MMP I
Final test

HS 2019
Prof. M. Grazzini

S. Devoto, M. Höfer, J. Yook

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https://www.physik.uzh.ch/en/teaching/PHY312/HS2019.html

This test consists of 8 questions. You are not expected to solve all of them. The various parts are often independent, so, for example, you can attempt to solve (b) even if you have not managed to solve (a).

Vorname/First name	
Name/Surname	
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1	2	3	4	5	6	7	8	Total

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Exercise 1: Fourier series (5 Pts.)

Let g be the periodic extension of f defined by:

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x \leq 0 \\ \sin x & \text{if } \quad 0 < x \leq \pi \end{cases}$$

a) Find the Fourier series for $f(x)$.

b) Discuss the convergence of the Fourier series you found in part a). Does the series converge to $f(x)$ everywhere? Justify your answer using the convergence theorem.

c) Use the result of part a) and b) to compute the following series:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \, .$$

Exercise 2: Variational calculus (6 Pts.)

A rope hangs between the two points $(x, y) = (\pm a, 0)$ in a curve $y = y(x)$, so as it minimises its potential energy

$$U = \int_{-a}^{a} m\, g\, y \sqrt{1 + y'^{\,2}}\, dx$$

while keeping constant its length:

$$L = \int_{-a}^{a} \sqrt{1 + y'^{\,2}}\, dx \, ,$$

with $L > 2a$.

a) Write the Euler equation.

b) Show that, when minimising a functional $F(y'(x), y(x))$ independent on x, the following relation holds:

$$F - y' \frac{\partial F}{\partial y'} = \text{constant} \, .$$

c) By using the result of part b), find the solution of the Euler equation in a).

Exercise 3: Differential equations (7 Pts.)

Find the general solutions to the following differential equations:

a) $y' = \left(1 - \frac{2}{x}\right) y + x^2 e^x$;

b) $y' = -\frac{x}{y}$;

c) $y' = (1 + y)\sin x$;

d) $y''' + y' = x$.

Consider the differential equation:

e) $x\,y\,dx + \left(x^2 + 2y^2\right) dy = 0$.

Show that

$$\frac{1}{\sqrt{x^2 + y^2}}$$

is an integrating factor and find the general solution $y(x)$ of the differential equation.

Exercise 4: Abstract spaces (3 Pts.)

Let X be a vector space and let $\mathcal{N}$ be the set of all norms on X. We define a relation $\sim$ on $\mathcal{N}$ by saying that $||\cdot||_1 \sim ||\cdot||_2$ if and only if there exist $m, M > 0$ such that

$$m||x||_1 \leq ||x||_2 \leq M||x||_1$$

for all $x \in X$. Show that $\sim$ is an equivalence relation.

Exercise 5: Linear operators (6 Pts.)

Let H_1 and H_2 two Hilbert spaces and $T : H_1 \to H_2$ an arbitrary linear operator.

a) Show that if T is bounded then it is also continuous.

b) Show that if H_1 is finite dimensional then T is bounded.

Exercise 6: Differential equation systems (6 Pts.)

Solve the following differential equation systems (write the solutions in their real representations in all cases):

a) $\begin{cases} \dot{x} = -x + 2y + 2z \\ \dot{y} = -2x + 3y + 2z \\ \dot{z} = 2x - 2y - z \end{cases}$ with $\begin{cases} x(0) = 1 \\ y(0) = 0 \\ z(0) = 0 \end{cases}$.

b) $\begin{cases} \dot{x} = x \\ \dot{y} = -x + 2y + 2z \\ \dot{z} = x + z \end{cases}$.

Exercise 7: Orthonormal bases (5 Pts.)

Consider the Hilbert space $L^2[0,1]$. Define the vectors $x_i = t^i$, $0 \leq t \leq 1$, $i = 0, 1, \ldots$

a) Show that the x_i are linearly independent.

b) Use the Gram-Schmid procedure to construct the first four vectors of the corresponding orthonormal system ϕ_i of $L^2[0,1]$.

c) Discuss the relation of the ϕ_i with the Legendre polynomials.

Exercise 8: Integral operators (6 Pts.)

Consider the Hilbert space $H = L^2[-1,1]$ and the linear operator

$$|x\rangle \to |y\rangle = T|x\rangle \sim Tx(t) = \int_{-1}^{1} K(t,s)x(s)ds\,,$$

where $K(t,s) = 1 + st$.

a) Show that if $x \in H$ then $y \in H$.

b) Determine all the eigenvalues and eigenvectors of T.

c) What is the dimension of the kernel of T? And of its range? (note: $\text{Ker}(T) = \{x \in H \,|\, Tx = 0\}$, $\text{Ran}(T) = \{y \in H \,|\, y = Tx \text{ with } x \in H\}$)