
Exercise 1 [Post-Newtonian Lagrangian for a binary system]

(i) The action is given by

$$S_1 = -m_1 \int \mathrm{d}t \left(-g_{00}c^2 - 2cg_{0i}v_1^i - g_{ij}v_1^i v_1^j\right)^{1/2} =$$
$$= -m_1 c^2 \int \mathrm{d}t \left(1 - {}^{(2)}g_{00} - {}^{(4)}g_{00} - 2\,{}^{(3)}g_{0i}\frac{v_1^i}{c} - \frac{v_1^2}{c^2} - {}^{(2)}g_{ij}\frac{v_1^i v_1^j}{c^2}\right)^{1/2} . \tag{1}$$

We can expand the big parenthesis in equation (1) as $(1+x)^{1/2} \simeq 1 + x/2 - x^2/8 + \ldots$, and keeping only terms up to ϵ^4 we have:

$$S_1 = -m_1 c^2 \int \mathrm{d}t \left\{ \left(1 - \frac{{}^{(2)}g_{00}}{2} - \frac{v_1^2}{2c^2}\right) + \left(-\frac{v_1^4}{8c^4} - \frac{{}^{(2)}g_{00}^2}{8} - \frac{{}^{(2)}g_{00}}{4}\frac{v_1^2}{c^2} - \frac{{}^{(4)}g_{00}}{2} - \frac{{}^{(3)}g_{0i}v_1^i}{c} - \frac{{}^{(2)}g_{ij}v_1^i v_1^j}{2c^2}\right)\right\} =$$
$$= -m_1 c^2 \int \mathrm{d}t + m_1 c^2 \int \mathrm{d}t \left\{ \left(\frac{{}^{(2)}g_{00}}{2} + \frac{v_1^2}{2c^2}\right) + \left(+\frac{v_1^4}{8c^4} + \frac{{}^{(2)}g_{00}^2}{8} + \frac{{}^{(2)}g_{00}}{4}\frac{v_1^2}{c^2} + \frac{{}^{(4)}g_{00}}{2} + \frac{{}^{(3)}g_{0i}v_1^i}{c} + \frac{{}^{(2)}g_{ij}v_1^i v_1^j}{2c^2}\right)\right\} . \tag{2}$$

The first integral in equation (2) is associated to the rest-mass energy and does not appear in any equation. We can neglect it. The second integral contains the actual Lagrangian of the particle m_1. The first term inside the second integral is the Newtonian part $\left(\mathcal{O}\left(\epsilon^2\right)\right)$, whereas the second term is the post-newtonian part $\left(\mathcal{O}\left(\epsilon^4\right)\right)$. Using the definitions in the exercise sheet for ${}^{(i)}g_{\mu\nu}$, we finally have:

$$\mathcal{L} = m_1 c^2 \left\{ \frac{1}{2}\left(\frac{v_1}{c}\right)^2 - \phi + \frac{1}{8}\left(\frac{v_1}{c}\right)^4 - \frac{\phi^2}{2} - \psi - \frac{3\phi}{2}\left(\frac{v_1}{c}\right)^2 + \zeta_i \frac{v_1^i}{c} + \frac{v_1^i}{c}\partial_i \partial_0 \chi \right\} . \tag{3}$$

(ii) The motion of particle m_1 is derived from the contribution of m_2 only to the metric perturbations ${}^{(i)}g_{\mu\nu}$. This means in turn that $T^{\mu\nu}$ from which ${}^{(i)}g_{\mu\nu}$ are derived depends only on m_2:

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\mathrm{d}t}{\mathrm{d}\tau_2} m_2 \frac{\mathrm{d}x_2^\mu}{\mathrm{d}t}\frac{\mathrm{d}x_2^\nu}{\mathrm{d}t}\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_2(t)\right) . \tag{4}$$

From the lecture notes, we have that $-g \simeq 1 - 4\phi$, and hence $(-g)^{-1/2} \simeq 1 + 2\phi$. We also have:

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}t} = \frac{1}{c}\left(-g_{\mu\nu}\frac{\mathrm{d}x_2^\mu}{\mathrm{d}t}\frac{\mathrm{d}x_2^\nu}{\mathrm{d}t}\right)^{1/2} \simeq 1 + \phi - \frac{v_2^2}{2c^2}, \quad \Rightarrow \quad \frac{\mathrm{d}t}{\mathrm{d}\tau_2} \simeq 1 - \phi + \frac{v_2^2}{2c^2}. \tag{5}$$

Putting everything together we get:

$$T^{\mu\nu} = \left(1 + \phi + \frac{v_2^2}{2c^2}\right) m_2 \frac{\mathrm{d}x_2^\mu}{\mathrm{d}t}\frac{\mathrm{d}x_2^\nu}{\mathrm{d}t}\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_2(t)\right), \tag{6}$$

which can be finally expanded as:

$$^{(0)}T^{00} = m_2 c^2 \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_2(t)\right), \tag{7}$$

$$^{(2)}T^{00} = m_2 c^2 \left(\phi + \frac{v_2^2}{2c^2}\right)\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_2(t)\right), \tag{8}$$

$$^{(1)}T^{0i} = m_2 c^2 \frac{v_2^i}{c}\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_2(t)\right), \tag{9}$$

$$^{(2)}T^{ij} = m_2 c^2 \frac{v_2^i v_2^j}{c^2}\delta^{(3)}\left(\mathbf{x} - \mathbf{x}_2(t)\right). \tag{10}$$

(iii) Let's start with ϕ. We have:

$$\phi(\mathbf{x}, t) = -\frac{G}{c^2}\int \frac{\mathrm{d}\mathbf{x}' m_2 \delta^{(3)}\left(\mathbf{x}' - \mathbf{x}_2(t)\right)}{|\mathbf{x}' - \mathbf{x}_2(t)|} = -\frac{G m_2}{|\mathbf{x} - \mathbf{x}_2(t)|\, c^2}. \tag{11}$$

For ζ_i we have:

$$\zeta_i(\mathbf{x}, t) = -\frac{4G}{c^2}\frac{v_2^i}{c}\int \frac{\mathrm{d}\mathbf{x}' m_2 \delta^{(3)}\left(\mathbf{x}' - \mathbf{x}_2(t)\right)}{|\mathbf{x}' - \mathbf{x}_2(t)|} = -\frac{4G m_2}{|\mathbf{x} - \mathbf{x}_2(t)|\, c^2}\frac{v_2^i}{c}, \tag{12}$$

or:

$$\zeta(\mathbf{x}, t) = -\frac{4G m_2}{|\mathbf{x} - \mathbf{x}_2(t)|\, c^2}\frac{\mathbf{v}_2}{c}. \tag{13}$$

χ is given by:

$$\chi(\mathbf{x}, t) = -\frac{G}{2c^2}\int \mathrm{d}\mathbf{x}' \left|\mathbf{x}' - \mathbf{x}_2(t)\right| m_2 \delta^{(3)}\left(\mathbf{x}' - \mathbf{x}_2(t)\right) = -\frac{G m_2}{2c^2}\left|\mathbf{x} - \mathbf{x}_2(t)\right|. \tag{14}$$

The time derivative of χ is:

$$\partial_0 \chi(\mathbf{x}, t) = \frac{1}{c}\partial_t \chi = \frac{G m_2}{2c^2}\frac{(\mathbf{x} - \mathbf{x}_2)}{|\mathbf{x} - \mathbf{x}_2|}\cdot\frac{\mathbf{v}_2}{c}. \tag{15}$$

The actual expression that we need is $v_1^i \partial_i \partial_0 \chi = (\mathbf{v}_1 \cdot \nabla)\,\partial_0\chi$ which, after a bit of algebra, gives:

$$(\mathbf{v}_1 \cdot \nabla)\,\partial_0\chi(\mathbf{x}, t) = \frac{G m_2}{2\,|\mathbf{x} - \mathbf{x}_2(t)|\, c^2}\left\{\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c} - \frac{\left[(\mathbf{x} - \mathbf{x}_2(t))\cdot\mathbf{v}_1\right]\left[(\mathbf{x} - \mathbf{x}_2(t))\cdot\mathbf{v}_2\right]}{c\,|\mathbf{x} - \mathbf{x}_2(t)|^2}\right\}. \tag{16}$$

Finally, we have to evaluate ψ, which is:

$$\psi(\mathbf{x},t) = -\frac{G}{c^2}\int \frac{\mathrm{d}\mathbf{x}' m_2 \delta^{(3)}\left(\mathbf{x}'-\mathbf{x}_2(t)\right)}{|\mathbf{x}-\mathbf{x}_2(t)|}\left(\frac{3}{2}\frac{v_2^2}{c^2}+\phi'\right) . \tag{17}$$

In this case, ϕ' is the external Newtonian potential that depends on the other particles that compose the system. Since for particle m_1 the only particle that contributes to the metric is m_2, the system is made by m_2 only and there are not any other particles, so $\phi' = 0$. Therefore, we finally have:

$$\psi(\mathbf{x},t) = -\frac{3Gm_2}{2\left|\mathbf{x}-\mathbf{x}_2(t)\right| c^2}\frac{v_2^2}{c^2} . \tag{18}$$

(iv) Putting all the definitions above into equation (3) and applying the generic position $\mathbf{x}$ to $\mathbf{x_1}$ we get the Lagrangian for m_1 up to 1 PN correction:

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}m_1 v_1^2 + \frac{Gm_1m_2}{r_{12}} + \frac{1}{8}m_1\frac{v_1^4}{c^2} - \frac{G^2 m_2^2 m_1}{2r_{12}^2 c^2} \\ &+ \frac{Gm_1m_2}{2r_{12}}\left\{3\left[\frac{v_1^2}{c^2}+\frac{v_2^2}{c^2}\right] - 7\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2} - \frac{\left(\hat{\mathbf{r}}_{12}\cdot\mathbf{v}_1\right)\left(\hat{\mathbf{r}}_{12}\cdot\mathbf{v}_2\right)}{c^2}\right\} ,\end{aligned} \tag{19}$$

where we defined $r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|$ and $\hat{\mathbf{r}}_{12} = \left(\mathbf{x}_1 - \mathbf{x}_2\right)/\left|\mathbf{x}_1 - \mathbf{x}_2\right|$. There are two purelly symmetric terms in the subscript 1 and 2, which are Gm_1m_2/r_{12} and $Gm_1m_2/\left(2r_{12}\right)\{\ldots\}$. Therefore, summing up the above Lagrangian with the equivalent one for m_2 without double counting the symmetrical terms gives finally:

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{Gm_1m_2}{r_{12}} + \frac{1}{8}m_1\frac{v_1^4}{c^2} + \frac{1}{8}m_2\frac{v_2^4}{c^2} - \frac{G^2 m_2 m_1\left(m_1+m_2\right)}{2r_{12}^2 c^2} \\ &+ \frac{Gm_1m_2}{2r_{12}}\left\{3\left[\frac{v_1^2}{c^2}+\frac{v_2^2}{c^2}\right] - 7\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{c^2} - \frac{\left(\hat{\mathbf{r}}_{12}\cdot\mathbf{v}_1\right)\left(\hat{\mathbf{r}}_{12}\cdot\mathbf{v}_2\right)}{c^2}\right\} .\end{aligned} \tag{20}$$