g_{\nu\mu}(\delta g^{\nu\lambda}_{;\lambda}) - g_{\lambda\alpha}g_{\mu\beta}(\delta g^{\alpha\beta;\lambda}))\Big] \qquad (18) \\ &= [g_{\mu\nu}(\delta g^{\mu\nu;\sigma}) - (\delta g^{\lambda\sigma}_{;\lambda})]_{;\sigma} \equiv X^{\sigma}_{;\sigma}. \qquad (19) \end{aligned}$$

For any vector field X, we have

$$X^{\sigma}_{;\sigma} = \frac{1}{\sqrt{-g}}(\sqrt{-g}X^{\sigma})_{,\sigma}. \tag{20}$$

Hence,

$$0 = \int d^4x \left(\sqrt{-g}\frac{\partial f}{\partial R}X^{\sigma}\right)_{,\sigma} = \int d^4x\sqrt{-g}\left[\left(\frac{\partial f}{\partial R}\right)_{,\sigma} X^{\sigma} + \frac{\partial f}{\partial R}X^{\sigma}_{;\sigma}\right] \tag{21}$$

and

$$\int d^4x\sqrt{-g}\frac{\partial f}{\partial R}g^{\mu\nu}\delta R_{\mu\nu} = -\int d^4x\sqrt{-g}\left(\frac{\partial f}{\partial R}\right)_{;\sigma}[g_{\mu\nu}(\delta g^{\mu\nu;\sigma}) - (\delta g^{\lambda\sigma}_{;\lambda})]. \tag{22}$$

Using

$$\delta g^{\lambda\sigma}_{;\lambda} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\delta g^{\lambda\sigma})_{,\lambda} + \Gamma^{\sigma}_{\lambda\beta}\delta g^{\lambda\beta}, \tag{23}$$

we find that

$$\int d^4x\sqrt{-g}f_{,R;\sigma}\delta g^{\lambda\sigma}_{;\lambda} = -\int d^4x\sqrt{-g}f_{,R;\sigma;\lambda}\delta g^{\lambda\sigma}. \tag{24}$$

Similarly,

$$-\int d^4x\sqrt{-g}f_{,R;\sigma}g_{\mu\nu}\delta g^{\mu\nu;\sigma} = \int d^4x\sqrt{-g}f^{;\sigma}_{,R;\sigma}g_{\mu\nu}\delta g^{\mu\nu}. \tag{25}$$

❸ Fortunately this term is already a variation of the metric.

Finally our efforts pay off and the terms ❶ (eq. (16)),❷, and ❸ are giving the action

$$\delta S = \frac{1}{16\pi}\left\{\int d^4x\sqrt{-g}g_{\beta\mu} \times \left[\frac{\partial f}{\partial R}R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}f + \left(\frac{\partial f}{\partial R}\right)^{;\sigma}_{;\sigma}\delta^{\mu}_{\nu} - \left(\frac{\partial f}{\partial R}\right)^{;\mu}_{;\nu}\right] \times \delta g^{\beta\nu}\right\} = 0. \tag{26}$$

This gives us the equation of motion:

$$f_{,R}R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}f + (f_{,R})^{;\alpha}_{;\alpha}\delta^{\mu}_{\nu} - (f_{,R})^{;\mu}_{;\nu} = 0. \tag{27}$$

(ii) The FRW metric with coordinates (x^0, x^1, x^2, x^3) for $k = 0$ is given by $g_{\mu\nu} = \mathrm{diag}\left(-1, a^2(t), a^2(t), a^2(t)\right)$. Therewith one can calculate the diagonal components of the Ricci tensor as well as the Ricci scalar:

$$\begin{aligned} R_{00} &= -\frac{3\ddot{a}}{a} \qquad (28) \\ R_{ii} &= a\ddot{a} + 2\dot{a}^2,\ i = 1,2,3 \qquad (29) \\ R &= \frac{6}{a^2}(a\ddot{a} + \dot{a}^2). \qquad (30) \end{aligned}$$

By plugging in the given expression $f(R) = R + F(R)$ into the equation of motion in vacuum (27), we obtain:

$$(1+F_{,R})R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R - \frac{1}{2}\delta^{\mu}_{\nu}F + (1+F_{,R})^{;\alpha}_{;\alpha}\delta^{\mu}_{\nu} - (1+F_{,R})^{;\mu}_{;\nu} = 0. \tag{31}$$

In the presence of matter, we need to take into account contributions from the energy-momentum tensor. From the variation of the matter action S_m, we know that

$$8\pi T_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{1}{2}\frac{\delta S_m}{\delta g^{\mu\nu}} = 8\pi[(\rho+p)u_\mu u_\nu + g_{\mu\nu}p]. \tag{32}$$

Thus eq. (31) for the $(0,0)$-component becomes:

$$\frac{3\ddot{a}}{a}(1+F_{,R}) - \frac{1}{2}(6\frac{\ddot{a}}{a} + 6H^2) - \frac{1}{2}F + (1+F_{,R})^{;\alpha}_{;\alpha} - (1+F_{,R})^{;0}_{;0} = -8\pi\rho \tag{33}$$

$$H^2 + \frac{F}{6} - \frac{\ddot{a}}{a}F_{,R} - \frac{1}{3}(1+F_{,R})^{;i}_{;i} = \frac{8\pi\rho}{3}, \tag{34}$$

where we used equations (28) to (30). The second order covariant derivative $(F_{,R})^{;i}_{;i}$ is

$$\underbrace{\frac{\partial(g^{i\nu}(F_{,R})_{,\nu})}{\partial x^i}}_{=0} + \Gamma^i_{i\sigma}g^{\sigma\nu}(F_{,R})_{,\nu}. \tag{35}$$

It is giving us only non zero terms for time derivatives, all spatial once are vanishing due to the independence of F from spatial coordinates. For the same reason the righthand term in (35) prevails for $\Gamma^i_{i0}g^{00}(F_{,R})_{,0}$. With $\Gamma^i_{i0} = \frac{\dot{a}}{a}$ and $g^{00} = -1$ follows

$$(1+F_{,R})^{;i}_{;i} = -3H\dot{F}_{,R},$$

where we took the sum over i. Consequently we obtain the first Friedmann equation from (34):

$$H^2 + \frac{F}{6} - \frac{\ddot{a}}{a}F_{,R} + H\dot{F}_{,R} = \frac{8\pi\rho}{3}. \tag{36}$$

The second Friedmann equation can be obtained by calculating the spatial (i,i)-component and using the first Friedmann equation to simplify the expression. Alternatively, we can use the relation $T_{00} - \frac{1}{2}g_{00}T = \frac{1}{2}(\rho + 3p)$ to infer the second Friedmann equation:

$$\frac{\ddot{a}}{a} + \frac{1}{6}F - H^2F_{,R} + \frac{1}{2}H\dot{F}_{,R} + \frac{1}{2}\ddot{F}_{,R} = -4\pi(p + \frac{\rho}{3}). \tag{37}$$