MMP I
Tutorial 3

HS 2017
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Exercise 1: Fourier transform and convolution of a specific function (4 Pts.)
Let f be the following function:

$$f(x) = e^{-a|x|} \ . \ a > 0. \tag{1.1}$$

a) Evaluate the Fourier transform of f, $\hat{f}(k)$.

b) Find the function g such that $\hat{g}(k) = (\frac{2}{1+k^2})^2$.
Hint: Recall

$$\mathcal{F}(f * g)(k) = \mathcal{F}f(k)\mathcal{F}g(k). \tag{1.2}$$

Exercise 2: Heat equation in $\mathbb{R}$ (5 Pts.)
Consider the homogeneous heat equation:

$$\partial_t u(x,t) - a^2 \Delta_x u(x,t) = 0 \ , \ t > 0 \tag{2.1}$$

with the initial condition $u(0,x) = f(x)$ where f is a bounded function, i.e. $|f(x)| \leq M < +\infty$ for $x \in \mathbb{R}$.

a) Find the bounded solution $u(x,t)$ to the heat equation in terms of the Gaussian function:

$$K_t(x-y) = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4ta^2}}, \ t > 0 \ . \tag{2.2}$$

Note: $\mathcal{F}(\partial_x^\alpha f)(k) = i^{|\alpha|} k^\alpha (\mathcal{F}f)(k)$.

b) Take f to be $f(x) = \delta(x)$ and find the corresponding solution $u(x,t)$. Graphically represent how $u(x,t)$ evolves as a function of $t > 0$.

– please turn over –

Exercise 3: Wave equation in $\mathbb{R}$ (5 Pts.)
Consider the wave equation defined on $\mathbb{R} \times \mathbb{R}$:

$$\frac{1}{c^2}\partial_t^2 u(x,t) - \Delta_x u(u,t) = 0 \tag{3.1}$$

with the initial conditions $u(x,0) = f(x)$ and $\partial_t u(x,0) = g(x)$.

a) Use Fourier transform to obtain the following solution to (3.1):

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\hat{f}(k)\cos\left(|k|ct\right) + \frac{\hat{g}(k)}{|k|c}\sin\left(|k|ct\right)\right]\mathrm{d}k. \tag{3.2}$$

b) Check that the above solution indeed does satisfy the wave equation (3.1).

c) Show that the solution satisfies the initial conditions.