Exercise 1. ***Perturbed Harmonic Oscillator***

Consider the Hamiltonian describing an anharmonic oscillator

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + \lambda x^4 \ , \tag{1}$$

where the term λx^4 is treated as a perturbation.

(a) Write down the Hamiltonian in terms of creation and annihilation operators.

(b) Compute the correction to the energy of the n-th excited state up to first order in λ.

Exercise 2. ***Perturbation theory applied to a three-level system***

Non-degenerate perturbation theory fails when some of the energy levels of the unperturbed system become degenerate. In order to illustrate how to correctly handle this case we consider the following Hamiltonian

$$H = -DS_z^2 + \lambda B S_x \tag{2}$$

acting on a spin-1 particle. The purpose of this exercise is to find the spectrum of this Hamiltonian, treating the second term as a small perturbation.

(a) What are the eigenvalues and eigenvectors of the unperturbed Hamiltonian $H_{\lambda=0}$?

(b) Find the correction to the energy of the $|m_s = 0\rangle$ state up to second order in the perturbation parameter λ. Compute also the corresponding eigenvector up to first order in perturbation theory.

(c) In order to obtain the corrections to the energy of the two other states $|m_s = -1\rangle$ and $|m_s = +1\rangle$ one needs to use degenerate perturbation theory. Show that the first order corrections vanish.

(d) Which matrix needs to be diagonalised in order to obtain the second order corrections? Calculate these corrections and find the associated eigenvectors.

(e) Find the exact spectrum of the Hamiltonian 2. Compare your results with the ones obtained using perturbation theory.

Exercise 3. ***Exact solution vs perturbation theory***

Consider the Hamiltonian of a two dimensional harmonic oscillator with an xy coupling $H = H_0 + \lambda V$, where

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) \ , \qquad V = m\omega^2 xy \ . \tag{3}$$

The purpose of this exercise is to find the spectrum of this Hamiltonian exactly and then compare to the results obtained using perturbation theory, by treating the xy coupling as a perturbation.

Exact solution

(a) To find the spectrum one needs to make a change of variables to decouple the oscillators. Rewrite the Hamiltonian in terms of $x + y$ and $x - y$, then introduce two new pairs of canonically conjugated operators (X, P_X) and (Y, P_Y) and write H as a sum of two independent harmonic oscillators.

(b) What are the eigenvalues of H ? In particular, what are the energies E_0 and E_1 of the two lowest lying states ?

Perturbation theory

(c) Write down H_0 in terms of ladder operators $(a_x, a_x^\dagger)$ and $(a_y, a_y^\dagger)$ corresponding to the x and y directions. Find the energies and eigenstates of the unperturbed system. Determine the degeneracy of the first energy levels.

(d) Compute the correction to the energy of the ground state up to second order in perturbation theory.

(e) Compute the correction to the energy of the first and second excited states up to first order in perturbation theory.

(f) Compare your results to the exact spectrum obtained in (b).