(N=1) SUPER YANG-MILLS ACTION

We now want to build an action for the vector field V, aiming to include the YM action for it's v^μ component

Starting point (Abelian case):

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\,D_\alpha V \qquad \bar{W}_{\dot\alpha} = -\frac{1}{4}DD\,\bar{D}_{\dot\alpha} V$$

* These are clearly superfields, actually they are chiral (anti chiral), since $\bar{D}^3 = 0$ $(D^3=0)$

* They are also invariant under the (Abelian) gauge transformation

$$V \longrightarrow V + (\Phi + \bar\Phi)$$

$$W_\alpha \longrightarrow W_\alpha - \frac{1}{4}\bar{D}\bar{D}\,D_\alpha(\Phi+\bar\Phi) = W_\alpha - \frac{1}{4}\bar{D}^2 D_\alpha \Phi$$

$$= W_\alpha + \frac{1}{4}\bar{D}^{\dot\beta}\{\bar{D}_{\dot\beta}, D_\alpha\}\Phi = W_\alpha + \frac{i}{2}\sigma^\mu_{\alpha\dot\beta}\,\partial_\mu\,\bar{D}^{\dot\beta}\Phi$$

$$= W_\alpha$$

* Being gauge-invariant, we can stick to the WZ gauge, where V has a particularly simple structure $\{\lambda, v^\mu, D\}$
In this gauge, using $(y, \theta, \bar\theta)$ coordinates,

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha\cdot D + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta\,(\sigma^\mu\partial_\mu\bar\lambda)_\alpha$$

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$$

$\Rightarrow$ Good candidate to build the action

(recall $y^\mu = x^\mu + i\,\theta^\alpha\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}$)

$\Rightarrow$ Natural guess for the action

$$+\frac{1}{2}\int d\vartheta^2\, W^\alpha W_\alpha = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Abelian Y.M.}} \underbrace{- i\lambda\sigma^\mu\partial_\mu\bar\lambda}_{\text{Weyl fermion}} \underbrace{+\frac{1}{2}D^2}_{\text{aux. (e.o.m } D=0} \underbrace{+\frac{i}{8}\tilde F_{\mu\nu}F^{\mu\nu}}_{\text{non-hermitian term}}$$

Full Abelian SUSY Y.M. :

$$\frac{1}{4}\left(\int d\vartheta^2 W^\alpha W_\alpha + \int d\bar\vartheta^2\, \bar W^{\dot\alpha}\bar W_{\dot\alpha}\right) =$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar\lambda + \frac{1}{2}D^2$$

<u>N.B.:</u> Despite we have written the action as an integral on $\int d\vartheta^2$, it is not an F-term, it can indeed be re-written as

$$\int d^2\vartheta\, \underbrace{W^\alpha W_\alpha}_{-\frac{1}{4}\bar D\bar D D^\alpha V} = \int d^2\vartheta\, d^2\bar\vartheta\, \underbrace{(D^\alpha V)}_{\text{antichiral}}\, \underbrace{W_\alpha}_{\text{chiral}}$$

$$\int d^2\vartheta\, d^2\bar\vartheta\, f(\vartheta,\bar\vartheta) = \int d^2\vartheta\left(\bar D^2 f + \text{total deriv.}\right)$$

<u>Non-Abelian case</u> :

* Non-Abelian group G $\rightarrow$ generators T^a

$\Rightarrow$ we introduce V_a vector superfield and define

$\hat V = V_a T^a$

(*) Non-Abelian gauge transf :

$$e^{\hat V} \to e^{i\bar\Lambda} e^{\hat V} e^{-i\Lambda} \qquad \Lambda = \phi_a T^a$$

↳ it's clear that expanding in Λ this is equivalent to

$$V \to V - i(\Lambda - \bar\Lambda) = V + (\phi_\Lambda + \bar\phi_\Lambda) \qquad \phi_\Lambda = -i\Lambda$$

Note that in the WZ gauge

$$e^{\hat V} = 1 + \hat V + \frac{1}{2}\hat V^2 \qquad (\text{no more terms since } V^3|_{WZ} = 0)$$

(*) The analog of W_α for the Abelian case is

$$\hat W_\alpha = -\frac{1}{4}\bar D\bar D\left(e^{-\hat V} D_\alpha e^{\hat V}\right) \qquad \hat{\bar W}_{\dot\alpha} = -\frac{1}{4} DD\left(e^{+\hat V}\bar D_{\dot\alpha} e^{-\hat V}\right)$$

we can indeed show that $\hat W_\alpha$ transforms linearly under gauge transformations :

$$\hat W_\alpha \to -\frac{1}{4}\bar D\bar D\left[e^{i\Lambda} e^{-\hat V} e^{-i\bar\Lambda} D_\alpha\left(e^{+i\bar\Lambda} e^{V} e^{-i\Lambda}\right)\right]$$

$$D_\alpha e^{i\bar\Lambda} = \phi$$
$$\bar D_{\dot\alpha} e^{i\Lambda} = \phi$$

$$= -\frac{1}{4}\bar D\bar D\left[e^{i\Lambda} e^{-\hat V}\left(D_\alpha e^{V}\right) e^{-i\Lambda} + e^{i\Lambda}\left(D_\alpha e^{-i\Lambda}\right)\right]$$

$$= -\frac{1}{4} e^{i\Lambda}\left[\bar D\bar D\left(e^{-\hat V} D_\alpha e^{V}\right)\right] e^{-i\Lambda}$$

$$= -\frac{1}{4} e^{+i\Lambda} W_\alpha\, e^{-i\Lambda}$$

* With a little bit of Algebra, using $(y, \theta, \bar\theta)$ coordinates and the WZ gauge, one finds

$$\hat{W}_\alpha = -i\hat{\lambda}_\alpha + \theta_\alpha \hat{D} + i(\sigma^{\mu\nu}\theta)_\alpha \hat{F}_{\mu\nu} + \theta\theta(\sigma^\mu \hat{D}_\mu \hat{\bar\lambda})_\alpha$$

gauge covariant derivative

$\hat\lambda = \lambda_a T^a$, etc...

$$\left(\partial_\mu v_\nu^a - \partial_\nu v_\mu^a - \tfrac{1}{2} f^{abc}[v_\mu^b, v_\nu^c]\right) T^a$$

From now on I will omit the $\wedge$ over non-Abelian fields (contraction with generators is understood)

$$\Downarrow$$

* In order to recover the Y.M. Lagrangian is convenient to re-define the field, introducing the coupling g

$$V \to 2gV \quad \Rightarrow \quad \begin{cases} v_\mu \to 2g v_\mu & D \to 2gD \\ \lambda \to 2g\lambda \end{cases}$$

$$F_{\mu\nu} \to g\left(\partial v_\nu - \partial_\nu v_\mu - ig[v_\mu, v_\nu]\right)$$

$$\mathcal{L}_{SYM} = \frac{1}{4g^2}\left[\int d^2\theta \, \mathrm{Tr}(W^\alpha W_\alpha) + \int d^2\bar\theta \, \mathrm{Tr}(\bar W^{\dot\alpha} \bar W_{\dot\alpha})\right]$$

$$= \mathrm{Tr}\left[-\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu \bar\lambda + \frac{1}{2}D^2\right]$$

$\mathrm{Tr}(T^aT^b) = \delta^{ab}$

<u>N.B:</u> possible CP-odd term $(F\tilde F)$ introduced if g is complex (not discussed in this lectures)

SYM with matter (or chiral fields)

We now want to couple the gauge field to some chiral field → As first step, we need to define the gauge transf. of the chiral field:

$$\phi \to \phi' = e^{i\Lambda}\phi \qquad \Lambda = \Lambda_a T^a_R$$

↑ chiral superfield (gauge term)

Easy to show that this is equivalent to ordinary gauge transformations on φ & ψ

⇓

The Kähler potential $\bar\phi\phi$ is not gauge invariant.

$$\bar\phi\phi \to \bar\phi\, e^{-i\bar\Lambda} e^{i\Lambda}\phi \neq \bar\phi\phi$$

but is then clear that $\bar\phi\, e^{V}\phi$ is gauge inv. and it reduces to $\bar\phi\phi$ in the limit $V \to \emptyset$

⇓

natural candidate for the action (kinetic term)

$$\bar\phi\, e^{V}\phi\Big|_{WZ} = \bar\phi\phi + \bar\phi V\phi + \tfrac{1}{2}\bar\phi V^2\phi$$

$$\int d^2\theta\, d^2\bar\theta\ \bar\phi\, e^{V}\phi = \underbrace{\bar\phi\phi_{\theta^2\bar\theta^2}}_{\mathrm{I}} + \underbrace{\bar\phi V\phi_{\theta\bar\theta^2}}_{\mathrm{II}} + \underbrace{\tfrac{1}{2}\bar\phi V^2\phi_{\theta\bar\theta^2}}_{\mathrm{III}}$$

$\mathrm{I} \to \mathcal{L}_{kin}$

$\mathrm{II} \to \frac{i}{2}\bar\varphi\, v^\mu\partial_\mu\varphi - \frac{i}{2}\partial_\mu\bar\varphi\, v^\mu\varphi - \frac{1}{2}\bar\psi\,\bar\sigma^\mu v_\mu\psi + \frac{i}{\sqrt2}\bar\varphi\lambda\psi$
$\qquad - \frac{i}{\sqrt2}\bar\psi\bar\lambda\varphi + \frac{1}{2}\bar\varphi D\varphi$

$\mathrm{III} \to \frac{1}{2}\bar\varphi\, v^\mu v_\mu\varphi$

$$\int d^2\vartheta\, d^2\bar\vartheta\; \bar\Phi e^V \Phi \Big|_{WZ} = \overline{D_\mu\varphi}\, D^\mu\varphi - i\bar\psi\bar\sigma^\mu D_\mu\psi + \bar F F + \tfrac{1}{2}\bar\varphi D\varphi$$

$$+ \tfrac{i}{\sqrt{2}}\,\bar\varphi\lambda\psi - \tfrac{i}{\sqrt{2}}\,\underbrace{\bar\psi\bar\lambda\varphi}_{\bar\psi^i (T^a_R)_{ij}\bar\lambda^a\varphi^j}$$

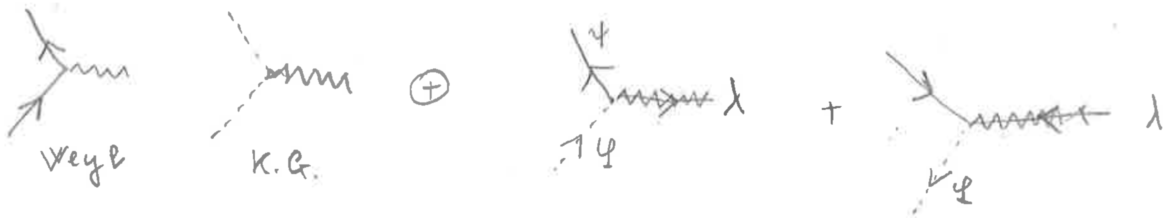
where $D_\mu = \partial_\mu - \frac{i}{2} v_\mu^a T^a_R$

performing the re-scaling $V \to 2gV$ we finally get

$$\int d^2\vartheta\, d^2\bar\vartheta\; \bar\Phi e^V \Phi \Big|_{WZ} = \underbrace{\overline{D_\mu\varphi}\, D^\mu\varphi}_{\text{K.G.}} \underbrace{- i\bar\psi\bar\sigma^\mu D_\mu\psi}_{\text{Weyl}} + \underbrace{\bar F F + g\,\bar\varphi D\varphi}_{\text{aux.}}$$

$$\underbrace{+ i\sqrt{2}\, g\left(\bar\varphi\lambda\psi - \bar\psi\bar\lambda\varphi\right)}_{\text{"gaugino's" interactions}}$$

$$D_\mu = \partial_\mu - ig\, v_\mu^a T^a_R$$



The Fayet-Iliopoulos term

In principle, for Abelian gauge fields V^A, we have an alternative option since the corresponding $\vartheta^2\bar\vartheta^2$ term transforms as a total derivative

$$V^A \to V^A - i\Lambda + i\bar\Lambda \qquad V^A\big|_{\vartheta^2\bar\vartheta^2} \doteq \tfrac{1}{2}\left(D^A + \tfrac{1}{2}\partial^2 C\right) \to V^A\big|_{\vartheta^2\bar\vartheta^2}$$

+ tot. deriv.

$$\Downarrow$$

$\mathcal{L}_{FI} = \int d^2\vartheta\, d^2\bar\vartheta\; \xi_A V^A$ is a good term for the Action

$$\| $$

$\frac{1}{2}\xi_A D^A \longrightarrow \sum_A g\,\xi_A D^A$ after re-scaling ($V \to 2gV$)

We can now put all together, assuming N chiral field, and a generic gauge group G that include some $U(1)$ terms (not semi-simple, as in the SM)

$$\mathcal{L} = \mathcal{L}_{SYM} + \mathcal{L}_{\substack{\text{chiral} \\ + \\ \text{gauge}}} + \mathcal{L}_{FI}$$

$$= \mathrm{Tr}\left[-\frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} - i \lambda \bar{\sigma}_\mu D^\mu \bar{\lambda} + \frac{1}{2} D^2 \right]_G + g \sum_A \xi_A D^A$$

$$\sum_i^N \left(\bar{D}_\mu \bar{\phi}_i D^\mu \phi_i - i \bar{\psi}_i \bar{\sigma}^\mu D_\mu \psi_i \right) + \sum_i^N \bar{F}_i F_i$$

$$+ \sum_i^N i\sqrt{2} g \left[\bar{\varphi}_i (\lambda \psi)_i - (\bar{\psi}\bar{\lambda})_i \varphi_i \right] + g \sum_i \bar{\varphi}_i (D\varphi)_i$$

$$- \frac{\partial W}{\partial \varphi_i} F^i - \frac{\partial \bar{W}}{\partial \bar{\varphi}_i} \bar{F}_i - \frac{1}{2} \left(\frac{\partial^2 W}{\partial \varphi_i \partial \varphi_j} \psi^i \psi^j + \text{h.c.} \right)$$

$\Rightarrow$ e.o.m for F & D:

$$\bar{F}_i = \frac{\partial W}{\partial \varphi_i} \qquad F_i = \frac{\partial \bar{W}_i}{\partial \bar{\varphi}_i} \qquad \text{(as before)}$$

$$D^a = -g \sum_i \bar{\varphi}_i T^a \varphi_i - g\, \xi_A \delta^{Aa}$$

N.B: we rescaled all gauge fields by the same coupling g to simplify the notation

$$\mathrm{Tr}\left[-\frac{1}{4g^2} F^{\mu\nu} F^{\mu\nu} \right]_{G = G_1 \times G_2} = \mathrm{Tr}\left[-\frac{1}{4g_1^2} F_1^{\mu\nu} F_1^{\mu\nu} \right] + \mathrm{Tr}\left[-\frac{1}{4g_2^2} F_2^{\mu\nu} F_2^{\mu\nu} \right]$$

$$g T^a \rightarrow g_1 T_1^a + g_2 T_2^a$$

Once we eliminate the auxiliary fields, the on-shell Lagrangian assumes the form

$$\mathcal{L}_s = \mathcal{L}_{kin}(\varphi_i, \psi_i, \lambda^a) + \mathcal{L}_{YM}(F_a^{\mu\nu}) + \mathcal{L}_{\text{"Yukawa"}} - V(\varphi_i, \bar{\varphi}_i)$$

$$\mathcal{L}_{\text{"Yukawa"}} = i\sqrt{2}\, g\, \bar{\varphi}_i (\lambda\psi)_i - \frac{1}{2} \frac{\partial^2 W}{\partial\varphi_i \partial\varphi_j} \psi^i \psi^j + h.c.$$

$$V(\varphi_i, \bar{\varphi}_j) = \sum_i \left| \frac{\partial W}{\partial \varphi_i} \right|^2 + \frac{g^2}{2} \sum_a \sum_{ij} \left| \bar{\varphi}_i (T^a)_{ij} \varphi_j + \xi_A \delta^{Aa} \right|^2$$

<u>N.B.</u> contribution to the scalar potential from gauge interactions