Applications of General Relativity

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Solutions 10

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Exercise 1 [Post-Newtonian Lagrangian for a binary system] 10 points

Let's work on particle m_1's lagrangian only.

(i) The lagrangian is:

$$\mathcal{L} = -m_1 c \int \mathrm{d}t \left(-g_{00}c^2 - 2cg_{0i}v_1^i - g_{ij}v_1^i v_1^j\right)^{1/2} =$$
$$= -m_1 c^2 \int \mathrm{d}t \left(1 - {}^{(2)}g_{00} - {}^{(4)}g_{00} - 2\,{}^{(3)}g_{0i}\frac{v_1^i}{c} - \frac{v_1^2}{c^2} - {}^{(2)}g_{ij}\frac{v_1^i v_1^j}{c^2}\right)^{1/2}. \tag{1}$$

We can expand the big parenthesis in equation (1) as $(1+x)^{1/2} \simeq 1 + x/2 - x^2/8 + \ldots$, and keeping only terms up to ϵ^4 we have:

$$\mathcal{L} = -m_1 c^2 \int \mathrm{d}t \left\{\left(1 - \frac{{}^{(2)}g_{00}}{2} - \frac{v_1^2}{2c^2}\right) + \right.$$
$$\left.\left(-\frac{v_1^4}{8c^4} - \frac{{}^{(2)}g_{00}^2}{8} - \frac{{}^{(2)}g_{00}}{4}\frac{v_1^2}{c^2} - \frac{{}^{(4)}g_{00}}{2} - \frac{{}^{(3)}g_{0i}v_1^i}{c} - \frac{{}^{(2)}g_{ij}v_1^i v_1^j}{c^2}\right)\right\} =$$
$$= -m_1 c^2 \int \mathrm{d}t + m_1 c^2 \int \mathrm{d}t \left\{\left(\frac{{}^{(2)}g_{00}}{2} + \frac{v_1^2}{2c^2}\right) + \right.$$
$$\left.\left(+\frac{v_1^4}{8c^4} + \frac{{}^{(2)}g_{00}^2}{8} + \frac{{}^{(2)}g_{00}}{4}\frac{v_1^2}{c^2} + \frac{{}^{(4)}g_{00}}{2} + \frac{{}^{(3)}g_{0i}v_1^i}{c} + \frac{{}^{(2)}g_{ij}v_1^i v_1^j}{c^2}\right)\right\}. \tag{2}$$

The first integral in equation (2) is associated to the rest-mass energy and does not appear in any equation. We can neglect it. The second integral is the actual lagrangian of the particle m_1. The first term inside the second integral is the newtonian part ($\mathcal{O}(\epsilon^2)$), whereas the second term is the post-newtonian part ($\mathcal{O}(\epsilon^4)$). Using the definitions in the exercise sheet for ${}^{(i)}g_{\mu\nu}$, we finally have:

$$\mathcal{L} = m_1 c^2 \left\{\frac{1}{2}\left(\frac{v_1}{c}\right)^2 - \phi + \frac{1}{8}\left(\frac{v_1}{c}\right)^4 - \frac{\phi^2}{2} - \psi - \frac{3\phi}{2}\left(\frac{v_1}{c}\right)^2 + \zeta_i\frac{v_1^i}{c} + \frac{v_1^i}{c}\partial_i\partial_0\chi\right\}. \tag{3}$$

(ii) The motion of particle m_1 is derived from the contribution of m_2 only to the metric perturbations ${}^{(i)}g_{\mu\nu}$. This means in turn that $T^{\mu\nu}$ from which ${}^{(i)}g_{\mu\nu}$ are derived depends only on m_2:

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\mathrm{d}t}{\mathrm{d}\tau_2}m_2\frac{\mathrm{d}x_2^\mu}{\mathrm{d}t}\frac{\mathrm{d}x_2^\nu}{\mathrm{d}t}\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_2(t)\right). \tag{4}$$

From the lecture notes, we have that $-g \simeq 1 - 4\phi$, and hence $(-g)^{-1/2} \simeq 1 + 2\phi$. We also have:

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}t} = \frac{1}{c}\left(-g_{\mu\nu}\frac{\mathrm{d}x_2^\mu}{\mathrm{d}t}\frac{\mathrm{d}x_2^\nu}{\mathrm{d}t}\right)^{1/2} \simeq 1 + \phi - \frac{v_2^2}{2c^2}, \quad \Rightarrow \quad \frac{\mathrm{d}t}{\mathrm{d}\tau_2} \simeq 1 - \phi + \frac{v_2^2}{2c^2}. \tag{5}$$

Putting everything together we get:

$$T^{\mu\nu} = \left(1 + \phi + \frac{v_2^2}{2c^2}\right) m_2 \frac{\mathrm{d}x_2^\mu}{\mathrm{d}t}\frac{\mathrm{d}x_2^\nu}{\mathrm{d}t}\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_2(t)\right), \tag{6}$$

which can be finally expanded as:

$${}^{(0)}T^{00} = m_2 c^2 \delta^{(3)}(\mathbf{x} - \mathbf{x}_2(t)), \tag{7}$$

$${}^{(2)}T^{00} = m_2 c^2 \left(\phi + \frac{v_2^2}{2c^2}\right)\delta^{(3)}(\mathbf{x} - \mathbf{x}_2(t)), \tag{8}$$

$${}^{(1)}T^{0i} = m_2 c^2 \; \frac{v_2^i}{c} \; \delta^{(3)}(\mathbf{x} - \mathbf{x}_2(t)), \tag{9}$$

$${}^{(2)}T^{ij} = m_2 c^2 \; \frac{v_2^i v_2^j}{c^2} \; \delta^{(3)}(\mathbf{x} - \mathbf{x}_2(t)). \tag{10}$$

(iii) Let's start with ϕ. We have:

$$\phi(\mathbf{x}, t) = -\frac{G}{c^2}\int \frac{\mathrm{d}\mathbf{x}' \; m_2 \; \delta^{(3)}(\mathbf{x}' - \mathbf{x}_2(t))}{|\mathbf{x}' - \mathbf{x}_2(t)|} = -\frac{Gm_2}{|\mathbf{x} - \mathbf{x}_2(t)|c^2}. \tag{11}$$

For ζ_i we have:

$$\zeta_i(\mathbf{x}, t) = -\frac{4G}{c^2}\frac{v_2^i}{c}\int \frac{\mathrm{d}\mathbf{x}' \; m_2 \; \delta^{(3)}(\mathbf{x}' - \mathbf{x}_2(t))}{|\mathbf{x}' - \mathbf{x}_2(t)|} = -\frac{4Gm_2}{|\mathbf{x} - \mathbf{x}_2(t)|c^2}\frac{v_2^i}{c}, \tag{12}$$

or:

$$\boldsymbol{\zeta}(\mathbf{x}, t) = -\frac{4Gm_2}{|\mathbf{x} - \mathbf{x}_2(t)|c^2}\frac{\mathbf{v}_2}{c} \tag{13}$$

χ is given by:

$$\chi(\mathbf{x}, t) = -\frac{G}{2c^2}\int \mathrm{d}\mathbf{x}'|\mathbf{x}' - \mathbf{x}_2(t)| \; m_2 \; \delta^{(3)}(\mathbf{x}' - \mathbf{x}_2(t)) = -\frac{Gm_2}{2c^2}|\mathbf{x} - \mathbf{x}_2(t)|. \tag{14}$$

The time derivative of χ is:

$$\partial_0\chi(\mathbf{x}, t) = \frac{1}{c}\partial_t\chi = \frac{Gm_2}{2c^2}\frac{(\mathbf{x} - \mathbf{x}_2)}{|\mathbf{x} - \mathbf{x}_2|}\cdot\frac{\mathbf{v}_2}{c}. \tag{15}$$

The actual expression that we need is $v_1^i\partial_i\partial_0\chi = (\mathbf{v}_1 \cdot \nabla)\partial_0\chi$ which, after a bit of algebra, gives:

$$(\mathbf{v}_1 \cdot \nabla)\partial_0\chi(\mathbf{x}, t) = \frac{Gm_2}{2|\mathbf{x} - \mathbf{x}_2(t)|c^2}\left\{\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c} - \frac{[(\mathbf{x} - \mathbf{x}_2(t)) \cdot \mathbf{v}_1][(\mathbf{x} - \mathbf{x}_2(t)) \cdot \mathbf{v}_2]}{c|\mathbf{x} - \mathbf{x}_2(t)|^2}\right\}. \tag{16}$$

Finally, we have to evaluate ψ, which is:

$$\psi(\mathbf{x}, t) = -\frac{G}{c^2}\int \frac{\mathrm{d}\mathbf{x}' \; m_2 \; \delta^{(3)}(\mathbf{x}' - \mathbf{x}_2(t))}{|\mathbf{x} - \mathbf{x}_2(t)|}\left(\frac{3}{2}\frac{v_2^2}{c^2} + \phi\right). \tag{17}$$

In this case, $\phi = \phi'$ is the newtonian potential that depends *on the other particles* that compose the system. Since for particle m_1 the only particle that contribute to the metric is m_2, the system is made by m_2 only and there are not any other particles, so $\phi = 0$. Therfore, we finally have:

$$\psi(\mathbf{x}, t) = -\frac{3Gm_2}{2|\mathbf{x} - \mathbf{x}_2(t)|c^2} \frac{v_2^2}{c^2}. \tag{18}$$

(iv) Putting all the definitions above into equation (3) and applying the generic position $\mathbf{x}$ to $\mathbf{x}_1$ we get the lagrangian for m_1 up to 1PN correction:

$$\mathcal{L} = \frac{1}{2} m_1 v_1^2 + \frac{G m_1 m_2}{r_{12}} + \frac{1}{8} m_1 \frac{v_1^4}{c^2} - \frac{G^2 m_2^2 m_1}{2 r_{12}^2 c^2} + \\ + \frac{G m_1 m_2}{2 r_{12}} \left\{ 3 \left[\frac{v_1^2}{c^2} + \frac{v_2^2}{c^2} \right] - 7 \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2} - \frac{(\hat{\mathbf{r}}_{12} \cdot \mathbf{v}_1)(\hat{\mathbf{r}}_{12} \cdot \mathbf{v}_2)}{c^2} \right\}, \tag{19}$$

where we defined $r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|$ and $\hat{\mathbf{r}}_{12} = (\mathbf{x}_1 - \mathbf{x}_2)/|\mathbf{x}_1 - \mathbf{x}_2|$. There are two purelly symmetric terms in the subscript 1 and 2, which are $G m_1 m_2 / r_{12}$ and $G m_1 m_2 / (2 r_{12}) \{\ldots\}$. Therefore, summing up the above lagrangian with the equivalent one for m_2 without double counting the symmetrica terms gives finally:

$$\mathcal{L} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{G m_1 m_2}{r_{12}} + \frac{1}{8} m_1 \frac{v_1^4}{c^2} + \frac{1}{8} m_2 \frac{v_2^4}{c^2} - \frac{G^2 m_2 m_1 (m_1 + m_2)}{2 r_{12}^2 c^2} + \\ + \frac{G m_1 m_2}{2 r_{12}} \left\{ 3 \left[\frac{v_1^2}{c^2} + \frac{v_2^2}{c^2} \right] - 7 \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2} - \frac{(\hat{\mathbf{r}}_{12} \cdot \mathbf{v}_1)(\hat{\mathbf{r}}_{12} \cdot \mathbf{v}_2)}{c^2} \right\}. \tag{20}$$