**QFT methods for BSM theories**
**Problem Set 2.**

HS 2019
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**Exercise 1.** ***The scalar sector of the Standard Model***

The scalar and the Yukawa sectors of the Standard Model Lagrangian are given by,

$$\mathcal{L} = \bar{Q}_L i\not{D} Q_L + \bar{d}_R i\not{D} d_R + \bar{u}_R i\not{D} u_R + |D_\mu \Phi|^2 + \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 - y_u \bar{Q}_L \tilde{\Phi} u_R - y_d \bar{Q}_L \Phi d_R + \text{h.c.} \quad (1)$$

where $\Phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(h - i\phi^0) \end{pmatrix}$ is the Higgs scalar $SU(2)_L$ doublet, $\phi^\pm = \frac{1}{\sqrt{2}}(\phi_1 \pm i\phi_2)$ and $\tilde{\Phi} = i\tau_2 \Phi^*$. The $D_\mu$ is the $SU(2)_L \otimes U(1)_Y$ gauge-covariant derivative is,

$$D_\mu = \partial_\mu + ig\frac{\vec{\tau}}{2}\vec{W}_\mu - ig'\frac{\tau_3}{2}B_\mu. \quad (2)$$

where $\tau^i$ are the Pauli-matrices.

1. In the limit of vanishing Yukawa and hypercharge interactions, which is the symmetry of the fermion sector?
2. Let us define the scalar field,
$$\Sigma = h + i\vec{\tau}\cdot\vec{\phi}. \quad (3)$$
where $\vec{\phi} = (\phi^2 \ \phi^1 \ \phi^0)^T$.
Show that the scalar sector in terms of $\Sigma$,
$$\mathcal{L} = \frac{1}{4}\text{Tr}(D_\mu \Sigma^\dagger D^\mu \Sigma) + \frac{\mu^2}{4}\text{Tr}(\Sigma^\dagger \Sigma) - \frac{\lambda}{16}(\text{Tr}(\Sigma^\dagger \Sigma))^2, \quad (4)$$
is equivalent to the scalar sector in Eq. (1).
3. Under which condition is the Yukawa sector of Eq. (1) equivalent to the term $-\frac{y}{\sqrt{2}}\bar{Q}_L \Sigma\,(u_R \ d_R)^{\text{T}}$? How should $\Sigma$ transform if we wish to keep this term invariant under the symmetry? How do the fields $h$ and $\vec{\phi}$ transform in this case? Show that the scalar sector is then also invariant under the symmetry.
4. It is convenient to work with fields with nonzero vacuum expectation value. To this end we perform the shift,
$$h \to h + v \quad \text{and} \quad \vec{\phi} \to \vec{\phi}, \quad (5)$$
where $v = \sqrt{\frac{\mu^2}{\lambda}}$. Which is the symmetry of the vacuum? This symmetry is called *custodial symmetry.*
5. Expand the Lagrangian after the shift (5) and identify the mass spectrum. For simplicity, neglect the electroweak interactions.

6. Compute the amplitude for the scattering $\phi^+\phi^- \to \phi^+\phi^-$ at tree-level. Confirm that a basic property of Goldstone interactions is manifested. Then examine the behaviour of each diagram in the low-energy limit.

We define a more convenient realisation of the model by performing a field redefinition of the form,

$$\Sigma(x) = (v + H(x))U(G(x)), \quad U(G(x)) = \exp\left(i\vec{\tau} \cdot \vec{G}(x)/v\right) \tag{6}$$

where $\vec{G} = (G^2\ G^1\ G^0)^T$.

7. How do $H$ and $U$ transform under the symmetry? How do the fields $\vec{G}$ transform under the custodial symmetry?

8. Show that using Eq. (6) the Lagrangian (neglecting the Yukawa interaction) takes the form,

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu H)^2 - 2\lambda v^2 H^2\right] + \frac{(v+H)^2}{4}\mathrm{Tr}(D_\mu U^\dagger D^\mu U) - \lambda v H^3 - \frac{\lambda}{4}H^4. \tag{7}$$

9. Verify explicitly that the amplitude for the scattering $G^+G^- \to G^+G^-$ at tree-level is the same with the full result of part 6. In the low-energy limit, what difference do you observe with what you found in part 6?

10. Consider now a theory described solely by the Lagrangian,

$$\mathcal{L} = \frac{v^2}{4}\mathrm{Tr}(D_\mu U^\dagger D^\mu U), \tag{8}$$

Calculate the cross-section of the process $G^+G^- \to G^+G^-$ at tree-level, take the high energy limit and determine at which point the theory loses unitarity. Can you comment on the necessity of the field $H$ in Eq. (7) with respect to the unitarization of the theory.