

2. The Lagrangian Formalism

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2.1 Generalities

In the previous lectures we have seen how the eq. of motion get modified in the case of conservative systems with holonomic constraints & generalised coordinates:

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0} \quad (2.1)$$

where

$$L = T(q_i, \dot{q}_i, t) - U(q_i, t) \quad (2.2)$$

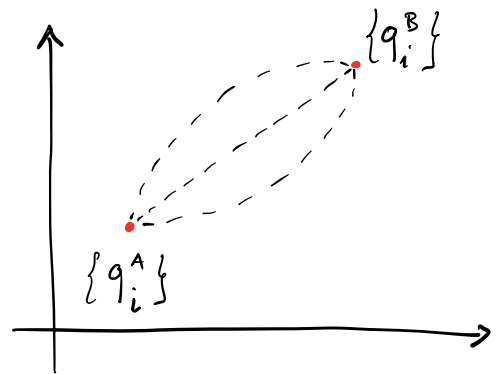
Today we will prove that these equations follows from a variational principle, known as the Principle of Least Action (or Hamilton's Principle).

To state this principle, let's first define the action (S) associated to a given trajectory describing the motion from an initial configuration $q_i^A \equiv q_i(t_A)$ to a final configuration $q_i^B \equiv q_i(t_B)$

$$S[q_i(t)] = \int_{t_A}^{t_B} L(q_i(t), \dot{q}_i(t), t) dt \quad (2.3)$$

↑
"functional"

(= function of a given function)



The principle states that

The trajectory followed by the system is such that S reaches a stationary point (correct $q(t) \leftrightarrow$ extremum of the action).

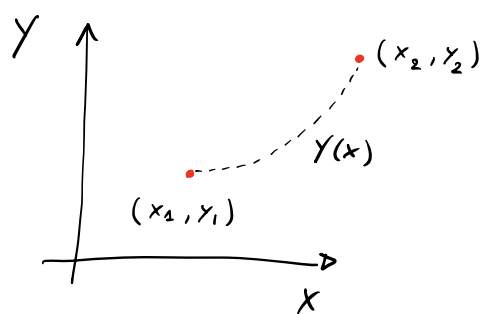
Before showing that this principle leads to (2.1), let us first note some points:

- I. This principle focuses on the whole trajectory, rather than on the instantaneous evolution of the system. In physical terms, we can say that the system evolves in such a way to minimize the transfer of energy from kinetic to potential (and viceversa).
- II. This formulation is, by construction, independent from the coordinates (and the reference frame) used to describe the motion of the system. It is also suitable to be generalised to systems where the q_i have nothing to do with 3D space coordinates.
- III. As far as we know, all "fundamental interactions" in Nature can be described by this principle (using the proper action) and by its generalization when we consider quantum phenomena.
- IV. Using this principle, the transition from classical mechanics to quantum mechanics is particularly simple [$\rightarrow$ discussion].

2.2 Elements of Variational Calculus

To understand what it means to extremise a functional, let's consider a simple example.

Let's start from a 1D system:



path: $y(x)$

$$\text{functional: } J_F[y] = \int_{x_1}^{x_2} F(y, \dot{y}, x) dx \quad (2.4)$$

$$\dot{y} = \frac{dy}{dx}$$

We want to determine the form of $y(x)$ that corresponds to an extremum of J_F , keeping fixed $y(x_1) = y_1$ and $y(x_2) = y_2$.

To this purpose, let's consider a generic (well-behaved) function $z(x)$, satisfying $z(x_1) = z(x_2) = 0$, and a continuous parameter α .

We can then build a class of paths depending on α :

$$y(x, \alpha) = y(x) + \alpha z(x) \quad (2.5)$$

Finding the extrema of J_F with respect to this class of path is obtained solving $dJ/d\alpha = 0$. The specific path $y(x)$ is a stationary point of the functional if

$$\left. \frac{dJ}{d\alpha} \right|_{\alpha=0} = 0 \quad (2.6)$$

This implies

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \alpha} \right) \bigg|_{\alpha=0} dx = 0 \quad (2.7)$$

(4)

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \alpha} dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial \dot{y}} \frac{\partial^2 y}{\partial x \partial \alpha} dx = \frac{\partial F}{\partial \dot{y}} \frac{\partial y}{\partial \alpha} \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial \dot{y}} \right) \frac{\partial y}{\partial \alpha} dx \quad (2.8)$$

(after integrating by parts)

$$\frac{\partial y}{\partial \alpha} \Big|_{x_1}^{x_2} = y(x_2) - y(x_1) = 0$$

 $\Downarrow$

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial \dot{y}} \right) \frac{\partial y}{\partial \alpha} dx \quad (2.9)$$

 $\Downarrow$

$$\frac{dJ}{d\alpha} \Big|_{\alpha=0} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial \dot{y}} \right) y(x) dx = 0 \quad (2.10)$$

Since this equation must be satisfied for any $y(x)$, we reach the conclusion that the path $y(x)$ corresponding to a stationary point of $J_F[y]$ must satisfy

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial \dot{y}} = 0 \quad (2.11)$$

The introduction of the continuous parameter α is an auxiliary step we can skip introducing the concept of **functional derivative**:

$$\delta J_F[y] = J_F[y + \delta y] - J_F[y] \quad (2.12)$$

$$= \int_{x_1}^{x_2} \left\{ F(y + \delta y, \dot{y} + \delta \dot{y}, x) - F(y, \dot{y}, x) \right\} dx$$

$$= \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial \dot{y}} \frac{d}{dx}(\delta y) \right\} dx$$

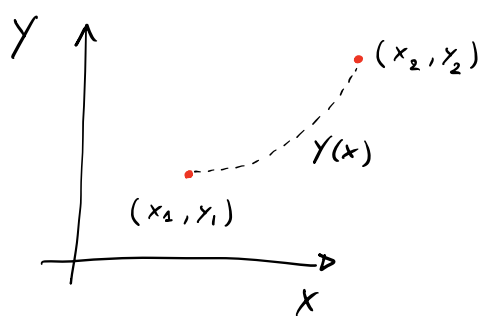
(5)

$$= \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial \dot{y}} \right) \right\} \delta y \, dx + \frac{\partial F}{\partial \dot{y}} \delta y \Big|_{x_2}^{x_1} \quad (2.13)$$

$\uparrow$
 vanishes if $\delta y(x_1) = \delta y(x_2) = 0$

$$\boxed{\delta J_F[y] = 0} \Rightarrow \boxed{\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial \dot{y}} \right) = 0} \quad (2.14)$$

Example: Shortest distance between two points on a plane



$$dS = \sqrt{dx^2 + dy^2} = (1 + \dot{y}^2)^{\frac{1}{2}} dx \quad (2.15)$$

$$I_S = \int_{x_1}^{x_2} dS = \int_{x_1}^{x_2} (1 + \dot{y}^2)^{\frac{1}{2}} dx$$

$\uparrow$
 $F(y, \dot{y}, x)$

$$\frac{\partial F}{\partial y} = 0 \quad \frac{\partial F}{\partial \dot{y}} = \dot{y} (1 + \dot{y}^2)^{-\frac{1}{2}} \quad (2.16)$$

The minimization condition (2.14) reads:

$$\frac{d}{dx} \left[\dot{y} (1 + \dot{y}^2)^{-\frac{1}{2}} \right] = 0 \Rightarrow \dot{y} = \text{const} \quad (2.17)$$

(or $\frac{d^2 y}{dx^2} = 0$)

which is nothing but the path corresponding to a straight line.

2.3 Euler-Lagrange's Eq.s from Hamilton's principle

⑥

Generalizing the 1D case discussed above, we can easily prove that (2.1), i.e. the Euler-Lagrange Eq.s for a system described by N coordinates $q_i(t)$, $i=1 \dots N$, follow from the minimization of the action defined in (2.3)

$$\delta S[q_i] = \int_{t_A}^{t_B} [L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i, t) - L(q_i, \dot{q}_i, t)] dt \quad (2.18)$$

Considering each of the $\delta q_i(t)$ as independent, and satisfying $\delta q_i(t_A) = \delta q_i(t_B) = 0$, we arrive to

$$\delta S[q_i] = 0 \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad i=1 \dots N \quad (2.19)$$

It is interesting to check explicitly the statement II. on pag. 4, namely the independence from the coordinate system of the Euler-Lag. equations.

Let's consider a new set of variables y_i $i=1 \dots N$, such that

$$y_i(t) = y_i(q_1(t), \dots, q_N(t), t) \quad (2.20)$$

The only condition we require is that the change of coordinate is invertible, namely

$$\det \left(\frac{\partial y_i}{\partial q_k} \right) \neq 0 \quad (2.21)$$

$\nwarrow$ $N \times N$ matrix J_{ik} (Jacobian of the transformation)

We have

(7)

$$\dot{Y}_i = \frac{d}{dt} Y_i = \frac{\partial Y_i}{\partial q_k} \dot{q}_k + \frac{\partial Y_i}{\partial t}$$

$$\dot{q}_i = \frac{d}{dt} q_i = \frac{\partial q_i}{\partial Y_k} \dot{Y}_k + \frac{\partial q_i}{\partial t}$$

$$\text{N.B.: } \frac{\partial q_i}{\partial \dot{Y}_k} = \frac{\partial Y_k}{\partial \dot{q}_i} = 0$$

$$\hookrightarrow \frac{\partial \dot{q}_i}{\partial \dot{Y}_k} = \frac{\partial q_i}{\partial Y_k}$$

(2.22)

Let's now compute $\partial L / \partial Y_i$ and $\partial L / \partial \dot{Y}_i$

$$\begin{aligned} \frac{\partial L}{\partial Y_i} &= \frac{\partial L}{\partial q_k} \cdot \frac{\partial q_k}{\partial Y_i} + \frac{\partial L}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial Y_i} \\ &= \frac{\partial L}{\partial q_k} \frac{\partial q_k}{\partial Y_i} + \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt} \left(\frac{\partial q_k}{\partial Y_i} \right) \end{aligned}$$

$$\frac{\partial L}{\partial \dot{Y}_i} = \frac{\partial L}{\partial \dot{q}_k} \cdot \frac{\partial \dot{q}_k}{\partial \dot{Y}_i} = \frac{\partial L}{\partial \dot{q}_k} \frac{\partial q_k}{\partial Y_i} \quad (\text{because of (2.22)})$$

Hence

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{Y}_i} \right) - \frac{\partial L}{\partial Y_i} = \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} \right] \times \underbrace{\frac{\partial q_k}{\partial Y_i}}_{(J^{-1})_{ki}} \quad (2.23)$$

Since $\det(J) \neq 0$, the Euler-Lagrange equations on the q_i variables imply equivalent equations for the Y_i variables,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{Y}_i} - \frac{\partial L}{\partial Y_i} = 0 \quad i = 1 \dots N$$

We have thus proved that these equations are indeed coordinate independent.

⑧

To conclude, let's finally verify that the action principle is equivalent to Newton's second law for a conservative system of N particles

$$q_i(t) \leftrightarrow \vec{x}_k(t) \equiv x_k^j(t)$$

$k = 1 \dots N$ n. of particles

$j = 1 \dots 3$ space directions

$i = 1 \dots 3N$

$$T = \sum_k \frac{1}{2} m_k^2 \dot{\vec{x}}_k^2 = \sum_{k,j} \frac{1}{2} m_k^2 (\dot{x}_k^j)^2$$

$$U = U(\vec{x}_i)$$

$$\vec{F}_k = -\vec{\nabla}_{\vec{x}_k} U(\vec{x}_i) \rightarrow F_k^j = -\frac{\partial U(x_i^m)}{\partial x_k^j} = \frac{\partial \mathcal{L}(\vec{x}_i, \dot{\vec{x}}_i)}{\partial x_k^j} \quad (2.24)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \Rightarrow \boxed{m_k \ddot{x}_k - \vec{F}_k(\vec{x}_i) = 0} \quad (2.25)$$