PHY127 FS2026

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Lecture 11

May 15th, 2026

from PHY 117:
angular momentum
torque
magnetic moment

NMR → MRI
(spin precession of nuclei)

Angular momentum reminder

Linear motion

momentum $\bar{p} = m\bar{v}$

$\frac{d\bar{p}}{dt} = m\frac{d\bar{v}}{dt} = m\bar{a}$ (mass constant)

Newton's 2nd law

$\Sigma \bar{F} = m\bar{a} = \frac{d\bar{p}}{dt}$

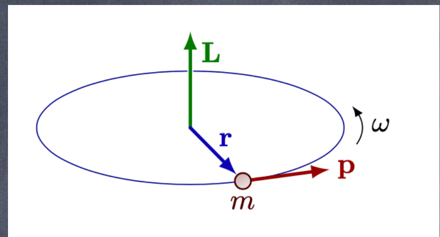
Angular motion

angular momentum $\bar{L} = \bar{r} \times \bar{p} = \bar{r} \times m\bar{v}$

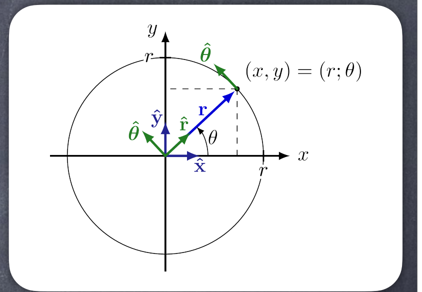
Newton's 2nd Law for rotation:

$\underbrace{\Sigma \bar{\tau}}_{\text{torques}} = \frac{d\bar{L}}{dt}$ $\quad \bar{\tau} = \bar{r} \times \bar{F}$

polar coordinates



(a) Uniform circular motion.



$\hat{\theta}$ is counter-clockwise

picture (a)

when $\bar{r} \perp \bar{V}$, then

$\bar{L} = \bar{r} \times m\bar{v} = rmv = rm\omega r$

$(rmv \sin\theta)$

$\omega = \frac{V}{r}$

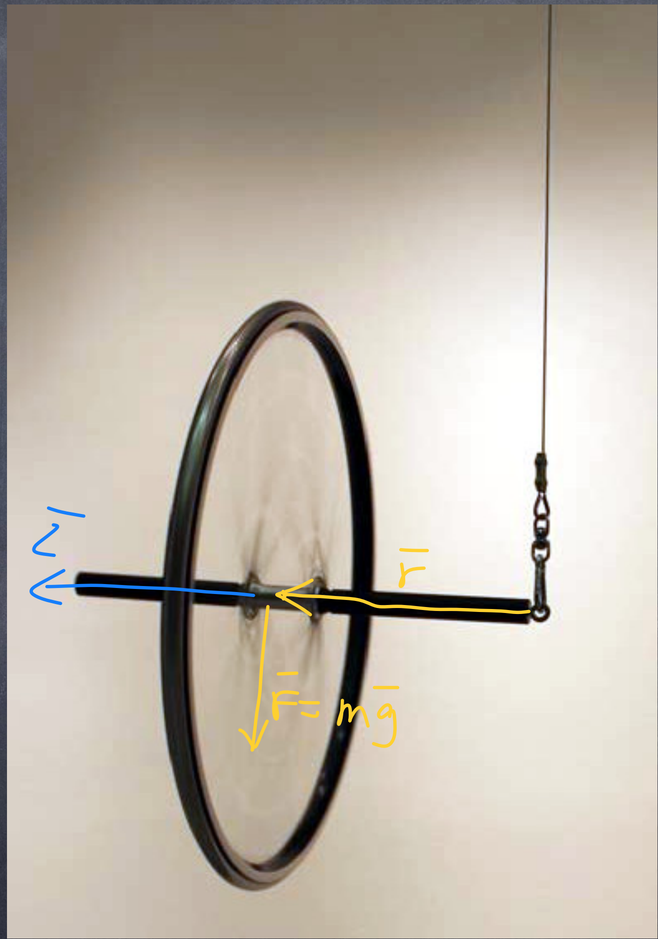
① $L = mr^2\omega$

$$\boxed{② \quad \frac{d\bar{L}}{dt} = \bar{\tau}}$$

$d\bar{L} = \bar{\tau}\, dt = (\bar{r} \times \bar{F})\, dt$

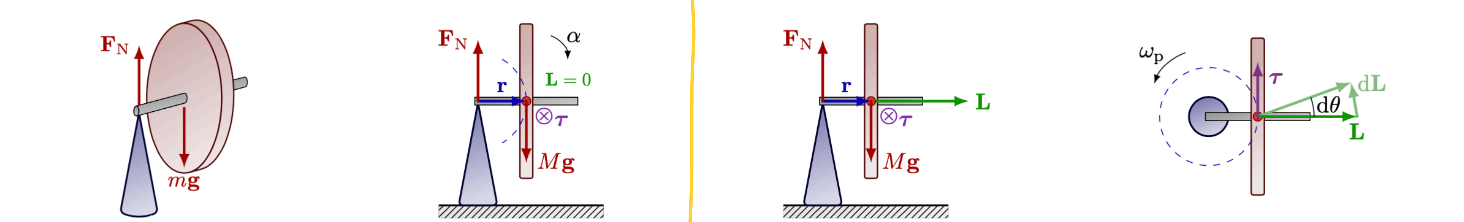
Interesting consequences

Interesting consequences



static | rotating

from above



(a) The handle allows the disk to spin around its axis and around the pivot.

(b) The disk does not spin, and it will fall down due to an unbalanced torque $\boldsymbol{\tau}$.

(c) The disk spins, creating an angular momentum **L**. Torque $\boldsymbol{\tau}$ will cause a precession.

(d) Torque $\boldsymbol{\tau}$ perpendicular to angular momentum **L**, will only change its direction.

θ changing

(a)+(b) Forces balance, torque $\bar{\tau} = \bar{r} \times m\bar{g}$, causes wheel to rotate (falls) when $\bar{L}=0$

(c) wheel spins with angular speed $\omega = \frac{v}{r}$, we have $L = mr^2\omega$ ①

(d) torque from $\bar{r}\times\bar{F}$, causes $\bar{L}$ to change direction.

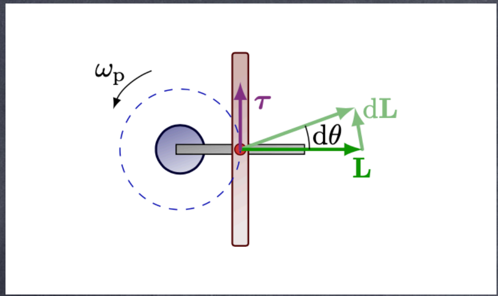
from ② $\frac{d\bar{L}}{dt} = \bar{\tau}$ $\quad d\bar{L} = \bar{\tau}\,dt = \boxed{rmg\,dt\,\hat{\theta}}$ ③

Looking at (d), $dL = L\,d\theta$ $\quad \boxed{d\theta = \frac{dL}{L}}$ ④

③ → ④, $d\theta = \frac{rmg\,dt}{L}$, so $\frac{d\theta}{dt} = \frac{rmg}{L}$

⑥ we define $\frac{d\theta}{dt} \equiv \omega_p$ (precession)

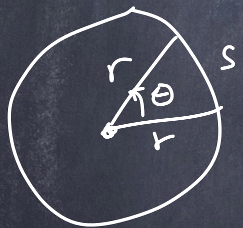
⑤ $\omega_p = \frac{rmg}{L}$ $\quad$ put ① → ⑤, $\omega_p = \frac{rmg}{mr^2\omega} = \frac{g}{r\omega}$ ← spinning wheel



Aside: why is $dL = L\,d\theta$?

Circumference: $C = 2\pi r$

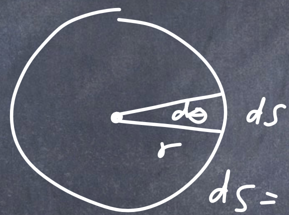
A part of the circumference, s



$s = r\theta$

$\rightarrow$

For small angles, $d\theta$

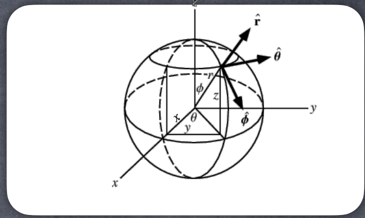


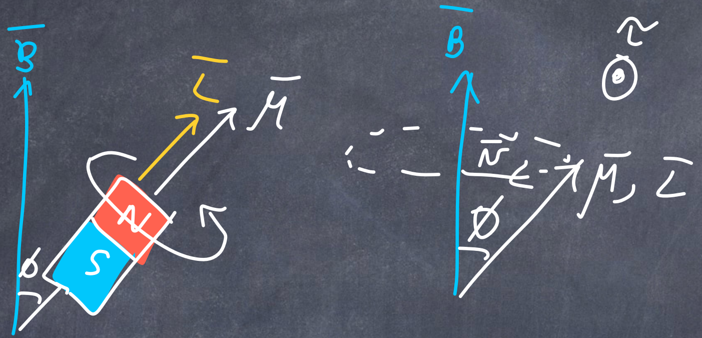
$ds = r\,d\theta$

we can use the same arguments to get $dL = L\,d\theta$

$dL = L\,d\theta$

The same effect occurs for a magnet spinning in a magnetic field





magnet has angular momentum and magnetic moment, which are parallel vectors

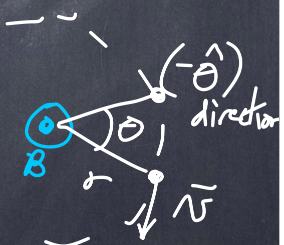
when a magnetic moment and an external magnetic field are not parallel, there is a torque on the magnetic moment.

$$\bar{\tau} = \bar{\mu} \times \bar{B} = \mu B \sin\phi \,(\hat{\theta})$$

Now, when magnet is spinning, it has angular momentum, and the τ causes $\bar{L}$ to change $\rightarrow$ $\bar{\tau} = \dfrac{d\bar{L}}{dt}$

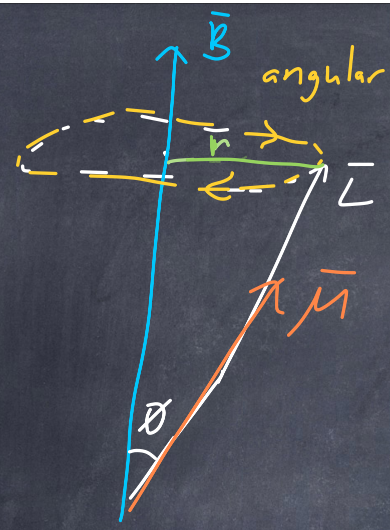
from above:

clockwise



velocity of precession in direction of $-\hat{\theta}$

$\bar{\nu} = r\omega_p(-\hat{\theta})$



$\tau \odot$

$\bar{L}$ has a component in r direction: $L\sin\phi$

$r = \bar{L}\sin\phi$

$v = r\omega_p = \omega_p L \sin\phi$

what is ω_p?

$$|\bar{\tau}| = \left|\frac{d\bar{L}}{dt}\right|$$

$$|d\bar{L}| = (L\sin\phi)\, d\theta$$

$$\frac{|d\bar{L}|}{dt} = L\sin\phi \frac{d\theta}{dt}$$

$$\frac{|d\bar{L}|}{dt} = L\sin\phi\, \omega_p$$

$$\frac{d\bar{L}}{dt} = L\sin\phi\, \omega_p$$

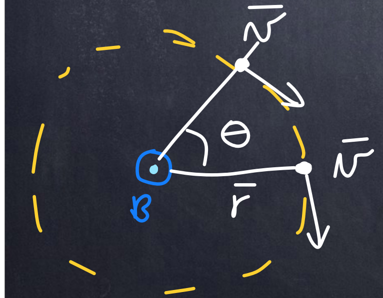
$$\text{so } \omega_p = \frac{|\bar{\tau}|}{L\sin\phi} = \frac{|\bar{\mu}\times\bar{B}|}{L\sin\phi} = \frac{\mu B \cancel{\sin\phi}}{L\cancel{\sin\phi}} = \frac{\mu B}{L}$$

$$\boxed{\omega_p = \frac{\mu B}{L}}$$

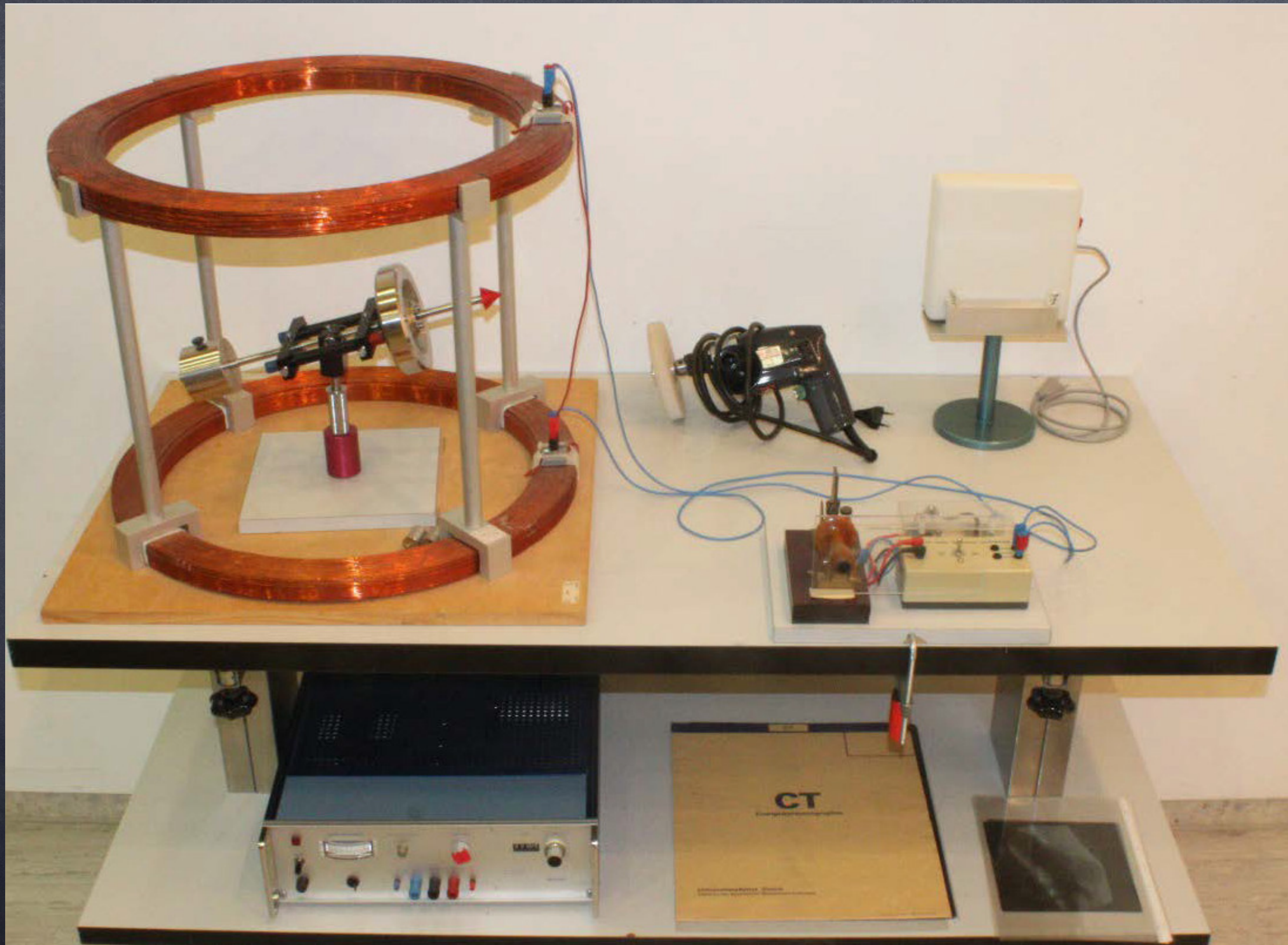
constant for all angles ϕ!

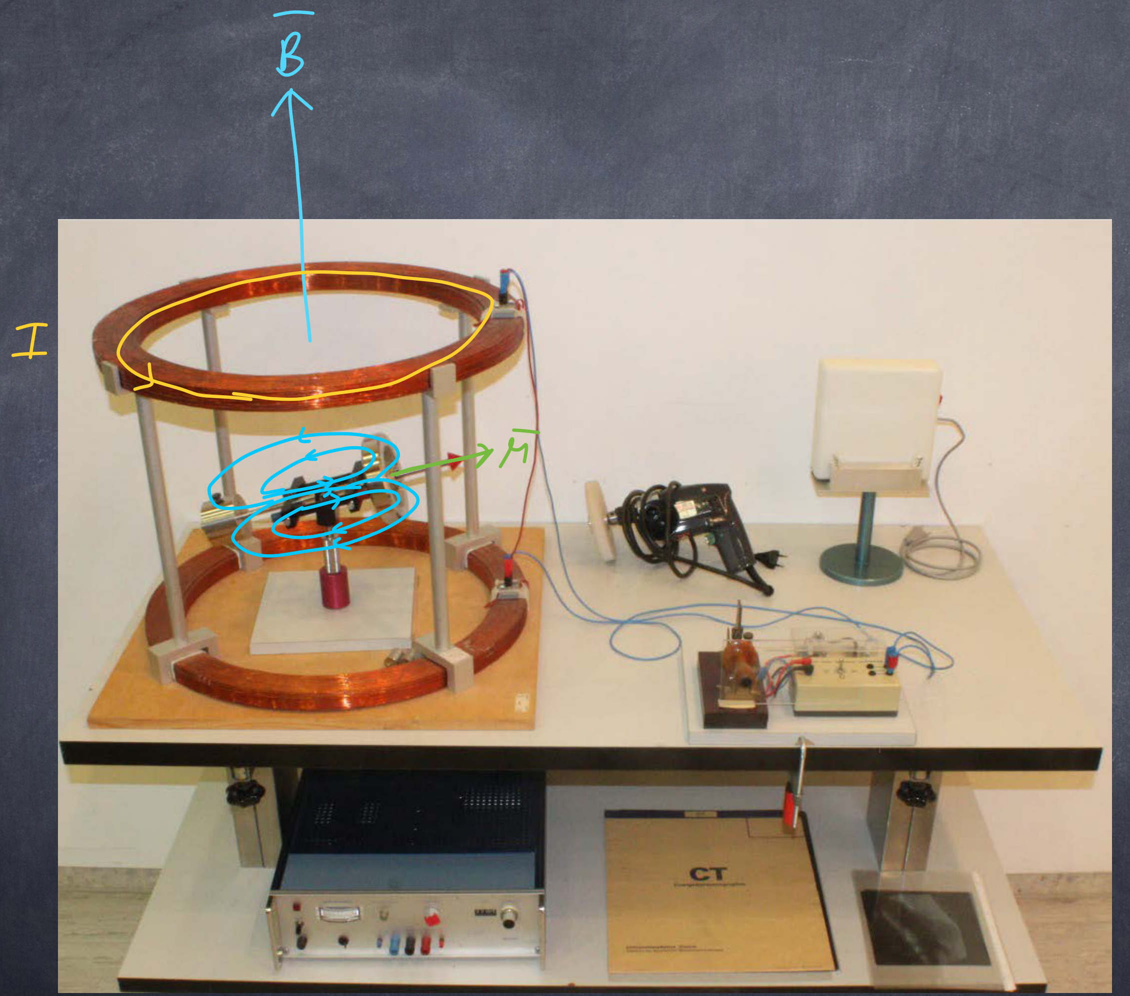
This is the angular velocity of precession of a spinning magnet in a magnetic field.

From above:

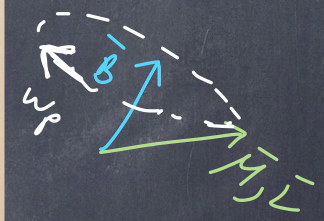


$$v = r\frac{d\theta}{dt} = r\omega_p \quad (-\hat{\theta})$$


CT



$\bar{B}$
I
$\bar{M}$
CT



ω_p
$\bar{B}$
$\bar{M}, \bar{L}$

But wait... we know that subatomic particles have angular momentum and electric charge (they have a magnetic moment)

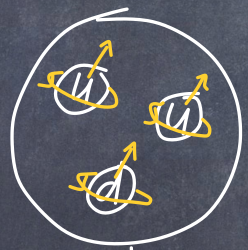
Could this precession happen at a quantum level?

Intrinsic angular momentum (spin)

① protons + neutrons are composed of quarks.

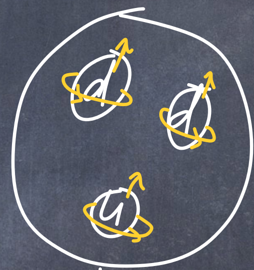
$q(u) = +\frac{2}{3}e$

$q(d) = \frac{-1}{3}e$



proton

$q(p) = +1e$



neutron

$q(n) = 0$

quarks are spin-$\frac{1}{2}$ particles, with electric charge

The spin of quarks in protons + neutrons give them spin as well.

angular momentum S_z quantized by an integer number of $\frac{1}{2}\hbar$ (h: Planck's constant, $\hbar = \frac{h}{2\pi}$

reminder:

$S_z = m_s \hbar$, $m_s = \pm\frac{1}{2}$

$|\bar{S}| = \sqrt{s(s+1)}\,\hbar$

S_z is z-component of S vector

nucleus	(s) spin quantum number	(S_z) spin projection	($\lvert\bar{S}\rvert$) spin vector magnitude
proton	$\frac{1}{2}$	$\pm\frac{1}{2}\hbar$	$\frac{\sqrt{3}}{2}\hbar$
neutron	$\frac{1}{2}$	$\pm\frac{1}{2}\hbar$	$\frac{\sqrt{3}}{2}\hbar$
deuteron (^2H)	1	$\pm\hbar$	$\sqrt{2}\,\hbar$
Helium (He)	0	0	0
^{12}C	0	0	0
^{13}C	$\frac{1}{2}$	$\pm\frac{1}{2}\hbar$	$\frac{\sqrt{3}}{2}\hbar$
^{14}N	1	$\pm\hbar$	$\sqrt{2}\,\hbar$
^{16}O	0	0	0
^{19}F	$\frac{1}{2}$	$\pm\frac{1}{2}\hbar$	$\frac{\sqrt{3}}{2}\hbar$
^{31}P	$\frac{1}{2}$	$\pm\frac{1}{2}\hbar$	$\frac{\sqrt{3}}{2}\hbar$

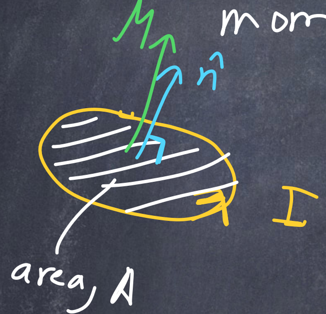
If spinning charged particle, you have a magnetic moment.

① Rotating particle carries angular momentum

$$\bar{L} = \bar{r} \times m\bar{v}$$

② An electric current generates a magnetic field.

A current-carrying loop will generate a magnetic moment, $\bar{\mu}$.



$\hat{n}$: normal vector perpendicular to the loop

$$\bar{\mu} = IA\hat{n}$$

$$\bar{\mu} = \frac{qL}{2m} \quad \text{(classically)}$$

derivation: see script 3, § 11.1

③ In QM, a particle with intrinsic spin and charge has a magnetic moment related to the angular momentum:

$$\bar{\mu} = g\,\frac{q}{2m}\,\bar{L}$$

$\left.\begin{array}{l} q\text{: electric charge} \\ m\text{: mass} \end{array}\right]$ of particle

g: strength parameter

$g = 1$ if charge + mass have the same distrib

The atomic nuclei have nuclear magnetic moments that depend on their nuclear spin.

$$\boxed{\bar{\mu} = g\frac{q}{2m}\bar{I} \quad = \gamma \bar{S}} \quad (8)$$

γ: gyromagnetic ratio

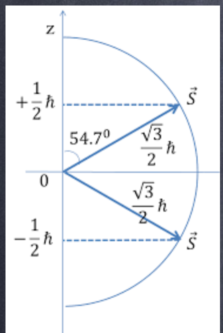
$\gamma = \frac{gq}{2m}$

S: spin (intrinsic angular momentum)

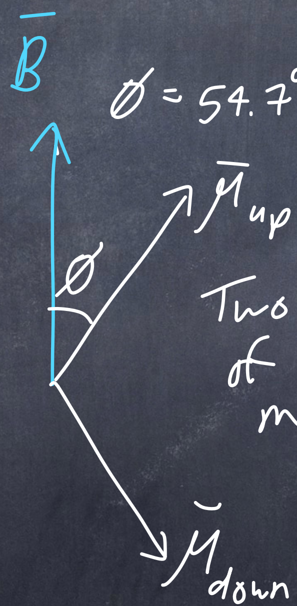
$\bar{\mu} = \gamma \bar{S}$

$\mu_z = \gamma S_z$

$\bar{S}$ is quantized, so $\bar{\mu}$ is quantized.



$\emptyset = 54.7^\circ$



Two possible orientations of the magnetic moment, $\bar{\mu}$ is quantized

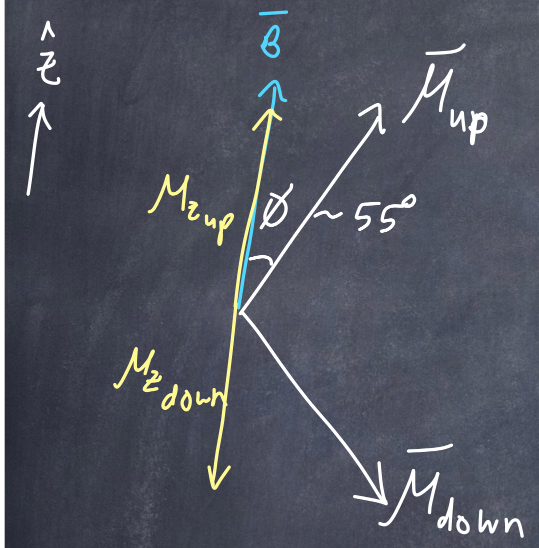
z-component μ_z is quantized & points up or down relative to $\bar{B}$.

For quantum particles, we see the same precession effects. We refer to the angular frequency of precession as the Larmor frequency, ω_L

$$\boxed{\omega_L = \omega_p = \frac{\mu B}{L} = |\gamma| B}$$

Larmor frequency

The fact that μ is not aligned with $\bar{B}$ (external) gives the nucleus a potential energy, U.

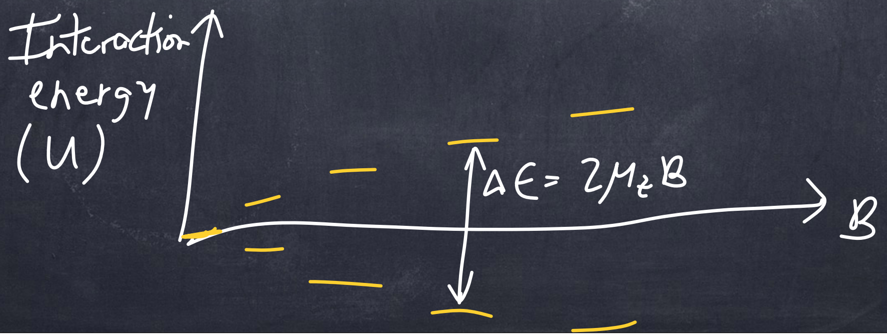


$$U = -\bar{\mu}\cdot\bar{B} = -\mu B\cos\phi = -B(\mu\cos\phi)$$
$$= -B\mu_z$$

The energy difference between the up & down states ($\bar{\mu}_{up}$ + $\bar{\mu}_{down}$) is

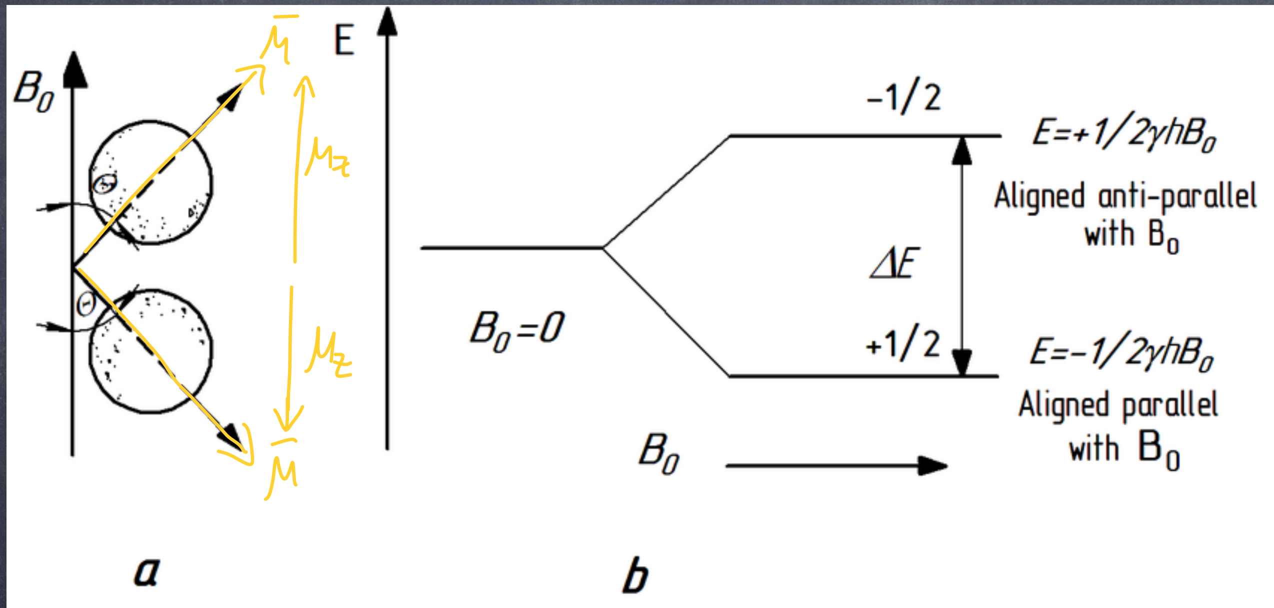
$$\Delta E = \mu_z B - (-\mu_z B) = 2\mu_z B$$

The energy difference increase with increasing magnetic field, B.



$$\Delta E = 2\mu_z B$$

from ⑧, $\mu_z = \gamma S_z \qquad S_z = \pm \frac{1}{2}\hbar$



so $\Delta E = 2 \cdot \gamma \cdot \frac{1}{2} B = \gamma \hbar B$

If a photon with energy ΔE was absorbed, it would flip the spin and $\bar{\mu}$ vector

Some numbers: For $B \simeq 1\,T$ + nucleus of hydrogen (1 proton) with nuclear spin of $\frac{1}{2}\hbar$

we would get $\Delta\epsilon \sim 2\epsilon{-}7\,eV$

we can compare this to the thermal energy of a proton (Hydrogen) at room temperature

thermal energy $\sim K_B T \cong 2.5\epsilon{-}2\,eV$

$\Rightarrow$ The magnetic potential energy is small compared to the thermal energy.

According to the Boltzmann factor for the ratio of spin-up nuclei to spin-down nuclei

$$\frac{n_{up}}{n_{down}} = e^{-\frac{\Delta\epsilon}{K_B T}} = e^{-\frac{2\epsilon-7}{2.5\epsilon-2}} = 0.999992$$

$\rightarrow$ diff. between n_{up} + n_{down} is only a few parts per million (at room temperature)

NMR (nuclear magnetic resonance)

involves adding electromagnetic radiation in units of photon energy, $\epsilon = h\nu$, and then measuring the total absorption of the photons.

$$\Delta\epsilon = 2\mu_z B = h\nu$$

we need very low frequency photons ~ radio frequency (RF) (MHz)

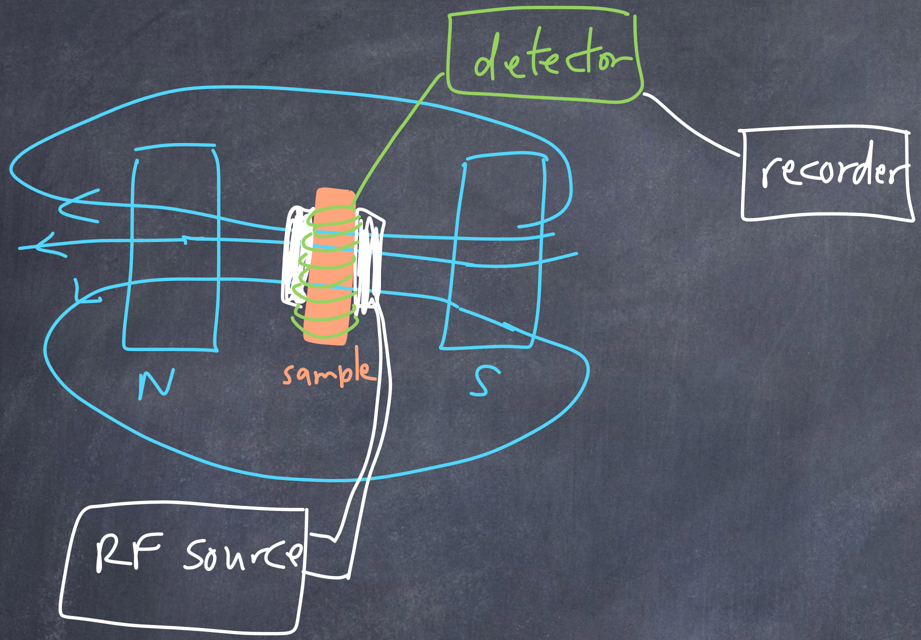
Look at formula, we see that by either varying ν + fix $\bar{B}$, or fixing the ν and varying $\bar{B}$, we can generate a resonance condition where there will be a net absorption of photon energy causing the nuclei to flip spins to a higher energy state.

In NMR, RF (radio frequency) is fixed and $\bar{B}$ is varied by small amounts while scanning through for resonance conditions. (see video on wikipedia page)

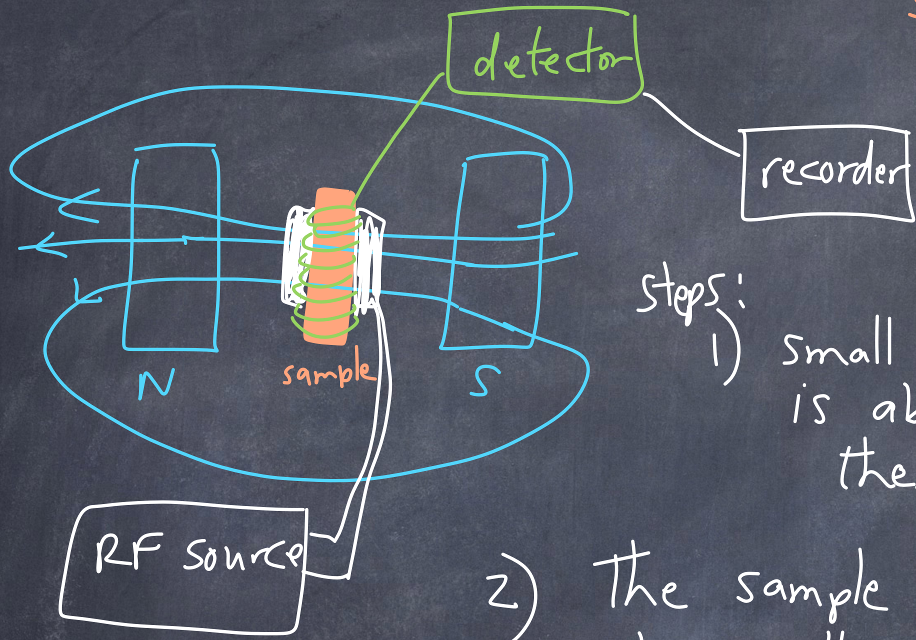
Einstein showed that the same RF photons that can be absorbed, flipping spins to a higher energy state, can with equal probability, flip the nuclear spin to a lower energy state, emitting a second RF photon with energy ΔE.

If n_{up} & n_{down} were equal, there would be no net absorption. But, since there are slightly more nuclei in the n_{down} state than the n_{up} state, there is a slight net absorption of photons. This is our signal for NMR.

How to detect NMR radiation.



How to detect NMR radiation.



sample is placed in a magnetic field

detector is a solenoid that can measure changes in electric current

steps:

1) Small amount of RF radiation is absorbed by our sample, then we turn off the RF.

2) The sample returns to equilibrium, by emitting RF energy.

The net magnetic moment of the sample changes, and this can be detected in the solenoidal coil. (This comes from PHY 117 (script 2))

The changing magnetism of the sample induces a electric current in the solenoid. (Faraday's Law)

Next time: how to use NMR to determine molecular structure