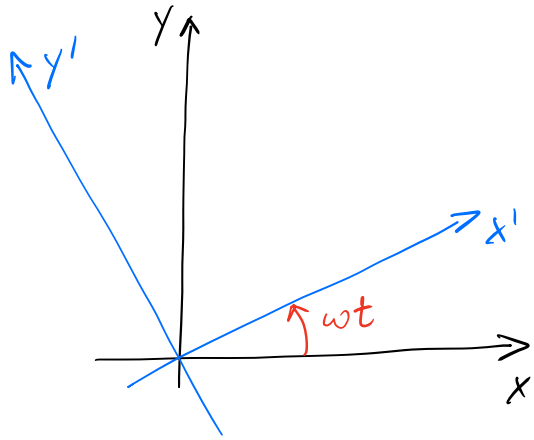


2.4 Rotating coordinate systems & the "fictitious forces"

(9)

As an example of the usefulness of the Lagrangian formalism and its independence from the choice of coordinate systems, let's consider the case of a free particle in a rotating (hence non-inertial) coordinate system.



Inertial system: (x, y, z)



Rotating system

$$\begin{cases} x' = x \cos(\omega t) + y \sin(\omega t) \\ y' = -x \sin(\omega t) + y \cos(\omega t) \\ z' = z \end{cases}$$

(Rotation with angular velocity ω around the z axis)

$$L = \frac{1}{2} m \dot{\vec{v}}^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$



changing coordinates:

$$L = \frac{1}{2} m [(\dot{x}' - \omega y')^2 + (\dot{y}' + \omega x')^2 + \dot{z}^2] = \frac{1}{2} m (\dot{\vec{r}}' + \vec{\omega} \times \vec{r}')^2$$

where $\vec{r} = (x', y', z')$

$\vec{\omega} = (0, 0, \omega)$

This implies:

$$\frac{\partial L}{\partial \vec{r}'} = m (\dot{\vec{r}}' + \vec{\omega} \times \vec{r}') \times \vec{\omega}$$

(2.26)

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vec{r}}'} \right) = m \frac{d}{dt} (\dot{\vec{r}}' + \vec{\omega} \times \vec{r}') = m (\ddot{\vec{r}}' + \vec{\omega} \times \dot{\vec{r}}')$$

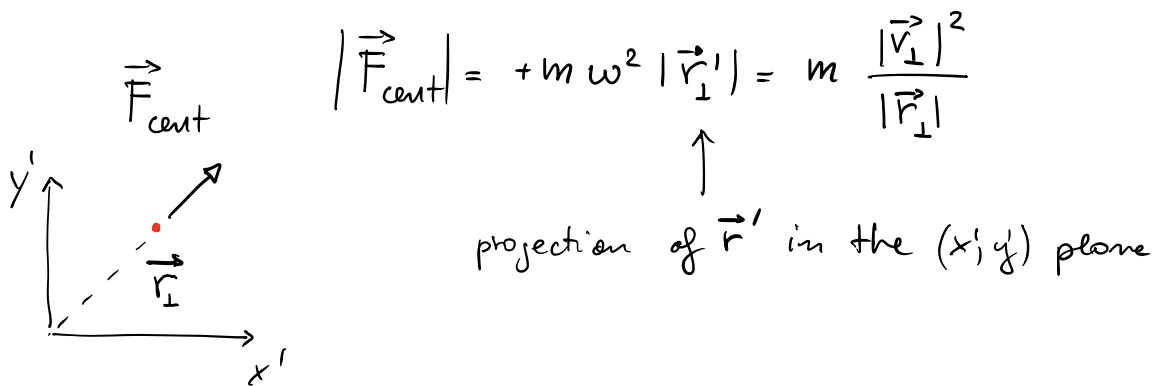
The Euler-Lagrange eq.^s in the new (rotating) coordinates

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vec{r}}'} \right) - \frac{\partial L}{\partial \vec{r}'} = 0$$

leads to

$$m \ddot{\vec{r}}' = -m \vec{\omega} \times (\vec{\omega} \times \vec{r}') - 2m \vec{\omega} \times \dot{\vec{r}}' \quad (2.27)$$

$\uparrow$ centrifugal force $\uparrow$ Coriolis force



$$|\vec{F}_{cent}| = +m \omega^2 |\vec{r}'_{\perp}| = m \frac{|\vec{v}_{\perp}|^2}{|\vec{r}'|}$$

$\uparrow$
projection of $\vec{r}'$ in the (x', y') plane

$$\vec{F}_{cor} = -2m \vec{\omega} \times \dot{\vec{r}}' \quad \left(\begin{array}{l} \text{force responsible for the large-scale} \\ \text{circulation of oceans and the atmosphere} \end{array} \right)$$

$\Rightarrow$ We recover the known expressions of the so-called "fictitious" forces.

2.5 Holonomic constraints & Lagrange multipliers

In sect. 1.6 we derived Euler-Lagrange Eq.^s starting from D'Alembert principle

$$\begin{cases} \vec{x}_i & i=1 \dots N & \rightarrow 3N \text{ variables} \\ f_a(\vec{x}_i, t) = 0 & a=1 \dots K & \rightarrow K \text{ constraints} \end{cases}$$

I. q_i $i=1 \dots 3N-K$ generalised coord. such that
 $f_a(\vec{x}_i(q_j)) = 0$ for any value of q_i

(II) D'Alembert principle
(constraints produce no work) $\left(\frac{\partial \vec{x}_i}{\partial q_j} \cdot \vec{F}_i = 0 \right)$

(11)

(I.) + (II.) $\Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad L = L(\dot{q}_i, q_i, t) \quad (2.28)$

We will now show that we can arrive to the same conclusion following a different path, which does not require this principle (indeed D'Alembert principle is a consequence of Hamilton's principle for systems with holonomic constraints).

Key step: introduce the so-called Lagrange multipliers λ_a

$$L'(\dot{\vec{x}}_i, \vec{x}_i, \lambda_a, t) = L(\dot{\vec{x}}_i, \vec{x}_i, t) + \sum_a \lambda_a f_a(\vec{x}_i, t) \quad (2.29)$$

$\uparrow$
 Lagrangian of the system in
 absence of constraints

$\uparrow$
 constraints

The λ_a are auxiliary variables that force the system to satisfy the constraint and do not modify the action

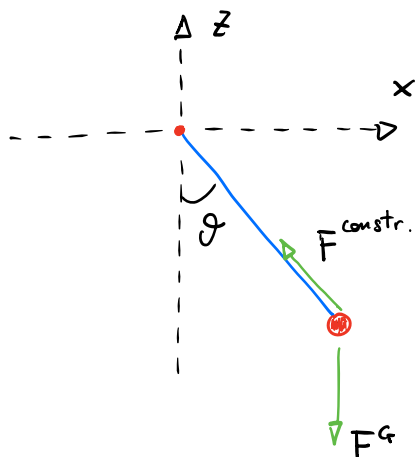
$$x_i \oplus \lambda_a \Rightarrow 3N + K \text{ variables}$$

By construction, the minimization with respect to the λ_a gives back the eq.s of the constraints:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\lambda}_a} = 0 = \frac{\partial L}{\partial \lambda_a} = f_a(\vec{x}_i, t) \Rightarrow f_a(\vec{x}_i, t) = 0$$

The minimization of $S' = \int dt L'$ gives rise to the correct e.o.m. in the original variables, taking into account the effect of the constraints.

Example: back to the pendulum



$$T = \frac{1}{2} m (\dot{x}^2 + \dot{z}^2)$$

$$U = mgz$$

constraint: $x^2 + z^2 = L^2$

$$L' = T - U = \frac{1}{2} m (\dot{x}^2 + \dot{z}^2) - mgz + \frac{\lambda}{2} (x^2 + z^2 - L^2)$$

Lagrange eq.s:

$$\begin{cases} m \ddot{x} = \lambda x \\ m \ddot{z} = -mg + \lambda z \\ x^2 + z^2 = L^2 \end{cases} \quad \begin{array}{l} \swarrow \text{effect of the constraint} \\ \searrow \end{array} \quad \begin{cases} F_x^{\text{const}} = \lambda x \\ F_z^{\text{const}} = \lambda z \end{cases}$$

Solving these eq.s we determine both the eq.s of motions in the original variables and the structure of the forces exerted by the constraints:

$$\ddot{z} = -g + \frac{\ddot{x}}{x} z \quad \left(F_z^{\text{constr.}} = m \ddot{x} \frac{z}{x} \right)$$

which is equivalent to $\ddot{\theta} = -\frac{g}{L} \sin \theta$

once we express x and z as $\begin{cases} x = L \sin \theta \\ z = -L \cos \theta \end{cases}$

In this case the introduction of the Lagrange multipliers does not lead to a simplification of the problem. However, it represents a powerful method to treat constraints in general terms.

At this point we are ready to show why we end up in (2.28) in the limit where we are not interested in determining the forces exerted by the constraints (making use of a proper set of generalised coordinates q_i).

The structure of the Lagrangian describing the motion in presence of constraints is:

$$L' = L(\dot{x}_i, x_i, t) + \lambda_a f_a(x_i) \quad \Rightarrow \quad (\dot{x}_i, \lambda_a)$$

$\uparrow \quad \quad \uparrow$
 $3N \quad + \quad K$ variables

Let's consider the following change of variables:

$$\left. \begin{array}{ll} x_i \rightarrow \begin{cases} q_i & i = 1 \dots 3N-K \\ f_a & a = 1 \dots K \end{cases} \\ \lambda_a \rightarrow \lambda_a & a = 1 \dots K \end{array} \right\} \quad 3N+K \text{ variables}$$

With such choice, the e.o.m. for the q_i get separated:

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{q}_i} \right) = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right)$$

$$\frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial q_i} + \lambda_a \underbrace{\frac{\partial f_a}{\partial q_i}}_{\rightarrow 0 \text{ by construction}} = \frac{\partial L}{\partial q_i} \quad \left. \vphantom{\frac{\partial L}{\partial q_i}} \right\} \rightarrow (2.28)$$