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# Applications of General Relativity

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# Solutions 4

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**Exercise 1** [Metric of a static star] (8 points)

We start from the general static, spherical symmetric metric

$$\mathrm{d}s^2 = \exp[2\alpha(r)]\mathrm{d}t^2 - \exp[2\beta(r)]\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{1}$$

From the normalization of the velocity in the fluid restframe we obtain $u_t = \sqrt{g_{tt}}$, which then straightforwardly leads to the energy-momentum tensor

$$T_{\mu\nu} = \mathrm{diag}\left\{\exp[2\alpha]\,\rho, \exp[2\beta]\,p, r^2\,p, r^2\sin^2(\theta)\ p\right\}. \tag{2}$$

The Ricci scalar for the above metric ansatz read as

$$R = 2\exp[-2\beta]\left(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}(\partial_r\alpha - \partial_r\beta) + \frac{1}{r^2}(1-\exp[2\beta])\right). \tag{3}$$

For the time component of the Einstein equations we thus have

$$R_{tt} - \frac{1}{2}Rg_{tt} = 8\pi G T_{tt} \tag{4}$$

$$-\exp[2(\alpha-\beta)]\left(-\frac{2}{r}\partial_r\beta + \frac{1}{r^2}(1-\exp[2\beta])\right) = 8\pi G\rho\exp[2\alpha] \tag{5}$$

$$\frac{1}{r^2}\partial_r m(r) = 4\pi\rho, \tag{6}$$

where $m(r)$ is defined in Eq. (3) of the exercise sheet. This differential equation can be integrated to obtain the mass within a given radius $r$

$$m(r) = 4\pi\int_0^r \rho(r')r'^2\mathrm{d}r'. \tag{7}$$

Note that it is different from the integrated energy density, which is obtained integrating the density over the volume element $\sqrt{\gamma}\mathrm{d}^3x$. Here $\gamma$ is the determinant of the spatial metric. In contrast to the above, the integrated energy density accounts for the gravitational binding energy as well.
With the replacement of $m(r)$ we can immediately write down the radial metric element

$$\exp[2\beta] = \left(1 - \frac{2Gm(r)}{r}\right)^{-1}, \tag{8}$$

which has remarkable similarity with the metric element of the Schwarzschild solution. For the radial component we have

$$R_{rr} - \frac{1}{2}Rg_{rr} = 8\pi G T_{rr} \tag{9}$$

$$\frac{2}{r}\partial_r\alpha - \frac{2G}{r^3}\exp[2\beta]m(r) = 8\pi G p \tag{10}$$

thus

$$\frac{\mathrm{d}\alpha}{\mathrm{d}r} = \frac{4\pi G r^3 p + G m(r)}{r\left[r - 2Gm(r)\right]} \tag{11}$$

The Bianchi identy or energy-momentum conservation reads as

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\rho} T^{\rho\nu} + \Gamma^\nu_{\mu\rho} T^{\mu\rho}. \tag{12}$$

While the $\nu = t$ equation is trivial, the $\nu = r$ equation leads to

$$\begin{aligned}
\nabla_\mu T^{\mu r} =& \partial_r (\exp\left[-2\beta\right] p) + T^{rr}(\Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r} + \Gamma^t_{tr}) && (13)\\
&+ (\Gamma^r_{tt} T^{tt} + \Gamma^r_{rr} T^{rr} + \Gamma^r_{\phi\phi} T^{\phi\phi} + \Gamma^r_{\theta\theta} T^{\theta\theta}) && (14)\\
=& \exp\left[-2\beta\right]\left\{(\partial_r p - 2p\partial_r \beta) + p(\partial_r\beta + 2/r + \partial_r \alpha) + (\rho \partial_r \alpha + \partial_r \beta p - 2p/r)\right\} && (15)\\
=& \exp\left[-2\beta\right]\left\{(\rho + p)\partial_r \alpha + \partial_r p\right\} \overset{!}{=} 0 && (16)
\end{aligned}$$

From this we obtain

$$(\rho + p)\frac{\mathrm{d}\alpha}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r}. \tag{17}$$

Plugging in Eq. (11) we finally have

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)\left[Gm(r) + 4\pi G r^3 p\right]}{r\left[r - 2Gm(r)\right]}. \tag{18}$$

Now we assume $\rho = \rho_* =$ const. out to the surface of the star yielding

$$m(r) = \frac{4\pi}{3}\rho_* r^3 = M\frac{r^3}{R^3}, \tag{19}$$

where $M$ is the total mass of the star and from now on $R$ will denote the radius of the star. From Eq. (18) we obtain by separation of variables

$$\begin{aligned}
\frac{\mathrm{d}p}{p^2 + \frac{4}{3}\rho_* p + \frac{\rho_*^2}{3}} =& \frac{4\pi G r \mathrm{d}r}{\frac{8\pi G}{3} r^2 \rho_* - 1} && (20)\\
\frac{\mathrm{d}p}{\left(p + \frac{2}{3}\rho_*\right)^2 - \frac{\rho_*^2}{9}} =& \frac{3}{4\rho_*}\frac{\frac{16\pi G}{3} r \rho_* \mathrm{d}r}{\frac{8\pi G}{3} r^2 \rho_* - 1} && (21)
\end{aligned}$$

Using that the pressure vanishes at the surface of the star we can integrate to obtain

$$\begin{aligned}
-\frac{1}{2}\frac{3}{\rho_*}\ln\left[\frac{p' + \frac{2}{3}\rho_* + \frac{1}{3}\rho_*}{p' + \frac{2}{3}\rho_* - \frac{1}{3}\rho_*}\right]^0_{p'=p} =& \frac{3}{4\rho_*}\ln\left[\frac{8\pi G}{3} r'^2 \rho_* - 1\right]^R_{r'=r} && (22)\\
\ln\left[\frac{p + \rho_*}{3p + \rho_*}\right] =& \frac{1}{2}\ln\left[\frac{2GMR^2 - R^3}{2GMr^2 - R^3}\right]. && (23)
\end{aligned}$$

Solving for $p(r)$ yields the radial pressure distribution

$$p(r) = \rho_* \frac{\sqrt{R^3 - 2GMR^2} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3\sqrt{R^3 - 2GMR^2}}. \tag{24}$$

For the star of maximum mass the pressure diverges at $r = 0$ and there is no static solution, i. e. the star collapses. The maximum mass corresponds to the case where the denominator vanishes at $r = 0$

$$R = 9(R - 2GM), \tag{25}$$

leading to

$$M < \frac{4}{9}\frac{R}{G}. \tag{26}$$

This result remains true for more general equations of state and is known as Buchdahls theorem.
Now it remains to calculate the missing metric coefficients. From Eq. (11) we have

$$\frac{\mathrm{d}\alpha}{\mathrm{d}r} = -\frac{\mathrm{d}\ln(p+\rho_*)}{\mathrm{d}r}. \tag{27}$$

The boundary conditions can be set up at the surface of the star, where the interior solution goes over into the exterior Schwarzschild solution, thus $\rho + p|_{r=R} = \rho_*$ and

$$\exp\left[\alpha(R)\right] = \sqrt{1 - \frac{2GM}{R}}. \tag{28}$$

In summary, we have for the metric elements

$$\exp\left[\alpha\right] = \frac{3}{2}\sqrt{1 - \frac{2GM}{R}} - \frac{1}{2}\sqrt{1 - \frac{2GMr^2}{R^3}}, \tag{29}$$

$$\exp\left[-\beta\right] = \sqrt{1 - \frac{2Gm(r)}{r}} = \sqrt{1 - \frac{2GMr^2}{R^3}}. \tag{30}$$

**Exercise 2** [Out of plane precession of S2 orbit] (5 points)

The time derivative of the orbital momentum $\boldsymbol{l} = \boldsymbol{r} \wedge \boldsymbol{v}$ is

$$\frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t} = \boldsymbol{r} \wedge \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \tag{31}$$

$$= -\boldsymbol{r} \wedge \nabla\phi + 2\boldsymbol{r} \wedge (\boldsymbol{\Omega} \wedge \boldsymbol{v}) \tag{32}$$

$$= 2\boldsymbol{r} \wedge (\boldsymbol{\Omega} \wedge \boldsymbol{v}) \tag{33}$$

We assume now $\boldsymbol{l} \perp \boldsymbol{S}$ and circular orbits. A good choice for the radius of the circular orbit may be the actual mean distance between focal point and S2 as obtained by integrating over the true anomaly[1]. This results in

$$\bar{r} = \frac{r_0(1-e^2)}{2\pi}\int \frac{\phi}{1 + e\cos\phi} = r_0\sqrt{1-e^2} = 2.2 \times 10^{-3}\ \mathrm{pc}, \tag{34}$$

where $r_0$ is the semi-major axis and $e$ is the ellipticity.
The orbital time for S2 is given by

$$T_{\mathrm{o}} \approx 2\pi\sqrt{\frac{r_0^3}{GM}} = 16.86\ \mathrm{yr}. \tag{35}$$

It is reasonable to assume that the frequency of the orbital plane precession is much smaller than the frequency of the orbital motion of S2 (this can be checked a posteriori). Under this assumption

[1]The true anomaly is the angle between the direction of periapsis and the current position of the body, as seen from the main focus of the ellipse.

we can calculate the precession integrating over the unperturbed orbit. With $\boldsymbol{S} = S\boldsymbol{e}_y$, $\boldsymbol{r} = \bar{r}\,[\cos(\varphi)\,\boldsymbol{e}_x + \sin(\varphi)\,\boldsymbol{e}_y]$ and $\boldsymbol{v} = \bar{r}\omega\,[-\sin(\varphi)\boldsymbol{e}_x + \cos(\varphi)\boldsymbol{e}_y]$ we obtain

$$\frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t} = \frac{4GS\omega}{c^2\bar{r}}\left(\sin^2(\varphi)\boldsymbol{e}_x - \sin(\varphi)\cos(\varphi)\boldsymbol{e}_y\right) \tag{36}$$

After one orbit the change in angular momentum is

$$\Delta\boldsymbol{l} = \int \frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t}dt = \frac{T_\mathrm{o}}{2\pi}\int \frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t}d\varphi = 4\pi\frac{G\,S}{c^2\bar{r}}\boldsymbol{e}_x \tag{37}$$

The total change in the orbital angular momentum is perpendicular to the angular momentum, so the gravitomagnetic field does not change the magnitude of the angular momentum vector, but just its direction. The maximal angular frequency of the precession can now be estimated as

$$\Omega_\mathrm{max} = \frac{\Delta l}{lT_\mathrm{o}} = 2\frac{G^2M^2}{c^3\bar{r}^3}a, \tag{38}$$

where $a \in [0, 1]$ is the spin parameter of the central black hole. A comparison with the orbital frequency $\Omega_\mathrm{o} = \frac{2\pi}{T_\mathrm{o}}$ gives, setting $a = 1$,

$$\frac{\Omega_\mathrm{max}}{\Omega_\mathrm{o}} \sim 10^{-6} \ll 1, \tag{39}$$

which is in agreement with our a priori assumption. Estimating the "real" out-of-plane precession as $\Omega_\mathrm{real} = \frac{75}{360}\frac{2\pi}{t_\mathrm{S2}}$, we obtain the constraint on the black hole spin parameter:

$$a \geq \left(4\pi\frac{G^2M^2}{c^3\bar{r}^3}\right)^{-1}\Omega_\mathrm{real} \approx 0.40. \tag{40}$$

Since the lower bound lies within the interval allowed by General Relativity, the estimation we have performed is not able to exclude "gravitomagnetic" precession as a source of S2 inclination. Remember, however, that the approximations we have done are rather drastic, and we are not able, with these simple calculations, to say how realistic is the hypotesis of gravitomagnetic precession.