MMP I
Solution Sheet 3

HS 21
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Exercise 1 [Properties of Fourier transform (5 points)]

$f, g, \hat{f}, \hat{g} \in L^1(\mathbb{R}^n)$ where $L^p := ||f||_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{1/p}$, so $L^1 = \int_{-\infty}^{\infty} |f(x)| dx < \infty$

$\alpha, \beta \in \mathbb{C}$, $\lambda \in \mathbb{R}\backslash\{0\}$,

D_λ: $(D_\lambda f)(x) = f(\lambda x)$

T_α: $T_\alpha f(x) = f(x + \alpha)$, $\alpha \in \mathbb{R}^n$

a)

$$\begin{aligned}
(\widehat{\alpha f + \beta g}) &= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} (\alpha f(x) + \beta g(x)) e^{-ikx} d^n x \\
&= \frac{1}{(2\pi)^{n/2}} \alpha \int_{-\infty}^{\infty} f(x) e^{-ikx} d^n x + \frac{1}{(2\pi)^{n/2}} \beta \int_{-\infty}^{\infty} g(x) e^{-ikx} d^n x \\
&= \underline{\underline{\alpha \hat{f}(k) + \beta \hat{g}(k)}}
\end{aligned}$$

b)

$$\begin{aligned}
\overline{\hat{f}(k)} &= \overline{\frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} f(x) e^{-ikx} d^n x} \\
&= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \overline{f(x)} e^{ikx} d^n x \\
&= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \overline{f(x)} e^{-i(-k)x} d^n x \\
&= \underline{\underline{\hat{\overline{f}}(-k)}}
\end{aligned}$$

c)

$$(\widehat{D_\lambda f}) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} (D_\lambda f(x))\, e^{-ikx} d^n x$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} f(\lambda x) e^{-ikx} d^n x$$

with $y = \lambda x,\ d^n y = |\det D_x y| = |\lambda|^n d^n x$

$$= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} f(y) e^{-i\frac{k}{\lambda} y} \frac{d^n y}{|\lambda|^n}$$

$$= \underline{\underline{\hat{f}\left(\frac{k}{\lambda}\right) \frac{1}{|\lambda|^n}}}$$

d)

$$\begin{aligned}(\widehat{T_\alpha f}) &= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \left(T_\alpha f(x)\right) e^{-ikx} d^n x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} f(x+\alpha) e^{-ikx} d^n x \\ &\text{with } y = x + \alpha,\ d^n y = d^n x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} f(y) e^{-ik(y-\alpha)} d^n y \\ &= \underline{\underline{e^{ik\alpha} \hat{f}(k)}}\end{aligned}$$

e)

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{f}(k) g(k) d^n k &= \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} f(x) e^{-ikx} d^n x g(k) d^n k \\ &= \int_{-\infty}^{\infty} f(x) \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} g(k) e^{-ikx} d^n k d^n x \\ &= \underline{\underline{\int_{-\infty}^{\infty} f(x) \hat{g}(x) d^n x}}\end{aligned}$$

Exercise 2 [Properties of Fourier transform (4 points)]

a) $\hat{f}(k) = \overline{\hat{f}}(-k) \Leftrightarrow f(x) \in \mathbb{R}$

① Proof: $f(x) \in \mathbb{R} \Rightarrow \hat{f}(k) = \overline{\hat{f}}(-k)$:

$$\begin{aligned}\overline{\hat{f}}(-k) &= \overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(-k)x} dx} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)} e^{-ikx} dx \\ &\text{with } \overline{f(x)} = f(x)\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$
$$= \underline{\underline{\hat{f}(k)}}$$

② Proof: $\hat{f}(k) = \overline{\hat{f}}(-k) \Rightarrow f(x) \in \mathbb{R}$ (i.e. $f(x) = \overline{f(x)}$):

$$\overline{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\hat{f}(k) e^{ikx}} dk$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\hat{f}}(k) e^{-ikx} dk$$
$$\text{with } \overline{\hat{f}}(k) = \hat{f}(-k)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(-k) e^{-ikx} dk$$
$$= \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \hat{f}(k') e^{ik'x} dk'$$
$$= \underline{\underline{f(x)}}$$

b) $\widehat{f'}(k) = ik\hat{f}(k)$ for $f(x)$ compactly supported:

$$\widehat{f'}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-ikx} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d}{dx}(f(x) e^{-ikx}) dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{d}{dx}(e^{-ikx}) dx$$
$$= \frac{1}{\sqrt{2\pi}} \left[\cancel{f(x)}^{0} \underbrace{e^{-ikx}}_{\text{oscillating}} \right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)(-ik) e^{-ikx} dx$$
$$= \underline{\underline{ik\hat{f}(k)}}$$

c) $\widehat{xf}(k) = i\frac{d}{dk}\hat{f}(k)$

$$\widehat{xf}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) e^{-ikx} dx$$
$$\text{with } x e^{-ikx} = \frac{d}{dk}(e^{-ikx}) \frac{1}{(-i)} = i \frac{d}{dk}(e^{-ikx})$$
$$= \frac{1}{\sqrt{2\pi}} i \frac{d}{dk} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$
$$= \underline{\underline{i \frac{d}{dk} \hat{f}(k)}}$$

$$\left(x^2 f = x(\underbrace{xf}_{g}) \to \widehat{xg} = i\tfrac{d}{dk}\hat{g}(k) = i\tfrac{d}{dk}(\widehat{xf}) = i\tfrac{d}{dk}\left(i\tfrac{d}{dk}\hat{f}\right) \right)$$

Exercise 3 [Fourier transform (5 points)]

a) $f(x) = \begin{cases} 1 & -3 \leq x \leq -1 \\ 1 & 1 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$

$$\begin{aligned} \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-3}^{-1} e^{-ikx} dx + \frac{1}{\sqrt{2\pi}} \int_{1}^{3} e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-ikx}}{-ik} \Big|_{-3}^{-1} + \frac{e^{-ikx}}{-ik} \Big|_{1}^{3} \right] \\ &= \frac{-2}{\sqrt{2\pi}} \frac{e^{ik} - e^{3ik} + e^{-3ik} - e^{-ik}}{2i} \\ &= \underline{\underline{\sqrt{\frac{2}{\pi}} \frac{1}{k} \left(\sin(3k) - \sin(k) \right)}}, \text{ since } \sin x = \frac{e^{ix} - e^{-ix}}{2i} \end{aligned}$$

b) $f(x) = e^{-a^2x^2}$, hint: use $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-a^2x^2} e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 + ikx)} dx \end{aligned}$$

$\rightarrow$ We want to write $-(a^2x^2 + ikx)$ as t^2:

$(ax + Ak)^2 = a^2x^2 + A^2k^2 + 2Akax \rightarrow ikx$ if $A = \frac{i}{2a}$

$$(ax + \tfrac{ik}{2a})^2 + \underbrace{\frac{k^2}{4a^2}}_{-A^2k^2} = a^2x^2 + ikx$$

$$\begin{aligned} \rightarrow \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} e^{-k^2/4a^2} \int_{-\infty}^{\infty} e^{-\underbrace{(ax + ik/2a)}_{t}^2} dx \\ &\text{with } t = ax + \frac{ik}{2a}, \ dt = a dx \\ &= \frac{1}{a\sqrt{2\pi}} e^{\frac{-k^2}{4a^2}} \underbrace{\int_{-\infty}^{\infty} e^{-t^2} dt}_{\sqrt{\pi}} \\ &= \underline{\underline{\frac{1}{a\sqrt{2}} e^{\frac{-k^2}{4a^2}}}} \end{aligned}$$

If $a = \frac{1}{\sqrt{2}}$ then $\begin{cases} f(x) = e^{-\frac{x^2}{2}} \\ \hat{f}(k) = e^{-\frac{k^2}{2}} \end{cases}$ $\Rightarrow f$ is the Fourier transform of itself.

c) $f(x) = \begin{cases} e^{-ax} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$

$$\begin{aligned} \hat{f}(k) &= \int_0^\infty e^{-ax} e^{-ikx} dx \\ &= \int_0^\infty e^{-(a+ik)x} dx \\ &= \left[\frac{e^{-(a+ik)x}}{-(a+ik)} \right]_0^\infty = \underline{\underline{\frac{1}{a+ik}}} \end{aligned}$$

Now we have to calculate $\widehat{x^2 f}(x)$ from exercise 2c)

We know that $x\hat{f}(k) = i\frac{d}{dk}\hat{f}(k)$.

Now: $x^2 f = x(\underbrace{xf}_{=g}) \to \widehat{xg} = i\frac{d}{dk}\hat{f}(k) = i\frac{d}{dk}(\widehat{xg}) = i\frac{d}{dk}\left(i\frac{d}{dk}\hat{f}\right) = -\frac{d^2}{dk^2}\hat{f}$

$$\begin{aligned} \widehat{x^2 f} &= -\frac{d^2}{dk^2}\hat{f} = -\frac{d^2}{dk^2}\left(\frac{1}{a+ik}\right) \\ &= -\frac{d}{dk}\left[(-1)\frac{1}{(a+ik)^2} i\right] \\ &= \frac{d}{dk}\left[\frac{i}{(a+ik)^2}\right] = \frac{-2i}{(a+ik)^3} i \\ &= \underline{\underline{\frac{2}{(a+ik)^3}}} \end{aligned}$$