Supersymmetric invariant actions: generalities

Re-cap from last lecture:

(1) SUSY $\equiv$ generalized "translation" symmetry in SUPERSPACE

$$x^\mu \longleftrightarrow e^{x^\mu P_\mu}$$

$$\Updownarrow$$

$$\{x^\mu, \vartheta_\alpha, \bar\vartheta_{\dot\alpha}\} \longleftrightarrow e^{x^\mu P_\mu}\, e^{(\vartheta Q + \bar\vartheta\bar Q)}$$

(SUSY Algebra equiv to a Lie Algebra in superspace)

$$[\vartheta Q, \bar\vartheta\bar Q] = 2\vartheta\sigma^\mu\bar\vartheta P_\mu$$

$$[\vartheta Q, \vartheta Q] = [\bar\vartheta\bar Q, \bar\vartheta\bar Q] = 0 \qquad + \text{Poincaré}$$

(2) SUPERFIELDS = fields in superspace (that naturally have both fermion & boson components)

$$Y(x,\vartheta,\bar\vartheta) = f(x) + \vartheta\psi(x) + \bar\vartheta\bar\chi(x) + \dots \qquad \vartheta\vartheta\bar\vartheta\bar\vartheta\, d(x)$$

ordinary translation:

$x^\mu \to x^\mu + a^\mu$

$$\delta_a\phi = \phi(x+a) - \phi(x) = i a^\mu P_\mu \phi = a^\mu \partial_\mu \phi$$

$$P_\mu \equiv -i\frac{\partial}{\partial x^\mu}$$

susy transformation:

$\vartheta^\alpha \to \vartheta^\alpha + \epsilon^\alpha$

$\bar\vartheta^{\dot\alpha} \to \bar\vartheta^{\dot\alpha} + \bar\epsilon^{\dot\alpha}$

$x^\mu \to x^\mu + i(\vartheta\sigma^\mu\bar\epsilon - \epsilon\sigma^\mu\bar\vartheta)$

$$\delta_{\epsilon,\epsilon'} Y = (i\epsilon Q + i\bar\epsilon\bar Q) Y$$

Diff. repr. of Q & $\bar Q$

$$\begin{cases} Q_\alpha \equiv -i\dfrac{\partial}{\partial\vartheta^\alpha} - \sigma^\mu_{\alpha\dot\beta}\bar\vartheta^{\dot\beta}\dfrac{\partial}{\partial x^\mu} \\[2ex] \bar Q_{\dot\alpha} \equiv +i\dfrac{\partial}{\partial\bar\vartheta^{\dot\alpha}} + \vartheta^\beta\sigma^\mu_{\beta\dot\alpha}\dfrac{\partial}{\partial x^\mu} \end{cases}$$

(3) SUSY invariant actions (general principle)

→ Key observation : the integral in superspace of any superfield is a SUSY invariant quantity
⇒ good candidate for an action

$$\int d^4x\, d^2\vartheta\, d^2\bar{\vartheta}\; Y(x,\vartheta,\bar{\vartheta}) = I$$

by construction, integrals on Grassman variables are invariant under a shift of variables

- $\int d\vartheta\, \vartheta = 1$
- $\int d\vartheta\, f(\vartheta+\xi) = \int d\vartheta' f(\vartheta') = 1$

$$\Downarrow$$

$$\delta_{\epsilon,\epsilon'} I = \int d^4x\, d^2\vartheta\, d^2\bar{\vartheta}\; \Big[\underbrace{Y(x',\vartheta',\bar{\vartheta}') - Y(x,\vartheta,\bar{\vartheta})}_{\delta_{\epsilon\epsilon'} Y(x,\vartheta,\bar{\vartheta})} \Big]$$

$$\delta_{\epsilon\epsilon'} Y(x,\vartheta,\bar{\vartheta}) = \underbrace{\epsilon^\alpha \partial_\alpha Y + \bar{\epsilon}_{\dot\alpha}\bar{\partial}^{\dot\alpha} Y}_{\emptyset \text{ in the integral (not enough powers of } \vartheta \,\&\, \bar{\vartheta})} + \underbrace{\partial_\mu\left[-i(\epsilon\sigma\bar{\vartheta} - \vartheta\sigma\bar{\epsilon})Y\right]}_{\text{total derivative}}$$

$$\Rightarrow \boxed{\delta_{\epsilon,\epsilon'} I = 0}$$

⇒ To form possible generic SUSY invariant actions we will consider products of superfields (again a superfield) and integrate on superspace

N.B.: Only the $\vartheta^2\bar{\vartheta}^2$ component of the product of SUSY field will finally contribute to the action

$$\int d^4x\, d^2\vartheta\, d^2\bar{\vartheta}\; Y(x,\vartheta,\bar{\vartheta}) = \int d^4x \; Y(x,\vartheta,\bar{\vartheta})\Big|_{\vartheta^2\bar{\vartheta}^2\text{-comp.}}$$

N.B.: Imposing appropriate SUSY INVARIANT CONSTRAINTS on the superfields we select irreducible repr. of the SUSY algebra, starting from which we build our actions.

2 - basic types of SUSY FIELDS $\nearrow$ CHIRAL FIELDS $(\phi, \Psi_\alpha, F) \Leftrightarrow (S=0)_{2B} + (S=\tfrac{1}{2})_{2F}$

$\searrow$ VECTOR FIELDS (REAL ″) $\Leftrightarrow (S=1)_{2B} + (S=\tfrac{1}{2})_{2F}$

Re-cap on chiral superfield:

I) Define
$$\begin{cases} D_\alpha = \partial_\alpha - i\,\sigma^\mu_{\alpha\dot\beta}\,\bar{\vartheta}^{\dot\beta}\,\partial_\mu \\ \bar{D}_{\dot\alpha} = \bar{\partial}_{\dot\alpha} + i\,\vartheta^\beta\,\sigma^\mu_{\beta\dot\alpha}\,\partial_\mu \end{cases}$$

$\hookrightarrow$
$$\{D_\alpha, \bar{D}_{\dot\beta}\} = 2i\,\sigma^\mu_{\alpha\dot\beta}\,\partial_\mu = \ominus\, 2\,\sigma^\mu_{\alpha\dot\beta}\, P^\mu$$

$$\{D_\alpha, D_\beta\} = \{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot\beta}\} = 0$$
$$\{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = \{\bar{D}_{\dot\alpha}, Q_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

$$\Downarrow$$

$$\delta_{\epsilon\epsilon'}(D_\alpha Y) = D_\alpha(\delta_{\epsilon,\epsilon'} Y)$$

$$\Downarrow$$

if Y is a superfield $\Rightarrow$ $D_\alpha Y$ & $\bar{D}_{\dot\alpha} Y$ are also superfields

II) Impose the constraint:

$$\boxed{\bar{D}_{\dot{\alpha}}\, \Phi = 0} \quad \Rightarrow \quad \Phi \equiv \text{CHIRAL SUPERFIELD}$$

$$\boxed{D_{\alpha}\, \bar{\Psi} = 0} \quad \Rightarrow \quad \bar{\Psi} = \text{ANTI-CHIRAL} \quad \text{"}$$

Since $(\bar{D}_{\dot{\alpha}}\, \phi)^{+} = D_{\alpha}\, \phi^{+} \equiv D_{\alpha}\, \bar{\Phi}$ $\Rightarrow$ $\bar{\Phi}$ is necessarily antichiral if ϕ is chiral

($\rightarrow$ exercise)

$\Downarrow$

With a little of algebra it turns out that only few components of the generic superfield survive

$$\Phi(y, \vartheta) = \underbrace{\varphi(y)}_{\text{scalar}} + \sqrt{2}\, \vartheta\, \underbrace{\psi(y)}_{\text{spinor}} - \vartheta\vartheta\, \underbrace{F(y)}_{\text{aux.}}$$

$$y^{\mu} = x^{\mu} + i\, \vartheta^{\alpha}\, \sigma^{\mu}{}_{\alpha\dot{\beta}}\, \bar{\vartheta}^{\dot{\beta}}$$

$$\left(\bar{\Phi}(\bar{y}, \bar{\vartheta}) = \bar{\varphi}(\bar{y}) + \sqrt{2}\, \bar{\vartheta}\, \bar{\psi}(y) - \bar{\vartheta}\bar{\vartheta}\, \bar{F}(\bar{y}) \right)$$

$$\begin{cases} \delta\varphi = \sqrt{2}\, \epsilon\, \psi \\ \delta\psi_{\alpha} = \sqrt{2}\, i\, (\sigma^{\mu}\bar{\epsilon})_{\alpha}\, \partial_{\mu}\varphi - \sqrt{2}\, \epsilon_{\alpha} F \\ \delta F = i\sqrt{2}\, \partial_{\mu}\, \psi\, \sigma^{\mu}\epsilon \end{cases}$$

$\leftarrow$ N.B. total derivative in ordinary Lorentz space

As we shall see when constructing explicitly the Lagrangian, F turns out to be a non-dynamical field

$\Downarrow$

N.B.: When expressing the field in terms of $x, \vartheta, \bar{\vartheta}$ we get

$$\begin{aligned} \phi(x, \vartheta, \bar{\vartheta}) = {} & \varphi(x) + i\, \vartheta\sigma^{\mu}\bar{\vartheta}\, \partial_{\mu}\varphi(x) - \frac{1}{4}\vartheta\vartheta\,\bar{\vartheta}\bar{\vartheta}\, \Box\varphi(x) \\ & + \sqrt{2}\, \vartheta\psi(x) - \frac{i}{\sqrt{2}}\, \vartheta\vartheta\, \partial_{\mu}\psi(x)\, \sigma^{\mu}\bar{\vartheta} \\ & - \vartheta\vartheta\, F(x) \end{aligned}$$

The VECTOR SUPER FIELD :

⇒ We want something that projects out the v_μ component ($s=1$) in the generic super field Y

$$Y = f(x) + \vartheta\psi(x) + \dots \quad \underbrace{\vartheta\sigma^\mu\bar\vartheta}_{\uparrow\ \text{hermitian term}}\, v_\mu(x) + \dots$$

$$\boxed{V = V^+} \Rightarrow \text{VECTOR (OR REAL) SUPER FIELD}$$

↓

$$V(x,\vartheta,\bar\vartheta) = C(x) + i\vartheta\chi(x) - i\bar\vartheta\bar\chi(x) + \vartheta\sigma^\mu\bar\vartheta\, v_\mu(x)$$
$$+ \frac{i}{2}\vartheta\vartheta\left[M(x)+iN(x)\right] - \frac{i}{2}\bar\vartheta\bar\vartheta\left[M(x)-iN(x)\right]$$
$$+ \vartheta\vartheta\bar\vartheta\left[\bar\lambda(x) + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi(x)\right] - i\bar\vartheta\bar\vartheta\vartheta\left[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right]$$
$$+ \frac{1}{2}\vartheta\vartheta\bar\vartheta\bar\vartheta\left[D(x) - \frac{1}{2}\partial^2 C(x)\right]$$

$C, M, N, D \oplus v^\mu$ = real = $4_B + 4_B = 8_B$

$\chi \oplus \lambda$ = Compl. Spinors = $4_F + 4_F = 8_F$

→ $4_B + 4_F$ will be eliminated by a suitable GAUGE-FIXING

↳ $2_B + 2_F$ " " going on-shell (e.o.m.)

Gauge transformations :

If ϕ is a chiral superfield ⇒ $\phi + \bar\phi$ is a vector superfield.

Let's now consider the transformation

$$V \longrightarrow V + (\phi + \bar\phi) \qquad (*)$$

Recall that

$$\phi(x,\theta,\bar\theta)\Big|_{\theta\sigma^\mu\bar\theta\text{-comp.}} = \varphi(\underset{\substack{\uparrow\\ x^\mu + i\theta\sigma^\mu\bar\theta}}{y})\Big|_{\theta\sigma^\mu\bar\theta\text{-comp.}} = i\theta\sigma^\mu\bar\theta\,\partial_\mu\varphi(x)$$

$\Rightarrow$ Then the gauge transformation implies:

$$v_\mu \to v_\mu - \partial_\mu(2\,\mathrm{Im}\,\varphi) \quad \Rightarrow \text{ like Abelian gauge transform.}$$

More generally, one finds:

$$\begin{aligned} C &\to C + 2\,\mathrm{Re}\,\varphi & D &\to D \\ \chi &\to \chi - i\sqrt{2}\,\psi & \lambda &\to \lambda \\ M &\to M - 2\,\mathrm{Im}\,F & v^\mu &\to v^\mu - 2\partial^\mu \mathrm{Im}\,\varphi \\ N &\to N + 2\,\mathrm{Re}\,F & & \end{aligned}$$

$\Rightarrow$ with a proper choice of Φ, we can "gauge-away" (i.e. set to 0) $C, \chi, M, N \oplus 1$ component of v^μ

This is the so-called Wess-Zumino gauge $(4_B + 4_F)$

$$V\Big|_{WZ} = \theta\sigma^\mu\bar\theta\, v_\mu(x) + i\theta\theta\,\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\,\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta\, D(x)$$

N.B.

1) The WZ gauge is NOT a SUSY-invariant condition
2) For practical purposes, it is useful to work in a modified WZ gauge where $C = M = N = \chi = 0$, but v^μ is not restricted (same decomposition as above)
3) Since V_{WZ} starts with terms linear in θ & $\bar\theta$, it follows that

$$V_{WZ}^2 = \frac{1}{2}\theta\theta\bar\theta\bar\theta\, v_\mu v^\mu \qquad W_{WZ}^{n>2} = \emptyset$$