



Mechanics

Exercise Sheet 2

HS 2026
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Exercise 1: Skier on a bump

- a) Consider spherical polar coordinates r, ϕ, θ ($r > 0, \phi \in (-\pi, \pi), \theta \in (0, \pi)$) referred to cartesian coordinates x, y, z , as in Fig. 1.

Express the velocity and the acceleration of a point particle using polar coordinates and the unit vectors $\mathbf{e}_r, \mathbf{e}_\phi$ and $\mathbf{e}_\theta$. Prove that

$$\begin{aligned}\mathbf{v} &= \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta + r \dot{\phi} \sin \theta \mathbf{e}_\phi, \\ \mathbf{a} &= (\ddot{r} - r \dot{\theta}^2 - r \dot{\phi}^2 \sin^2 \theta) \mathbf{e}_r + (r \ddot{\theta} + 2 \dot{r} \dot{\theta} - r \dot{\phi}^2 \sin \theta \cos \theta) \mathbf{e}_\theta \\ &\quad + (r \ddot{\phi} \sin \theta + 2 \dot{r} \dot{\phi} \sin \theta + 2 r \dot{\phi} \dot{\theta} \cos \theta) \mathbf{e}_\phi.\end{aligned}$$

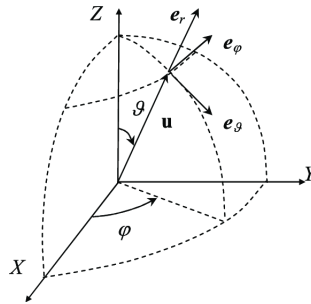


Figure 1: Spherical polar coordinates.

- b) A skier of mass m is skiing on a bump represented by a sphere of radius R (Fig. 2). He slides without initial speed from the top of the bump ($\theta = 0$).

Assuming the skier to be a point-like particle and neglecting friction, determine the speed of the skier as function of the angle θ .

Compute the reaction force $\mathbf{N}$ as a function of θ and find the angle at which the skier will take off.

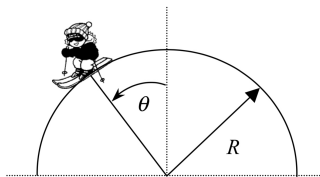


Figure 2: Skier on a bump.

Solution:

- a) The position of the particle in polar coordinates is $\mathbf{x}(t) = r(t) \mathbf{e}_r(t)$, hence velocity and acceleration are given by

$$\begin{aligned}\mathbf{v}(t) &= \dot{r} \mathbf{e}_r + r \dot{\mathbf{e}}_r, \\ \mathbf{a}(t) &= \ddot{r} \mathbf{e}_r + 2\dot{r} \dot{\mathbf{e}}_r + r \ddot{\mathbf{e}}_r.\end{aligned}\tag{1.1}$$

We need to find $\dot{\mathbf{e}}_r$, $\ddot{\mathbf{e}}_r$ in terms of $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$. In order to do this, it is convenient to do computations in terms of the cartesian unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$, which do not depend on time. From Fig. 1 we have the following relations

$$\begin{aligned}\mathbf{e}_r &= \sin \theta \cos \phi \mathbf{e}_x + \sin \theta \sin \phi \mathbf{e}_y + \cos \theta \mathbf{e}_z \\ \mathbf{e}_\theta &= \cos \theta \cos \phi \mathbf{e}_x + \cos \theta \sin \phi \mathbf{e}_y - \sin \theta \mathbf{e}_z \\ \mathbf{e}_\phi &= -\sin \phi \mathbf{e}_x + \cos \phi \mathbf{e}_y,\end{aligned}\tag{1.2}$$

which can be easily inverted by taking the transposed of the matrix representing 1.2. Then

$$\begin{aligned}\dot{\mathbf{e}}_r &= (\dot{\theta} \cos \theta \cos \phi - \dot{\phi} \sin \theta \sin \phi) \mathbf{e}_x + (\dot{\theta} \cos \theta \sin \phi + \dot{\phi} \sin \theta \cos \phi) \mathbf{e}_y - \dot{\theta} \sin \theta \mathbf{e}_z \\ &= \dot{\theta} \mathbf{e}_\theta + \dot{\phi} \sin \theta \mathbf{e}_\phi.\end{aligned}\tag{1.3}$$

Analogously,

$$\begin{aligned}\dot{\mathbf{e}}_\theta &= -\dot{\theta} \mathbf{e}_r + \dot{\phi} \cos \theta \mathbf{e}_\phi \\ \dot{\mathbf{e}}_\phi &= -\dot{\phi} \sin \theta \mathbf{e}_r - \dot{\theta} \cos \theta \mathbf{e}_\theta\end{aligned}\tag{1.4}$$

And

$$\begin{aligned}\ddot{\mathbf{e}}_r &= \ddot{\theta} \mathbf{e}_\theta + \dot{\theta} \dot{\mathbf{e}}_\theta + \ddot{\phi} \sin \theta \mathbf{e}_\phi + \dot{\phi} \dot{\theta} \cos \theta \mathbf{e}_\phi + \dot{\phi} \cos \theta \dot{\mathbf{e}}_\phi \\ &= -(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) \mathbf{e}_r + (\ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta) \mathbf{e}_\theta + (2\dot{\theta} \dot{\phi} \cos \theta + \ddot{\phi} \sin \theta) \mathbf{e}_\phi\end{aligned}\tag{1.5}$$

Substituting $\dot{\mathbf{e}}_r$, $\ddot{\mathbf{e}}_r$ in 1.1 we obtain the desired result, namely

$$\begin{aligned}\mathbf{v} &= \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta + r \dot{\phi} \sin \theta \mathbf{e}_\phi \\ \mathbf{a} &= (\ddot{r} - r \dot{\theta}^2 - r \dot{\phi}^2 \sin^2 \theta) \mathbf{e}_r + (r \ddot{\theta} + 2\dot{r} \dot{\theta} - r \dot{\phi}^2 \sin \theta \cos \theta) \mathbf{e}_\theta \\ &\quad + (r \ddot{\phi} \sin \theta + 2\dot{r} \dot{\phi} \sin \theta + 2r \dot{\theta} \dot{\phi} \cos \theta) \mathbf{e}_\phi.\end{aligned}\tag{1.6}$$

- b) Using the conservation of mechanical energy

$$\frac{1}{2}mv^2 + mgR \cos \theta = mgR,\tag{1.7}$$

we find the scalar velocity of the skier at an angle θ to be

$$v = \sqrt{2gR(1 - \cos \theta)}.\tag{1.8}$$

Simplifying the expression 1.6 by taking constant radius (the skier stays on the surface of the sphere) and, without loss of generality, constant ϕ , we find

$$\begin{aligned}\mathbf{r} &= R \mathbf{e}_r \\ \mathbf{v} &= R \dot{\theta} \mathbf{e}_\theta \\ \mathbf{a} &= -R \dot{\theta}^2 \mathbf{e}_r + R \ddot{\theta} \mathbf{e}_\theta.\end{aligned}\tag{1.9}$$

We can now use Newton's second law in $\mathbf{e}_r$ direction

$$N - mg \cos \theta = m(-R\dot{\theta}^2) = -\frac{mv^2}{R} \quad (1.10)$$

Since we know the velocity as function of θ from 1.8, we find the reaction force

$$N = mg(3 \cos \theta - 2) \quad (1.11)$$

which as expected is mg when $\theta = 0$. The condition for take-off is $N \leq 0$ which means

$$\cos \theta \leq \frac{2}{3} \quad \Rightarrow \quad \theta \geq 48.2^\circ \quad (1.12)$$

Exercise 2: Conservation laws for a system of particles.

The profile of Fig. 3 has mass M , curvature radius R and lies on a horizontal, frictionless plane. A point-like body of mass m can move without friction on the curved surface of the profile. At t_i the body is at rest on top of the profile.

- Are the total linear momentum and mechanical energy of the system (profile + body) conserved? Why? Find the x -component of the velocity of the body v_x and the profile V_x once the body has reached the basis of the profile.
- Find the velocity of the center-of-mass of the system as a function of the coordinate θ describing the position of the body on the profile. Check that $(v_{CM})_y = 0$ for $\theta = 0, \pi/2$.

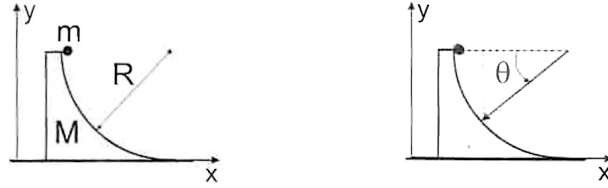


Figure 3: System of exercise 2.

Solution:

- The external forces acting on the system are the gravitational forces $M\mathbf{g}$ and $m\mathbf{g}$ and the normal reaction $\mathbf{N}$ that the horizontal plane exerts on the profile. Such forces are all directed along the y axis and their resultant is in general non-zero, hence the total linear momentum of the system is not conserved. However,

$$\sum_x F_x^{ext} = 0$$

hence the x -component of the momentum of the system is conserved. Since $P_x^i = 0$, we have the following relation among the x -components of the velocities once the body has reached the basis of the profile

$$P_x^f = MV_x^f + mv_x^f = 0, \quad (2.1)$$

The external forces acting on the system are either conservative ($M\mathbf{g}$, $m\mathbf{g}$) or do not do any work ($\mathbf{N}$). As to the internal forces, in absence of friction they are perpendicular to the curved surface of the profile, hence they do not work either. As a consequence the mechanical energy of the system is conserved

$$U_i + T_i = U_f + T_f,$$

yielding

$$mgR = \frac{1}{2}m(v^f)^2 + \frac{1}{2}M(V^f)^2. \quad (2.2)$$

From 2.1 and 2.2 (noting that $v_y^f = V_y^f = 0$), we have

$$(v^f)^2 = 2gR \frac{M}{m+M} \quad (V^f)^2 = 2gR \frac{m^2}{M(m+M)}.$$

b) The components of the CM velocity along x and y are given by

$$(v_{cm})_x = \frac{mv_x + MV_x}{m + M} = 0 \quad (v_{cm})_y = \frac{mv_y}{m + M}, \quad (2.3)$$

where we used the conservation of momentum along x . We see that the CM moves vertically. Let us study the behaviour of v_y as a function of θ . Using the conservation of mechanical energy we can equate the mechanical energy at t_i ($\theta = 0$) to the mechanical energy at a generic $\theta \in [0, \pi/2]$:

$$mgR = mgR(1 - \sin \theta) + \frac{1}{2}MV_x^2 + \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2. \quad (2.4)$$

Using the conservation of linear momentum along x we can express V_x as a function of v_x as

$$V_x = -\frac{m}{M}v_x, \quad (2.5)$$

finding

$$mgR \sin \theta = \frac{1}{2} \frac{m}{M} (m + M) v_x^2 + \frac{1}{2} m v_y^2. \quad (2.6)$$

In order to find a relation between v_x and v_y it is convenient to work in the reference frame where the profile is at rest. In this frame the x and y components of the velocity of the body are

$$v'_x = v_x - V = \left(1 + \frac{m}{M}\right) v_x \quad v'_y = v_y. \quad (2.7)$$

Since the body follows a circular trajectory with velocity tangent to the circumference of radius R , we have

$$\frac{v'_x}{v'_y} = \tan \theta \quad \rightarrow \quad v_x = v_y \tan \theta \frac{M}{m + M}. \quad (2.8)$$

Substituting this result in 2.6, we find

$$v_y^2 = 2gR \sin \theta \frac{m + M}{m + M(1 + \tan^2 \theta)}. \quad (2.9)$$

The final result is therefore

$$(v_{cm})_y^2 = \left(\frac{m}{m + M}\right)^2 v_y^2 = \frac{2gR m^2 \sin \theta}{(m + M)(m + M(1 + \tan^2 \theta))}, \quad (2.10)$$

which indeed vanishes for $\theta = 0$ and $\theta = \pi/2$.

Exercise 3: Atwood machine

Consider the following device under the action of gravity :

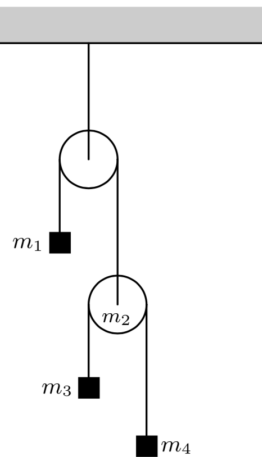
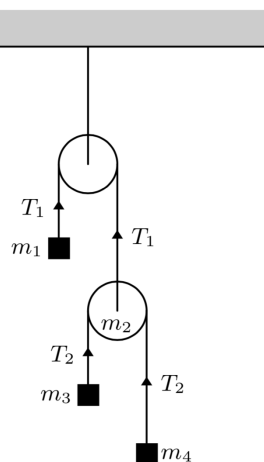


Figure 4: Atwood machine

- Write Newton's equations for the 4 masses and give the constraints of the system.
Hint: What do the finite lengths of the strings imply?
- Choose a_1 , a_3 , T_1 and T_2 as variables (where T_1 and T_2 are the tension of the cables) and solve their equation of motion.
Hint: Write $[(m_1 + m_2)(m_3 + m_4) + 4m_3m_4]v_i$ for each variable $v_i = a_1, a_3, T_1, T_2$.
- What can we tell about a_1 if m_1 and m_2 are the same? And if $m_1 = m_2 + m_3 + m_4$?
- What are the limits of this description?

Solution: Let us draw explicitly the tension of the cables T_1 and T_2 .



a) Choosing the z axis vertically pointing downwards, Newton's equations are

$$\begin{aligned} m_1 \ddot{z}_1 &= m_1 g - T_1 \\ m_2 \ddot{z}_2 &= m_2 g - T_1 + 2T_2 \\ m_3 \ddot{z}_3 &= m_3 g - T_2 \\ m_4 \ddot{z}_4 &= m_4 g - T_2. \end{aligned} \quad (3.1)$$

The strings being inextensible gives us two constraints on the position of the masses. Defining z_0 as the position of the first pulley and R_i the radius of the pulley i , the length of the strings can be written as

$$\begin{aligned} L_1 &= (z_1 - z_0) + \pi R_1 + (z_2 - z_0) \\ L_2 &= (z_3 - z_2) + \pi R_2 + (z_4 - z_2). \end{aligned} \quad (3.2)$$

We take the second derivative with respect to time to obtain the constraints on the accelerations

$$\begin{aligned} \ddot{z}_1 &= -\ddot{z}_2 \equiv a_1 \\ \ddot{z}_3 &\equiv a_3 \\ \ddot{z}_4 &= -2a_1 - a_3. \end{aligned} \quad (3.3)$$

We obtain the system of equations

$$\begin{aligned} m_1 a_1 &= m_1 g - T_1 \\ m_2 a_1 &= -m_2 g + T_1 - 2T_2 \\ m_3 a_3 &= m_3 g - T_2 \\ m_4 (2a_1 + a_3) &= -m_4 g + T_2. \end{aligned} \quad (3.4)$$

b) The solution of Eq. 3.4 are

$$\begin{aligned} a_1 &= \frac{(m_1 - m_2)(m_3 + m_4) - 4m_3 m_4}{(m_1 + m_2)(m_3 + m_4) + 4m_3 m_4} g \\ a_3 &= \frac{(m_1 + m_2 + 4m_4)m_3 + (m_2 - 3m_1)m_4}{(m_1 + m_2)(m_3 + m_4) + 4m_3 m_4} g \\ T_1 &= \frac{2m_1(4m_3 m_4 + m_2(m_3 + m_4))}{(m_1 + m_2)(m_3 + m_4) + 4m_3 m_4} g \\ T_2 &= \frac{4m_1 m_3 m_4}{(m_1 + m_2)(m_3 + m_4) + 4m_3 m_4} g. \end{aligned} \quad (3.5)$$

c) If $m_1 = m_2$ then $a_1 < 0$. The mass m_1 is pulled upwards.

If $m_1 = m_2 + m_3 + m_4$ then $a_1 \propto (m_3 - m_4)^2$. If $m_3 \neq m_4$ then $a_1 > 0$ and the mass m_1 is going down. However if $m_3 = m_4$, we have $a_1 = a_3 = 0$.

d) This description is valid for an "infinite" string length. It does not explain what happens when the mass does not hit the pulley.