Universität Zürich UZH

Master Thesis

Physics department

University of Zurich

Is the Universe Accelerating?

Shift in the Luminosity Distance from the Second-Order Relativistic Perturbation Theory

By

Stanislav Linchakovskyy

Supervisor: Prof. Dr. Jaiyul Yoo

August 2024

Is the Universe Accelerating?

Shift in the Luminosity Distance from the Second-Order Relativistic Perturbation Theory

Stanislav Linchakovskyy

Abstract

The luminosity distance is a key concept in cosmology, providing insights into the structure and evolution of the universe. Measurements of luminosity distances from distant Type Ia supernovae have shown that the universe is undergoing accelerated expansion. Recent studies suggest that second-order perturbations could contribute to this phenomenon, emphasizing the need for statistical analysis of luminosity distance in a universe influenced by such perturbations. This research outlines the theoretical framework, deriving observable quantities like redshift, source position, physical volume, and area, all in a gauge-invariant form. Using the zero-shear temporal gauge and spatial C-gauge, along with the Einstein-de Sitter universe limit, an analytical expression for luminosity distance is obtained. Finally, the study qualitatively explores methods to derive the luminosity distance monopole.

Contents

Introduction

The luminosity distance serves as a crucial tool in cosmology, enabling the extraction of significant information about the universe, particularly at low redshifts. This information, far from being trivial, has led to groundbreaking discoveries. Notably, measurements of luminosity distances to distant Type Ia supernovae have revealed one of the most profound cosmological findings of recent times: the universe is currently in a phase of accelerated expansion [1, 2]. The reason for this behavior is still a complete mystery.

If the universe is homogeneous and isotropic on large scales, all contributions to the cosmological energy-momentum tensor are characterized by their energy density $\rho(z)$ and pressure $P(z)$. Accelerated expansion requires that overall $\rho + 3P < 0$ today. This condition can be met by introducing a so-called 'dark energy' component with very negative pressure in addition to the usual pressureless matter. A natural choice for this dark energy is something that behaves like a cosmological constant in General Relativity (GR). Yet, its value does not align with expectations from quantum field theory. The predicted value exceeds the experimentally measured one by 120 orders of magnitude — "the worst prediction ever made by a scientific theory," according to Lee Smolin [3].

Thus, the luminosity distance is an excellent probe for cosmology, and accurate measurements and theoretical understanding are crucial for advancing our knowledge. To measure the luminosity distance, we need 'standard candles,' sources with known luminosity. Later in this thesis, we will discuss the construction and calibration of the distance ladder, starting from parallax and Cepheid variables, and explain why Type Ia supernovae are particularly effective as standard candles.

This discussion primarily applies to a homogeneous universe, where knowledge of fluctuations is not necessary to determine that the universe's expansion is accelerating. This is because deviations from homogeneity are minimal on very large scales. However, these fluctuations do exist and contain valuable information about the initial conditions of the universe and the behavior of gravity on large scales. Since the luminosity distance is well measured, we can use these measurements at late times to gain insights into the conditions of the very early universe.

Our theory of initial conditions, known as inflation, posits that primordial density

perturbations are generated by the enhancement of vacuum quantum fluctuations of the "inflaton" field [4, 5]. This leads to the observations we see in the sky being a realization of a random process. Therefore, while we cannot predict the exact outcome of any specific measurement, we can predict its statistical properties, such as the expectation value or mean. By comparing these predictions with actual observations, we can gain new insights into the Universe. In an idealized scenario where supernovae are present at every redshift and visible in all directions, the simplest observable statistic we can construct is the expected value of the luminosity distance at a given redshift.

In cosmological perturbation theory, the expectation for the luminosity distance fluctuations is zero at linear order. This arises from our theory of initial conditions, which posits that the mean of every fluctuation is zero [6]. Consequently, the mean value of the luminosity distance to the linear order is considered a background value. However, when we move beyond linear perturbation theory to second-order fluctuations, a non-zero contribution can emerge. Although this contribution is expected to be small, it is significant when compared to the zero prediction at linear order. Even a minuscule value, such as 10^{-10}, can be substantial and carry valuable information.

Since we only have one observable sky and cannot average over the ensemble of different realizations of the universe, the mean luminosity distance can be obtained by spatially averaging over all directions in the sky. The angular average of the luminosity distance includes both the background and fluctuations, known as monopole fluctuations because they are averaged over all angles. The expected value of this monopole fluctuation is not well known because it requires second-order perturbation theory, which is complex in its current form. Prior to the work of J. Yoo and M. Magi [7], the gauge invariance of the expressions had not been thoroughly checked, leading to the omission of many terms in the literature without a compelling reason.

Continuing the discussion about the acceleration of the Universe, it has been known for some time that locally measured cosmological parameters, such as the Hubble constant (H_0), may not reflect those of the underlying Friedmann universe. These parameters might be influenced by local fluctuations. Recently, it has even been argued that second-order perturbations could be responsible for the observed acceleration of the universe [8–11], suggesting that no cosmological constant or dark energy is necessary. This hypothesis underscores the importance of deriving statistics for the luminosity distance in a universe with perturbations to the second order.

In order to deal with luminosity distance monopole, we begin by introducing some theoretical concepts necessary to contextualize both cosmology in general and more specifically cosmological perturbation theory. After that, we present our calculations of luminosity distance perturbation to the second order in the Einstein-de Sitter

Universe.

The organization of this thesis is as follows: in section 1.1, we discussed the modern standard model of cosmology, whereas in section 1.1.4, special attention was given to homogeneous and isotropic spaces and Friedmann-Lemaître-Robertson-Walker Cosmology. This geometry will be used as the background solution, assuming a spatially flat case. Next, in sections 1.1.6 and 1.1.7, we presented various methods of determining distance in cosmology and emphasized the importance of luminosity distance in the context of observations.

However, it is quite obvious that the Universe is not completely homogeneous, which highlights the further necessity to consider in sections 1.2 the perturbed expression for the metric, along with the necessary mathematical apparatus for calculations in this model. Also, in 1.2.3 and 1.2.4, we specifically focused on the problem of gauge transformations and the choice of gauge.

At this stage, we are sufficiently equipped to obtain in section 1.3 a generic expression for cosmological quantities, such as time-lapse and spatial shift, redshift, source position, physical volume, the area occupied by the source, and finally, the expression for perturbations in luminosity distance.

Finally, at this stage, we can perform direct calculations for luminosity distance in terms of metric quantities. In Chapter 2, we obtained this expression in the linear order of perturbation theory for further use in Chapter 3 to derive the expression in the second order. After determining the gauge, we applied the Einstein-de Sitter limit to obtain a formula ready for further processing.

Chapter 1

Theory Overview

1.1 Standart Model of Cosmology

1.1.1 Homogeneity and Isotropy

The Universe is homogeneous and isotropic at large spatial scales. Extensive galaxy surveys, which have gathered data on over a million galaxies, confirm that the largest structures in the Universe, such as superclusters of galaxies and vast voids, span tens of megaparsecs. At scales greater than 200 Mpc, the Universe exhibits uniformity in all directions (homogeneity) and shows no preferential directions (isotropy).

Currently, around 20 superclusters have been identified [12–14]. The Local Group resides within the Laniakea Supercluster, encompassing approximately 100,000 galaxies spread over 160 Mpc. Superclusters are relatively sparse, with a galaxy density of only twice the average density of the Universe. One of the closest superclusters to Laniakea, the Shapley Supercluster, is located in the direction of the Centaurus constellation, about 200 Mpc away [15].

The most extensive catalog of galaxies and quasars to date is the Sloan Digital Sky Survey (SDSS) catalog [16]. Freely available, this catalog is the result of years of data collection by a dedicated telescope and includes millions of celestial objects. Most of the data has been processed, resulting in measurements of the spectra of nearly 2 million objects. These measurements have been used to create a three-dimensional map covering a significant portion of the visible Universe, spanning over a quarter of the sky.

Early results from the SDSS [17] are illustrated in Fig. 1.1, showing the positions of 40,000 galaxies and 4,000 quasars. This section of the celestial sphere spans 500 square degrees, revealing clusters of galaxies and voids. The isotropy and homogeneity of the Universe become apparent at spatial scales of around 100 Mpc and larger. The color of each dot indicates the type of astrophysical object, with certain types dominating due to the peculiarities of structure formation and evolu-

tion. Thus, observations partly reflect conditions from different epochs rather than a single moment in space.

Figure 1.1: Spatial distribution of galaxies and quasars based on early SDSS observational data. Green dots represent all galaxies within a specific solid angle with an apparent magnitude above a certain threshold. Red dots denote Large Red Galaxies, which are highly luminous and form a relatively uniform population. Light and dark blue points indicate the locations of ordinary quasars.

1.1.2 Expansion

The Universe expands as the distances between galaxies increase. Essentially, space, which is always homogeneous and isotropic, stretches. To describe this expansion, one introduces the scale factor $a(t)$ as a function of time. The distance between two distant objects in the Universe is proportional to $a(t)$. The rate of cosmological expansion, or the relative growth of distances per unit of time, is characterized by the Hubble parameter:

$$H(t) = \frac{\dot{a}(t)}{a(t)}. \tag{1.1}$$

The expansion of the Universe also causes the wavelength of a photon emitted in the distant past to increase [18, 19]. Similar to other distances, the photon wavelength increases in proportion to $a(t)$; thus, the photon experiences redshift. This redshift z is defined by the ratio of photon wavelengths at absorption and emission:

$$\frac{\lambda_{ab}}{\lambda_{em}} = 1 + z. \tag{1.2}$$

This ratio depends on the time of emission (assuming the photon is detected on Earth today), i.e., on the distance to the source. Redshift is a measurable quantity: the emission wavelength is determined by the physics of the emission process (e.g., the energy levels of an atom), while λ_{ab} is the measured wavelength. One measures the redshift by identifying the system of emission (or absorption) lines and measuring how much they are shifted to the red spectral region.

For $z \ll 1$, the distance to the source r and the redshift are related by the Hubble law as:

$$z = H_0 r. \tag{1.3}$$

Present value, according to our convention, is denoted by H_0. Notably, in 1929, Edwin Hubble first determined this constant to be 500 km/s per million parsecs [20]. Since then, the Hubble constant has been actively measured using various methods [21–24]. Early on, Hubble used bright stars and the light from nebulae to estimate redshift and determine the constant. Later, type Ia supernovae were identified as better standard candles for distant galaxies, leading to more precise redshift measurements. We will discuss this in more detail later.

High-accuracy measurements using the James Webb Space Telescope in 2023 [25] have confirmed earlier observations from the Hubble Space Telescope [26], yielding a Hubble constant of

$$H_0 = 73.0 \pm 1.0 \, \mathrm{km \, s^{-1} \, Mpc^{-1}}. \tag{1.4}$$

1.1.3   Pieces of General Relativity

In 1915, Albert Einstein presented the geometric theory of gravitation known as General Relativity (GR), which is considered a cornerstone of modern physics. This theory refines Newton's law of universal gravitation by providing a unified description of gravity as a geometric property of space and time. Specifically, the curvature of spacetime is directly related to the energy and momentum of the matter and radiation present. Wheeler [27] encapsulated this central idea of general relativity with the phrase, "Spacetime tells matter how to move; matter tells spacetime how to curve." Given that gravitation is the dominant force on large scales, GR is essential for adequately describing the Universe in cosmology.

The main mathematical instruments for describing curved spacetime are differential manifolds. By that, we mean a space consisting of patches that locally look like R^n and are smoothly connected together. A metric must be introduced to define geometrical properties on a manifold. The metric tensor $g_{\mu\nu}$ serves as a mathematical entity that defines distances and angles, similar to how the inner product in

Euclidean space does. With the quantities dx^μ interpreted as components of an infinitesimal coordinate displacement four-vector, the metric determines the invariant square of an infinitesimal line element, commonly known as an interval. This interval is represented by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{1.5}$$

The Christoffel symbol, derived from the metric, describes how basis vectors change from point to point within a coordinate system. It quantifies the rate of change of the covariant basis concerning the coordinate variable. The Christoffel symbol is given by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho}\right). \tag{1.6}$$

This symbol, also referred to as the connection, illustrates how the tangent spaces at different points in spacetime are interconnected. From this connection, we can derive the Riemann curvature tensor $R^\mu_{\nu\rho\sigma}$, which provides an intrinsic method for describing the curvature of a surface:

$$R^\mu_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\varepsilon_{\nu\sigma}\Gamma^\mu_{\rho\varepsilon} - \Gamma^\varepsilon_{\nu\rho}\Gamma^\mu_{\sigma\varepsilon}. \tag{1.7}$$

Contracting it, we get the Ricci tensor and the Ricci scalar.

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu}, \qquad R = R^\nu_\nu. \tag{1.8}$$

These quantities are of particular interest because they appear in the main equation of General Relativity, the Einstein field equations:

$$R_{\mu\nu} + \left(\Lambda - \frac{1}{2}R\right)g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.9}$$

where $T_{\mu\nu}$ is the stress-energy tensor that describes the density and flux of energy and momentum in spacetime. It is an attribute of matter, radiation, and non-gravitational force fields. Λ is the cosmological constant and G is the gravitational constant.

1.1.4 Homogeneous Isotropic Spaces and FLRW Metric

As we discussed before, to a very solid approximation, our Universe is homogeneous and isotropic at sufficiently large scales. This implies that, at a given moment in time, the geometry of space is that of a homogeneous and isotropic manifold. There are only three such manifolds: the 3D sphere, the 3D plane, and the 3D hyperboloid. The geometry of a 3-dimensional sphere is best understood by imagining it embedded

in a (fictitious) 4-dimensional Euclidean space and expressing the equation of the 3-sphere in its standard form:

$$\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2 = R^2, \tag{1.10}$$

where x^α are coordinates of the 4-dimensional Euclidean space and R is the radius of the 3-sphere. Let us introduce three angles χ, θ, ϕ so that

$$
\begin{aligned}
x^1 &= R \sin \chi \sin \theta \sin \phi, \\
x^2 &= R \sin \chi \sin \theta \cos \phi, \\
x^3 &= R \sin \chi \cos \phi, \\
x^4 &= R \cos \chi.
\end{aligned} \tag{1.11}
$$

With such parametrization, the distance between two points on the 3-sphere can be written as:

$$dl^2 = R^2 \left[d\chi^2 + \sin^2 \chi \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right]. \tag{1.12}$$

Similarly, we can write down the distance between two points on the hyperboloid, and its expression for the plane is just a 3-dimensional Pythagorean formula. However, 3-sphere, 3-hyperboloid, and 3-plane metrics can be written in a unified form. To do this, one may come back from using χ angle to radial coordinate r:

$$dl^2 = \frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right), \tag{1.13}$$

where

$$
K = \begin{cases}
\frac{1}{R^2}, & \text{3-sphere}, \\
0, & \text{3-plane}, \\
-\frac{1}{R^2}, & \text{3-hyperboloid}.
\end{cases} \tag{1.14}
$$

Expanding homogeneous and isotropic Universe is described by Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2 \left(t \right) \bar{g}_{ij} dx^i dx^j, \tag{1.15}$$

where $\bar{g}_{ij}$ is the metric of unit $(R = 1)$ 3-sphere, unit 3-hyperboloid or 3-plane.

It turns out useful to work with the conformal time, defined as $d\eta = adt$, such that (1.15) becomes

$$ds^2 = a^2 \left(\eta \right) \left(-d\eta^2 + \bar{g}_{ij} dx^i dx^j \right). \tag{1.16}$$

Usually, the spacetime is considered to be spatially flat, so from now on, when we refer to the FLRW metric, we consider the following:

$$ds^2 = a^2\left(\eta\right)\left(-d\eta^2 + dx^2 + dy^2 + dz^2\right). \tag{1.17}$$

1.1.5 Friedmann-Lemaître Equations

The dynamics of the expanding Universe appeared implicitly in the metric described in the previous subsection. To make this time dependence explicit, we must solve Einstein's field equations (1.9). The first consideration is that due to the symmetries arising from the Robertson-Walker metric, the energy-momentum tensor takes the form of a perfect fluid, the components dependent only on time coordinate. Hence, we know that the energy-momentum tensor for a perfect fluid is [28]:

$$T_{\mu\nu} = \left(\rho + p\right)u_\mu u_\nu + g_{\mu\nu}p, \tag{1.18}$$

and it fulfills the energy and momentum conservation

$$T_{\mu\nu;\nu} = 0. \tag{1.19}$$

Using the expression for metric (1.15) together with definitions (1.8) and (1.6), Einstein equations take a form:

$$\begin{aligned} \text{00-component:} \quad & \dot{H}\left(t\right) + H^2\left(t\right) = \frac{4\pi G}{3}\left(\rho + 3p\right) + \frac{\Lambda}{3}, \\ \alpha\beta\text{-component:} \quad & H^2\left(t\right) = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}. \end{aligned} \tag{1.20}$$

These are called Friedmann-Lemaître equations. Having these equations, together with equation of state $p = \omega\rho$ one may determine evolution of the Universe.

1.1.6 Distances in Cosmology

In cosmology, distance measures provide a natural concept of distance between two objects or events in the universe. These measures are commonly employed to link an observable quantity, such as the luminosity of a distant quasar or the redshift of a distant galaxy, to another, more calculation-friendly quantity that is not directly observable, such as the comoving coordinates of the quasar or galaxy.

- **Comoving distance:**

  The comoving distance between the source emitting light at time t_i and the observer at the Earth at time t_0 can be computed by the following formula

(derived using (1.15)):

$$\chi = \int_{t_i}^{t_0} \frac{dt}{a(t)} = \left[z(t) = \frac{a_0}{a(t)} - 1 \right] = \int_0^z \frac{dz'}{a_0 H(z')}. \tag{1.21}$$

- **Angular diameter distance:**

  In terms of an object's physical size, D, and its angular size, $\delta\theta$, as viewed from Earth, one can obtain angular diameter distance

  $$D_A = \frac{D}{\delta\theta}. \tag{1.22}$$

  As can be seen from the Fig. 1.2 $\delta\theta = \frac{D}{a\chi}$, and so

$(\chi, \theta + \delta\theta, \phi, t_i)$

$\delta\theta$

D

$(\chi, \theta, \phi, t_i)$

Figure 1.2: Simplified scheme of observation.

  $$D_A = \frac{\chi}{1+z}. \tag{1.23}$$

  This is commonly used to observe so called standard rulers, astronomical object for which the actual physical size is known.

- **Luminosity distance:**

  To express the luminosity distance, we will use the flux-luminosity relation:

  $$F = \frac{L}{4\pi D_L^2}. \tag{1.24}$$

  The number of photons crossing the unit surface at the observer's position is inversely proportional to the area A, while the energy of each photon differs from the energy at emission by the redshift factor $(1+z)^{-1}$. The same factor additionally enters the expression for the number of photons crossing the unit surface in unit time since the time intervals for the source and observer differ by factor $(1+z)^{-1}$. Using this, one can write down the expression of flux at

the observer's position in terms of absolute luminosity L as:

$$F = \frac{L}{(1+z)^2 A}.$$ (1.25)

In the flat universe, one may write $A = 4\pi\chi^2$, so one may find the relationship between angular and luminosity distance, known as Etherington's reciprocity theorem:

$$D_L = (1+z)\,\chi = (1+z)^2\,D_A.$$ (1.26)

The luminosity distance is important for measurements of standard candles like type Ia supernovae, which were first used to discover the acceleration of the expansion of the universe.

1.1.7 Cosmic Distance Ladder

The cosmic distance ladder is a series of techniques that astronomers use to measure the vast distances to celestial objects. Given the immense scales involved, there is no single method that can measure all these distances directly. Within our solar system, we can use direct measurements, such as radar, combined with straightforward trigonometry. However, these methods become impractical when dealing with larger scales. For instance, radar signals would take minutes or even hours to traverse the solar system, and the closest star is a staggering four light years away! This chapter will explore the ingenious methods astronomers have developed to overcome these challenges and map the universe.

- **Parallax:**

  The modern parallax method relies on observing the apparent shift in position of a star relative to the background stars as the Earth orbits the Sun. Due to the vast distances of stars, the perceived motion in the night sky is minuscule, and the parallax equation is simply:

  $$d = \frac{1}{p},$$ (1.27)

  where d is the distance in parsecs, and p is the parallax angle measured in arcseconds.

  Despite the high precision of parallax measurements, the method has significant limitations regarding the maximum measurable distance. The furthest distances currently measurable using parallax reach up to the center of the

Milky Way galaxy, achievable with advanced instruments such as the Gaia space telescope [29, 30].
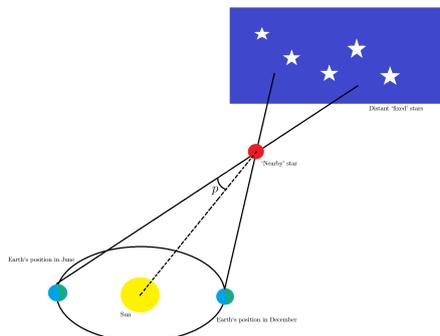

Figure 1.3: Stellar parallax motion

- **Standard candles:**

  In astronomical research, almost all objects used as physical distance indicators belong to a class with a known intrinsic brightness or luminosity. By comparing this intrinsic luminosity to the observed brightness of the object, the distance can be calculated using the formula from the Sec. 1.1.6. These objects with known luminosity are referred to as "standard candles", a term coined by Henrietta Leavitt [31].

  After measuring the periods and relative brightnesses of Cepheid variables, Henrietta Leavitt discovered a correlation between the period and apparent brightness of these stars. Specifically, stars with longer periods exhibited greater brightness than those with shorter periods. By reasonably assuming that the Small Magellanic Cloud is sufficiently distant to consider all stars within it at approximately the same distance from Earth, one can infer a relationship between the period and the absolute brightness of these stars as well. This finding was pivotal in the field of modern astrophysics, as it implied that by simply measuring the pulsation period of a Cepheid, one could determine a distance to it — an otherwise challenging task.

- **Supernovae of Type Ia:**

  The final step in the cosmic distance ladder involves Type Ia supernovae, which occur in binary systems where a white dwarf accretes material from its companion star. Once the white dwarf's mass reaches the Chandrasekhar limit, the electron degeneracy pressure can no longer support it, leading to a catastrophic collapse. This collapse triggers violent nuclear reactions in the core, releasing an enormous amount of energy and causing the star to explode, completely disrupting it. Because all Type Ia supernovae result from the same

physical process, they behave similarly and release a consistent amount of energy, making them ideal standard candles. Additionally, Type Ia supernovae are incredibly bright, often outshining all other stars in their galaxy combined. They can be detected in galaxies hundreds of millions of light years away, providing a powerful tool for measuring distances and calibrating redshifts far beyond the reach of Cepheid variables.

1.1.8 Universe is Accelerating. Or...

Soon after the expansion of the universe was firmly established, observational cosmologists began striving to detect variations in the expansion speed as a function of redshift. They were so convinced that gravitational interactions among galaxies would decelerate the expansion that they introduced the deceleration parameter q_0, which they believed to be positive.

The two teams [32, 33] aimed to measure the matter density parameter through the distance-redshift relation (1.26) of Type Ia supernovae. We can express χ via (1.21) as:

$$\chi = \frac{c}{a_0 H_0} \int\limits_0^z \frac{dz'}{\sqrt{\Omega_r \left(1 + z'\right)^4 + \Omega_m \left(1 + z'\right)^3}}, \tag{1.28}$$

where Ω_r and Ω_m are the normalized values of the present radiation energy density and matter density, respectively. Fitting these parameters leads to a paradox: if we assume a matter-dominated cosmology, the matter density parameter would have to be significantly negative to coincide with the observations, which favored an accelerated expansion.

To resolve this, a cosmological model mixing matter with some kind of fluid that acts as a source of static negative energy density has been introduced. Within the framework of general relativity, such a fluid may cause the acceleration of the expansion. The term "dark energy" used nowadays refers to this hypothetical fluid.

However, the assumption of dark energy is not entirely intuitive, and several researchers have questioned the majority opinion on the acceleration or even the assumption that the universe is homogeneous and isotropic. For example, a paper by Jacques Colin et al. [34] says that the cosmic acceleration deduced from supernovae may be an artifact of our being non-Copernican observers rather than evidence for a dominant component of "dark energy" in the Universe. A paper [9] argues that the concept of "dark energy" is unreliable because the assumption of a perfectly homogeneous background becomes invalid at later times due to the increasing influence of perturbations. This means that the standard use of the Friedmann equation to describe the average behavior of the universe at low redshifts is likely incorrect,

as it fails to account for the effects of inhomogeneities. Therefore, to continue the discussion, we need to enter the area of relativistic perturbation theory.

1.2 Relativistic Perturbation Theory

Our current understanding of the evolution of the universe is based on the FLRW (Friedmann-Lemaître-Robertson-Walker) cosmological model, which describes a spatially homogeneous and isotropic universe. Initially, the universe consisted of uniformly distributed matter. Over billions of years, this matter has coalesced into large-scale structures, including galaxies containing hundreds of billions of stars, galaxy clusters, superclusters, and vast filaments of matter.

Under the framework of general relativity, these denser regions and the voids between them influence the curvature of space-time, as matter determines the curvature of space-time. Consequently, the additional mass of galaxies and galaxy clusters (including dark matter, if particles of it are directly detected) causes the nearby space-time to curve more positively. Conversely, voids induce negative curvatures in the surrounding space-time. To account for these variations, the Relativistic Perturbation Theory has been introduced.

1.2.1 Metric Perturbations and Observed Four-Velocity

Using the FLRW coordinate system, wherein the temporal coordinate is denoted by cosmic time t or conformal time η, while the spatial coordinates are represented by comoving coordinates x^α. The components of the metric for the real inhomogeneous universe can be expressed as:

$$g_{\eta\eta}\left(\eta, \vec{x}\right) = -a^2\left(\eta\right)\left(1 + 2\mathcal{A}\left(\eta, \vec{x}\right)\right), \qquad g_{\eta\alpha}\left(\eta, \vec{x}\right) = -a^2\left(\eta\right)\mathcal{B}_\alpha\left(\eta, \vec{x}\right),$$
$$g_{\alpha\beta}\left(\eta, \vec{x}\right) = a^2\left(\eta\right)\left(\delta_{\alpha\beta} + 2\mathcal{C}_{\alpha\beta}\left(\eta, \vec{x}\right)\right); \tag{1.29}$$

where the spacetime fields $\mathcal{A}$, $\mathcal{B}_\alpha$, and $\mathcal{C}_{\alpha\beta}$ represent nonlinear perturbations to the background metric, based on the flat 3-metric $\delta_{\alpha\beta}$. These perturbations highlight the inhomogeneities of the real universe, and their absence yields the background FLRW metric.

It is also important to introduce a vector field u on the spacetime manifold describing the real inhomogeneous universe, expressed in coordinates via the chart $u = u^\mu\left(x\right)\partial_\mu$. This vector field describes the observer's four-velocity along the curve, satisfying the following equations at each point of the observer's world line:

$$u = \frac{d}{d\tau} = \frac{dx^\mu\left(\tau\right)}{d\tau}\frac{\partial}{\partial x^\mu}, \qquad u_\mu u^\mu = -1. \tag{1.30}$$

From the symmetries of the FLRW solution, one may notice that in the background, only the temporal component of the observer's four-velocity is non-vanishing. Perturbations from the background introduce nonlinear perturbations δu and $\mathcal{U}^\alpha$ to the observer's four-velocity:

$$u^\mu = \frac{1}{a}\left(1 + \delta u,\ \mathcal{U}^\alpha\right)^\mu. \tag{1.31}$$

1.2.2 Scalar-Vector-Tensor Decomposition

Given the splitting of the spatial hypersurface and the symmetry associated with it, we decompose the perturbation variables to all orders as:

$$\mathcal{A} = \alpha, \qquad \mathcal{B}_\alpha = \beta_{,\alpha} + B^{(v)}_\alpha \qquad \mathcal{C}_{\alpha\beta} = \phi\delta_{\alpha\beta} + \gamma_{\alpha|\beta} + C^{(v)}_{\alpha|\beta} + C^{(t)}_{\alpha\beta}, \qquad \mathcal{U}^\alpha = -v^{,\alpha} + v^{(v)\alpha}. \tag{1.32}$$

The decomposition into scalar, vector, and tensor components is interesting because it allows us to consider three types of perturbations separately. For example, scalar perturbations of the metric, which couple with density matter fluctuation ρ, lead to the large-scale structure we observe today. Tensor perturbations describe gravitational waves, while vector perturbations decay exponentially in an expanding universe, making them less interesting to study. At nonlinear order, these perturbations mix, complicating the evolution of the quantities.

1.2.3 Gauge Transformation

The general covariance of general relativity ensures that any coordinate system can be used to describe physical phenomena, and these descriptions must be independent of the coordinate system. This principle is known as diffeomorphism symmetry in general relativity. However, when we decompose the metric into a background and perturbations, we must evaluate both quantities at the same point for calculations. To achieve this, we introduce a diffeomorphism and its pull-back. In perturbation theory, this mapping is called a gauge choice, which establishes a one-to-one correspondence between points in the physical and background spacetimes to define perturbation quantities. The gauge choice is not unique, as different correspondences can be found for the same background spacetime due to general coordinate transformations on the perturbed spacetime.

To look at the gauge transformation using so-called passive approach [35], we consider the following transformation between two coordinates x^μ and $\hat{x}^\mu$:

$$\hat{x}^\mu \equiv x^\mu + \tilde{\xi}^\mu(x^\nu) \equiv x^\mu + \tilde{\zeta}^\mu + \frac{1}{2}\tilde{\zeta}^\mu_{,\rho}\tilde{\zeta}^\rho. \tag{1.33}$$

The variables $\tilde{\xi}^\mu$ and $\tilde{\zeta}^\mu$ are perturbed order quantities. To the second order, we may have the following:

$$\tilde{\xi}^\mu \equiv \tilde{\xi}^{(1)\mu} + \tilde{\xi}^{(2)\mu}, \tag{1.34}$$

and similarly for $\tilde{\zeta}^\mu$. For any tensor quantity, we use the tensor transformation property between x^μ and $\hat{x}^\mu$ spacetimes:

$$\tilde{\phi}(x^\sigma) = \hat{\tilde{\phi}}(\hat{x}^\sigma), \quad \tilde{v}_\mu(x^\sigma) = \frac{\partial \hat{x}^\rho}{\partial x^\mu} \hat{\tilde{v}}_\rho(\hat{x}^\sigma), \quad \tilde{t}_{\mu\nu}(x^\sigma) = \frac{\partial \hat{x}^\lambda}{\partial x^\mu} \frac{\partial \hat{x}^\rho}{\partial x^\nu} \hat{\tilde{t}}_{\lambda\rho}(\hat{x}^\sigma). \tag{1.35}$$

By comparing tensor quantities at the same spacetime point, x^a, we can derive the gauge transformation properties of the tensor quantity. Specifically, a tensor quantity $\mathbf{t}$ transforms as follows [36]:

$$\hat{\mathbf{t}}(x^c) = \mathbf{t}(x^c) - \mathcal{L}_{\tilde{\zeta}}\mathbf{t} + \frac{1}{2}\mathcal{L}_{\tilde{\zeta}}^2\mathbf{t}, \tag{1.36}$$

where $\mathcal{L}_{\tilde{\zeta}}$ is a Lie derivative along $\tilde{\zeta}^a$. One can show that [37]

$$\hat{\tilde{\phi}}(x^e) = \tilde{\phi}(x^e) - \tilde{\phi}_{,c}\tilde{\xi}^c + \tilde{\phi}_{,b}\tilde{\xi}^b{}_{,c}\tilde{\xi}^c + \frac{1}{2}\tilde{\phi}_{,bc}\tilde{\xi}^b\tilde{\xi}^c, \tag{1.37}$$

$$\hat{\tilde{v}}_a(x^e) = \tilde{v}_a(x^e) - \tilde{v}_{a,b}\tilde{\xi}^b - \tilde{v}_b\tilde{\xi}^b{}_{,a} + \frac{1}{2}\tilde{v}_{a,bc}\tilde{\xi}^b\tilde{\xi}^c + \tilde{v}_{a,b}\tilde{\xi}^b{}_{,c}\tilde{\xi}^c + \tilde{v}_{b,c}\tilde{\xi}^b{}_{,a}\tilde{\xi}^c + \tilde{v}_b\tilde{\xi}^b{}_{,ac}\tilde{\xi}^c + \tilde{v}_c\tilde{\xi}^c{}_{,b}\tilde{\xi}^b{}_{,a}, \tag{1.38}$$

$$\hat{\tilde{t}}_{ab}(x^e) = \tilde{t}_{ab}(x^e) - 2\tilde{t}_{c(a}\tilde{\xi}^c{}_{,b)} - \tilde{t}_{ab,c}\tilde{\xi}^c + 2\tilde{t}_{c(a}\tilde{\xi}^d{}_{,b)}\tilde{\xi}^c{}_{,d} + \tilde{t}_{cd}\tilde{\xi}^c{}_{,a}\tilde{\xi}^d{}_{,b} +$$
$$+ \tilde{\xi}^d\left(2\tilde{\xi}^c{}_{,(b}\tilde{t}_{a)c,d} + 2\tilde{t}_{c(a}\tilde{\xi}^c{}_{,b)d} + \frac{1}{2}\tilde{t}_{ab,cd}\tilde{\xi}^c + \tilde{t}_{ab,c}\tilde{\xi}^c{}_{,d}\right). \tag{1.39}$$

Following the approach in [37], one may define

$$\tilde{\xi}^0 \equiv \xi^0, \quad \tilde{\xi}^\alpha \equiv \xi^\alpha, \tag{1.40}$$

where ξ^α is based 3-metric on $\hat{g}_{\alpha\beta}^{(3)}$. Analogously, in terms of $\tilde{\zeta}^a$ we set $\tilde{\zeta}^0 \equiv \zeta^0$ and $\tilde{\zeta}^\alpha \equiv \zeta^\alpha$ where ζ^α is based on 3-metric $\hat{g}_{\alpha\beta}^{(3)}$. Thus, we have

$$\xi^0 = \zeta^0 + \frac{1}{2}\zeta^{0\prime}\zeta^0 + \frac{1}{2}\zeta^0{}_{,\alpha}\zeta^\alpha, \quad \xi^\alpha = \zeta^\alpha + \frac{1}{2}\zeta^{\alpha\prime}\zeta^0 + \frac{1}{2}\zeta^\alpha{}_{,\beta}\zeta^\beta. \tag{1.41}$$

Therefore, the metric quantities can be written as:

$$
\hat{\mathcal{A}} = \mathcal{A} - \left(\xi^{0\prime} + \frac{a'}{a}\xi^0\right) - \mathcal{A}'\xi^0 - 2\mathcal{A}\left(\xi^{0\prime} + \frac{a'}{a}\xi^0\right) - \mathcal{A}_{,\alpha}\xi^\alpha - \mathcal{B}_\alpha\xi^{\alpha\prime}
$$
$$
+ \frac{3}{2}\xi^{0\prime}\xi^{0\prime} + \xi^0_{,\alpha}\xi^{\alpha\prime} + \xi^\alpha\left(\xi^0_{,\alpha} + \frac{a'}{a}\xi^0_{,\alpha}\right) + \xi^0\left[\xi^{0\prime\prime} + 3\frac{a'}{a}\xi^{0\prime} + \frac{1}{2}\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)\xi^0\right] - \frac{1}{2}\xi^{\alpha\prime}\xi'_\alpha,
$$

$$(1.42)$$

$$
\hat{\mathcal{B}}_\alpha = \mathcal{B}_\alpha - \xi^0_{,\alpha} + \xi'_\alpha - 2\mathcal{A}\xi^0_{,\alpha} - \left(\mathcal{B}'_\alpha + 2\frac{a'}{a}\mathcal{B}_\alpha\right)\xi^0 - \mathcal{B}_\alpha\xi^{0\prime} - \mathcal{B}_{\alpha,\beta}\xi^\beta - \mathcal{B}_\beta\xi^\beta_{,\alpha} + 2\mathcal{C}_{\alpha\beta}\xi^{\beta\prime}
$$
$$
- \xi'_\alpha\xi^{0\prime} + 2\xi^{0\prime}\xi^0_{,\alpha} + \xi^0_{,\beta}\xi^\beta_{,\alpha} + \xi^\gamma\xi^0_{,\alpha\gamma} - \xi^0\left(\xi''_\alpha + 2\frac{a'}{a}\xi'_\alpha - \xi^{0\prime}_{,\alpha} - 2\frac{a'}{a}\xi^0_{,\alpha}\right)
$$
$$
- \xi^\beta_{,\alpha}\xi'_\beta - g^{(3)}_{\alpha\beta}\xi^\beta_{,\gamma}\xi^{\gamma\prime} - \xi^\gamma\left(g^{(3)}_{\alpha\beta,\gamma}\xi^{\beta\prime} + g^{(3)}_{\alpha\beta}\xi^{\beta\prime}_{,\gamma}\right), \quad (1.43)
$$

$$
\hat{\mathcal{C}}_{\alpha\beta} = \mathcal{C}_{\alpha\beta} - \frac{a'}{a}\xi^0 g^{(3)}_{\alpha\beta} - \frac{1}{2}g^{(3)}_{\alpha\beta,\gamma}\xi^\gamma - g^{(3)}_{\gamma(\alpha}\xi^\gamma_{,\beta)}
$$
$$
+ \mathcal{B}_{(\alpha}\xi^0_{,\beta)} - \left(\mathcal{C}'_{\alpha\beta} + 2\frac{a'}{a}\mathcal{C}_{\alpha\beta}\right)\xi^0 - \mathcal{C}_{\alpha\beta,\gamma}\xi^\gamma - 2\mathcal{C}_{\gamma(\alpha}\xi^\gamma_{,\beta)} + \xi'_{(\alpha}\xi^0_{,\beta)} - \frac{1}{2}\xi^0_{,\alpha}\xi^0_{,\beta} + \frac{a'}{a}g^{(3)}_{\alpha\beta}\xi^\gamma\xi^0_{,\gamma}
$$
$$
+ \xi^0\left[\frac{a'}{a}g^{(3)}_{\alpha\beta}\xi^{0\prime} + \frac{1}{2}\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)g^{(3)}_{\alpha\beta}\xi^0 + \left(\frac{1}{2}\xi^{\gamma\prime} + \frac{a'}{a}\xi^\gamma\right)g^{(3)}_{\alpha\beta,\gamma} + 2\frac{a'}{a}g^{(3)}_{\gamma(\alpha}\xi^\gamma_{,\beta)} + g^{(3)}_{\gamma(\alpha}\xi^{\gamma\prime}_{,\beta)}\right]
$$
$$
+ \xi^\delta_{,(\beta}g^{(3)}_{\alpha)\gamma}\xi^\gamma_{,\delta} + \frac{1}{2}g^{(3)}_{\gamma\delta}\xi^\gamma_{,\alpha}\xi^\delta_{,\beta} + \xi^\delta\left(\frac{1}{2}g^{(3)}_{\alpha\beta,\gamma}\xi^\gamma_{,\delta} + \xi^\gamma_{,(\beta}g^{(3)}_{\alpha)\gamma,\delta} + \frac{1}{4}g^{(3)}_{\alpha\beta,\gamma\delta}\xi^\gamma + g^{(3)}_{\gamma(\alpha}\xi^\gamma_{,\beta)\delta}\right).
$$

$$(1.44)$$

The Scalar-Vector-Tensor decomposed quantities transform as:

$$
\hat{\alpha} = \alpha - \frac{1}{a}\left(a\xi^0\right)' + \mathcal{A}_\xi, \tag{1.45}
$$

$$
\hat{\beta} = \beta - \xi^0 + \left(\frac{1}{a}\xi\right)' + \Delta^{-1}\nabla^\alpha\mathcal{B}_{\xi\alpha}, \tag{1.46}
$$

$$
\hat{B}^{(v)}_\alpha = B^{(v)}_\alpha + \xi^{(v)\prime}_\alpha + \mathcal{B}_{\xi\alpha} - \nabla_\alpha\Delta^{-1}\nabla^\beta\mathcal{B}_{\xi\beta}, \tag{1.47}
$$

$$
\hat{\gamma} = \gamma - \frac{1}{a}\xi + \frac{1}{2}\left(\Delta + \frac{1}{2}R^{(3)}\right)^{-1}\left(3\Delta^{-1}\nabla^\alpha\nabla^\beta\mathcal{C}_{\xi\alpha\beta} - \mathcal{C}_{\xi\alpha}{}^\alpha\right), \tag{1.48}
$$

$$
\hat{\varphi} = \varphi - \frac{a'}{a}\xi^0 + \frac{1}{3}\mathcal{C}_{\xi\alpha}{}^\alpha - \frac{1}{6}\Delta\left(\Delta + \frac{1}{2}R^{(3)}\right)^{-1}\left(3\Delta^{-1}\nabla^\alpha\nabla^\beta\mathcal{C}_{\xi\alpha\beta} - \mathcal{C}_{\xi\alpha}{}^\alpha\right), \tag{1.49}
$$

$$
\hat{C}^{(v)}_\alpha = C^{(v)}_\alpha - \xi^{(v)}_\alpha + 2\left(\Delta + \frac{1}{3}R^{(3)}\right)^{-1}\left(\nabla^\beta\mathcal{C}_{\xi\alpha\beta} - \nabla_\alpha\Delta^{-1}\nabla^\gamma\nabla^\beta\mathcal{C}_{\xi\gamma\beta}\right), \tag{1.50}
$$

$$\hat{C}_{\alpha\beta}^{(t)} = C_{\alpha\beta}^{(t)} - \mathcal{C}_{\xi\alpha\beta} - \frac{1}{3}\mathcal{C}_{\xi\gamma}{}^{\gamma}g_{\alpha\beta}^{(3)} - \frac{1}{2}\left(\nabla_\alpha\nabla_\beta - \frac{1}{3}g_{\alpha\beta}^{(3)}\Delta\right)\left(\Delta + \frac{1}{2}R^{(3)}\right)^{-1}\left(3\Delta^{-1}\nabla^\gamma\nabla^\delta\mathcal{C}_{\xi\gamma\delta} - \mathcal{C}_{\xi\gamma}{}^{\gamma}\right)$$

$$- \nabla_{(\alpha}\left(\Delta + \frac{1}{3}R^{(3)}\right)^{-1}\left(\nabla^\gamma\mathcal{C}_{\xi\beta)\gamma} - \nabla_{\beta)}\Delta^{-1}\nabla^\gamma\nabla^\delta\mathcal{C}_{\xi\gamma\delta}\right). \quad (1.51)$$

The quadratic parts of the gauge transformation equations (1.42), (1.43), and (1.44) are denoted by $\mathcal{A}_\xi$, $\mathcal{B}_{\xi\alpha}$, and $\mathcal{C}_{\xi\alpha\beta}$ respectively.

1.2.4 Gauge Conditions

Of course, the situation described above is not complete. To solve most problems, we also need to define the perturbation of the time-like four-vector u^μ, which describes the motion of an observer. Additionally, we must consider several fluid quantities related to the energy-momentum tensor, such as the energy density ρ, the isotropic pressure p, the energy flux q_α and the anisotropic pressure $\pi_{\alpha\beta}$.

Once we are all equipped, we may choose the gauge conditions. These are commonly separated into temporal gauge fixing conditions, fixing ξ^0, and spatial gauge fixing conditions, fixing ξ and $\xi_\alpha^{(v)}$.

- **Temporal gauge fixing condition** affects only scalar-type perturbation [37]. Depending on the task, one can impose any one of the following temporal gauge conditions to be valid at any spacetime point:

$$\begin{aligned} \alpha = 0 &\rightarrow \xi^0 = \xi^0(x), & \text{synchronous gauge,} \\ v = 0 &\rightarrow \xi^0 = 0, & \text{comoving gauge,} \\ \chi = a(\gamma' + \beta) = 0 &\rightarrow \xi^0 = 0, & \text{zero-shear gauge.} \end{aligned} \quad (1.52)$$

  Any variable under any gauge condition (except for the synchronous gauge) has a unique equivalent gauge-invariant combination. It is to say $\alpha_\chi = \alpha|_{\chi\equiv0}$ is the same as α in the zero-shear gauge.

- **Spatial gauge conditions** applies to both scalar-type (via fixing ξ) and the vector-type (via fixing $\xi_\alpha^{(v)}$) perturbations. Due to the spatial homogeneity of the background, we have natural spatial gauge fixing conditions to choose [38]:

$$\begin{aligned} \beta = 0, B_\alpha^{(v)} = 0 &\rightarrow \xi = \xi \propto a, \xi_\alpha^{(v)} = \xi_\alpha^{(v)}(x) & \text{B-gauge,} \\ \gamma = 0, C_\alpha^{(v)} = 0 &\rightarrow \xi = 0, \xi_\alpha^{(v)} = 0 & \text{C-gauge.} \end{aligned} \quad (1.53)$$

  The B-gauge conditions do not fully fix the spatial and rotational gauge modes, leaving some gauge modes remaining even after they are applied. In contrast, the C-gauge conditions successfully eliminate all gauge modes.

1.3 Gauge-Invariant Expression for Luminosity Distance

In the recent sections, we explored perturbations in a homogeneous and isotropic universe, which lay the groundwork for the complex structures we observe today. A completely homogeneous universe would not form such entities as planets, stars, galaxies, clusters, and superclusters of galaxies. To create these structures, primordial fluctuations are essential.

Cosmology is unique in its need for a distinct theory regarding its initial conditions. Currently, the theory of inflation stands as the most widely accepted explanation for these initial conditions. This theory suggests that initial perturbations arise from the amplification of vacuum quantum fluctuations of the "inflaton" field. As a result, the outcomes of our observations are inherently influenced by the realization of random processes. While we may not be able to predict the exact outcome of any specific measurement, we can predict its statistical properties.

The task of obtaining statistics for the observable Universe is theoretically complex. Unlike other fields, we cannot run multiple universes to observe their outcomes. Instead, we examine all directions in the sky to derive statistical quantities through averaging over all angles. Notably, linear-order perturbations do not produce any statistical differences, as the mean values of all fluctuations, according to inflationary theory, are zero. Therefore, we must derive an expression in the second order of perturbation theory to advance our understanding.

1.3.1 Metric Perturbations and Observer Four-Velocity

We will adopt the metric that is given in (1.29). And proceed further by lowering indices in (1.31), to obtain:

$$u_\eta = g_{\eta\mu} u^\mu = -a \left(1 + \delta u + 2\mathcal{A} + 2\mathcal{A}\delta u + \mathcal{B}_\alpha \mathcal{U}^\alpha \right), \tag{1.54}$$

$$u_\alpha = g_{\alpha\mu} u^\mu = a \left(\mathcal{U}_\alpha - \mathcal{B}_\alpha - \delta u \mathcal{B}_\alpha + 2\mathcal{C}_{\alpha\beta} \mathcal{U}^\beta \right). \tag{1.55}$$

The normalization condition constraints velocity perturbations to metric perturbations:

$$0 = \delta u + \mathcal{A}\left(1 + 2\delta u\right) + \frac{1}{2}\delta u^2\left(1 + 2\mathcal{A}\right) + \mathcal{B}_\alpha \mathcal{U}^\alpha\left(1 + \delta u\right) - \frac{1}{2}\mathcal{U}^\alpha\left(\mathcal{U}_\alpha + 2\mathcal{U}^\beta \mathcal{C}_{\alpha\beta}\right). \tag{1.56}$$

Assuming the observer's path is a geodesic, the observer's acceleration a^μ vanishes:

$$a^\mu := u^\nu \nabla_\nu u^\mu = u^\nu \partial_\nu u^\mu + \Gamma^\mu_{\rho\sigma} u^\rho u^\sigma = 0, \tag{1.57}$$

where ∇_μ is the covariant derivative compatible with g and $\Gamma^\mu_{\rho\sigma}$ are the Christoffel symbols. Utilizing (1.31), the geodesic can be explicitly expressed as:

$$
a^\eta = 0 = \left(\Gamma^\eta_{\eta\eta} + \frac{a'}{a} \right) (1 + \delta u)^2 + \left(2\Gamma^\eta_{\eta\alpha} \mathcal{U}^\alpha + \delta u' \right) (1 + \delta u) + \Gamma^\eta_{\alpha\beta} \mathcal{U}^\alpha \mathcal{U}^\beta + \mathcal{U}^\alpha \partial_\alpha \delta u,
$$

$$
a^\alpha = 0 = \Gamma^\alpha_{\eta\eta} (1 + \delta u)^2 + \left(2\Gamma^\eta_{\eta\beta} \mathcal{U}^\beta + \mathcal{U}'^\alpha - \frac{a'}{a} \mathcal{U}^\alpha \right) (1 + \delta u) + \Gamma^\alpha_{\beta\gamma} \mathcal{U}^\beta \mathcal{U}^\gamma + \mathcal{U}^\beta \partial_\beta \mathcal{U}^\alpha.
$$

$$(1.58)$$

Among all the Christoffel symbols, only $\Gamma^\eta_{\eta\eta}$ has a non-vanishing background contribution, such that $a^\mu = 0$ at the background level. Hence, terms involving $\Gamma^\gamma_{\alpha\beta} u^\alpha u^\beta$ are at least third order in perturbations.

1.3.2 Physical Observables in the Observer Rest Frame

Analogously to the observer four-velocity, a vector field k can be defined, expressed in coordinates as $k = k^\mu(x) \partial_\mu$. This vector field is constrained along the light path:

$$k = k^\mu(x_\Lambda) \partial_\mu, \tag{1.59}$$

and describes a null vector at each point:

$$k = \frac{d}{d\Lambda} = \frac{dx^\mu}{d\Lambda} \frac{\partial}{\partial x^\mu}. \tag{1.60}$$

This requirement uniquely specifies the affine parameter Λ, ensuring that the components of k are linked to physical observables.

The observers conduct cosmological observations within their rest frame, wherein the metric is Minkowski η_{ab}, and the time direction is determined by the four-velocity u^μ of the observer [39]. This frame is defined only within the infinitesimal neighborhood of a given spacetime point of the observer. Nevertheless, a tangent space orthogonal to the observer's time direction u^μ can be precisely defined by constructing three spacelike vectors e^μ_i. Together with the time direction $e^\mu_0 \equiv u^\mu$, these four vectors constitute an orthonormal basis for the observer ($\eta_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b$, where $a, b, c, .. = 0, 1, 2, 3$ denote the components of a tetrad and can be raised and lowered using the Minkowski metric).

In the observer's rest frame, cosmological measurements are made and geometrically described in terms of the tetrad basis at the point of observation. The components of the light wavevector in this reference frame are given by [40]:

$$k^a = e^a_\mu k^\mu = \omega \left(1, -n^i \right)^a, \tag{1.61}$$

where the components of the photon wavevector in the observer rest-frame are expressed in terms of the observable quantities: the angular frequency $\omega = 2\pi\nu$ of the photon and the angular position n^i of the source.

In the observer rest-frame, a set of angles (θ, ϕ) is assigned to the unit directional vector $n^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Notably, these cosmological observables (such as ω or n^i) remain independent of FLRW coordinates, while the components of the photon wavevector k^μ or the tetrad vectors e_a^μ are coordinate-dependent.

We equip the null light path with tetrad basis vectors at each point. However, numerous ways exist to achieve this. If $\{e_a\}$ is a tetrad basis at a point and we make a change of basis to arrive at $\{e_a'\}$, then $\{e_a'\}$ remains a tetrad basis only if the transformation consists of three independent boosts and three independent spatial rotations (Lorentz group). First of all, one can align the tetrad frame with the observer trajectory to fix the freedom in boosts. The observer moves with a timelike velocity u^a, defining the proper-time direction $e_0^\mu \equiv u^\mu$. We fix the residual rotation freedom by aligning the space-like tetrad vectors parallel to the FLRW spatial directions at the background level. Then, we can parameterize the spatial directions of the tetrad in the following way:

$$e_0^\mu \equiv u^\mu, \qquad e_i^\mu \equiv \frac{1}{a}\left(\delta e_i^\eta, \delta_i^\alpha + \delta e_i^\alpha\right), \tag{1.62}$$

where δe_i^η and δe_i^α are perturbations, capturing the deviation from the background. It is natural to further split δe_i^α into the symmetric part S_{ij} and the anti-symmetric part A_{ij} as:

$$\delta e_i^\alpha \equiv -\delta_j^\alpha p_i^j, \qquad S_{ij} \equiv \frac{1}{2}\left(p_{ij} + p_{ji}\right), \qquad A_{ij} \equiv \frac{1}{2}\left(p_{ij} - p_{ji}\right), \tag{1.63}$$

using orthonormality condition,

$$\eta_{ab} = g_{\mu\nu}e_a^\mu e_b^\nu, \qquad g^{\mu\nu} = \eta^{ab}e_a^\mu e_b^\nu, \qquad e_a^\mu e_\nu^a = \delta_\nu^\mu, \qquad e_a^\mu e_\mu^b = \delta_a^b. \tag{1.64}$$

We may constrain the time component of the spatial tetrads:

$$\delta e_i^\eta = \delta_i^\alpha\left(\mathcal{U}_\alpha - \mathcal{B}_\alpha\right), \tag{1.65}$$

and the symmetric part of the spatial component

$$S_{ij} = \delta_i^\alpha\delta_j^\beta\mathcal{C}_{\alpha\beta} = \mathcal{C}_{ij}. \tag{1.66}$$

The anti-symmetric part A_{ij} is, however, left unconstrained by the orthonormality

condition.

Now that we have the local tetrad vectors, we can proceed to establish a connection between the local observables and the photon wavevector k^μ within the FLRW frame. The observer measures both the photon frequency ω and the angular position n^i of the source. To compute the components of k in coordinates, we employ the expressions for tetrad components:

$$k^\mu = e_a^\mu k^a, \qquad e_0^\mu = u^\mu, \qquad e_i^\mu = \frac{\omega}{a}\left[\delta_i^\beta\left(\mathcal{U}_\beta - \mathcal{B}_\beta\right),\ \delta_i^\alpha - \delta_i^\beta \mathcal{C}_\beta^\alpha - \delta_j^\alpha A_i^j\right], \quad (1.67)$$

and photon propagation in an FLRW coordinate can be derived as:

$$n^\mu = n^i e_i^\mu = -\frac{k^\mu}{\omega} + u^\mu, \qquad n^\mu u_\mu = 0, \qquad n_\mu n^\mu = 1. \qquad (1.68)$$

In the absence of perturbations, the spatial components of the photon wavevector k^μ are directly proportional to the observed direction n^i in the observer's rest frame, as the spatial tetrad vectors are inherently aligned with an FLRW coordinate system.

However, the presence of perturbations alters their expressions within an FLRW coordinate framework. This modification is akin to a general relativistic extension of a Lorentz boost in special relativity, where an observer observes different frequencies and propagation directions in relative motion. It's crucial to note that all quantities mentioned are evaluated at the observer's position, and the expression remains valid solely at that position. The reason is that both the angular frequency ω and the propagation direction n^i are quantities measured by the observer rather than being universally defined fields.

1.3.3   Conformal Transformation

To simplify equations without altering physical conclusions, it is convenient to introduce a conformal transformation:

$$\hat{g}_{\mu\nu}\left(\eta, x^\alpha\right) = \frac{1}{a^2\left(\eta\right)} g_{\mu\nu}\left(\eta, x^\alpha\right). \qquad (1.69)$$

It is established that the photon wavevector must satisfy the null condition $k^\mu k_\mu = 0$ and the geodesic equation $k^\nu k^\mu_{\ ;\nu} = 0$ throughout its trajectory. However, under a conformal transformation, the geometry undergoes changes while preserving the manifold itself [41]. Consequently, the covariant derivatives do not remain identical to fulfill their metricity conditions: $\nabla_\sigma g_{\mu\nu} = 0$ and $\hat{\nabla}_\sigma \hat{g}_{\mu\nu} = 0$.

To relate covariant derivatives in the two metrics, it is convenient to use the

connecting tensor $C^{\mu}_{\nu\rho}$ as:

$$\hat{\nabla}_{\nu}k^{\mu} = \nabla_{\nu}k^{\mu} + C^{\mu}_{\nu\rho}k^{\rho}, \qquad C^{\mu}_{\nu\rho} = \mathcal{H}\left(g_{\nu\rho}g^{\mu\nu} - \delta^{\mu}_{\nu}\delta^{\eta}_{\rho} - \delta^{\mu}_{\rho}\delta^{\eta}_{\nu}\right). \tag{1.70}$$

Thus, it follows that the geodesic equation does not hold for the photon wavevector k^{μ} with respect to $\hat{\nabla}\mu$ in the conformally transformed metric. Nevertheless, through the re-parameterization of the photon path $x^{\mu}(\lambda)$ using a different affine parameter λ, we can deduce the conformally transformed wavevector $\hat{k}^{\mu}$ for the same null path, which satisfies the geodesic equation $\hat{k}^{\nu}\hat{\nabla}\nu\hat{k}^{\mu}$ within the conformally transformed metric. So, we define λ as:

$$k^{\mu} = \hat{k}^{\mu}\frac{d\lambda}{d\Lambda}, \qquad \hat{k}^{\mu} = \frac{dx^{\mu}}{d\lambda}. \tag{1.71}$$

One can show that $\frac{d\Lambda}{d\lambda}$ can be obtained up to a proportionality constant $\mathbb{C}$ that represents the additional degree of freedom from the conformal transformation:

$$\frac{d\Lambda}{d\lambda} = \mathbb{C}a^{2}. \tag{1.72}$$

We can now express a conformally transformed wavevector as:

$$\hat{k}^{\mu} = \mathbb{C}a^{2}k^{\mu} = \mathbb{C}a\omega\left(1, -n^{\alpha}\right)^{\mu}. \tag{1.73}$$

It proves convenient to choose the normalization constant $\mathbb{C}a_{obs}\omega_{obs} = 1$ at the observer position. To show that the product $\mathbb{C}a\omega$ is unity everywhere in the background universe, we firstly normalize the scale factor at the observer position as $a(\eta_{obs}) = a_{obs} = 1$. And then it is obvious that:

$$\mathbb{C}a_{obs}\omega_{obs} = \mathbb{C}a_{obs}\left(a_{p}\omega_{p}\right) = \mathbb{C}a_{p}\omega_{p}, \tag{1.74}$$

where p is any point along the background path. In the presence of perturbations, the combination $\mathbb{C}a\omega$ varies as a function of position. Now it is convenient to define the perturbation $\widehat{\Delta\nu}$ in the observed frequency in conformally transformed metric, in terms of the product:

$$\mathbb{C}a\omega = -\mathbb{C}a\left(u_{\mu}k^{\mu}\right) = -\hat{u}_{\mu}\hat{k}^{\mu} = 1 + \widehat{\Delta\nu} \tag{1.75}$$

because the four-velocity in the conformally transformed metric is $\hat{u}^{\mu} = au^{\mu}$ and $\hat{g}_{\mu\nu}\hat{u}^{\nu} = u_{\mu}/a$. Therefore our choice of the normalization as $\mathbb{C}a_{obs}\omega_{obs} = 1$ becomes $\widehat{\Delta\nu}_{0} = 0$ at the observer position.

Now we are at the point where we can discuss the components of the conformal

wavevector in nonlinear parametrization as:

$$\hat{k}^\mu = (1 + \delta\nu, -n^\alpha - \delta n^\alpha)^\mu, \tag{1.76}$$

and the perturbations of the wavevector at the observer position $\delta\nu_o = \delta\nu(x_o)$ and $\delta n_o^\alpha = \delta n^\alpha(x_o)$ can be written in terms of the tetrad components using $\hat{k}^\mu = \mathbb{C}a^2 e_a^\mu k^a$ as:

$$\delta\nu_o = \delta u_o - (\delta e_i^\eta)_o n^i, \quad \delta n_o^\alpha = n^i (\delta e_i^\alpha)_o - \mathcal{U}_o^\alpha = -\left(\mathcal{U}_o^\alpha + \delta_j^\alpha \left(S_i^j + A_i^j\right)_o n^i\right). \tag{1.77}$$

Then, one may obtain the exact expression of the wavevector in conformal metric:

$$\begin{aligned}
\hat{k}_\eta &= -\left(1 + 2\mathcal{A} + \delta\nu - \mathcal{B}_\alpha n^\alpha + 2\mathcal{A}\delta\nu - \mathcal{B}\delta n^\alpha\right) \quad, \\
\hat{k}_\alpha &= -\left(n_\alpha + \delta n_\alpha + 2\mathcal{C}_{\alpha\beta} n^\beta + \mathcal{B}_\alpha + \delta\nu\mathcal{B}_\alpha + 2\mathcal{C}_{\alpha\beta}\delta n^\beta\right).
\end{aligned} \tag{1.78}$$

It is convenient to use the notation that replaces contraction with n^α with $\|$, meaning the parallel to n^α. In this notation, the null condition can be written as:

$$\hat{k}_\mu \hat{k}^\mu = \delta\nu - \delta n_\| + \mathcal{A} - \mathcal{B}_\| - \mathcal{C}_\| - \delta\nu\mathcal{B}_\| - 2\mathcal{C}_{\|\alpha}\delta n^\alpha + 2\mathcal{A}\delta\nu - \mathcal{B}_\alpha\delta n^\alpha + \frac{1}{2}\delta\nu^2 -$$

$$-\frac{1}{2}\delta n_\alpha \delta n^\alpha + \mathcal{A}\delta\nu^2 - \delta\nu\mathcal{B}_\alpha\delta n^\alpha - \mathcal{C}_{\alpha\beta}\delta n^\alpha \delta n^\beta = 0, \tag{1.79}$$

and the background relation is trivially satisfied.

1.3.4 Geodesic Equation

Likewise, when considering the geodesic equation, the background can be easily eliminated, resulting in the propagation equation for the perturbation. Specifically, the temporal and spatial components of the geodesic equation are as follows:

$$\hat{k}^\mu \hat{\nabla}_\mu \hat{k}^\eta = \frac{d}{d\lambda}\delta\nu + \hat{\Gamma}_{\mu\nu}^\eta \hat{k}^\mu \hat{k}^\nu = 0, \qquad \hat{k}^\mu \hat{\nabla}_\mu \hat{k}^\alpha = -\frac{d}{d\lambda}\delta n^\alpha + \hat{\Gamma}_{\mu\nu}^\alpha \hat{k}^\mu \hat{k}^\nu. \tag{1.80}$$

where $\hat{\Gamma}$ are the Christoffel symbols of conformal metric $\hat{g}$ determined as:

$$\hat{\Gamma}_{\nu\rho}^\mu = \frac{1}{2}\hat{g}^{\mu\sigma}\left(\hat{g}_{\sigma\nu,\rho} + \hat{g}_{\rho\sigma,\nu} - \hat{g}_{\nu\rho,\sigma}\right), \tag{1.81}$$

and the derivative with respect to the affine parameter is obtained from (1.76):

$$\frac{d}{d\lambda} = \hat{k}^\mu \frac{\partial}{\partial x^\mu} = (1 + \delta\nu)\frac{\partial}{\partial\nu} - (n^\alpha + \delta n^\alpha)\frac{\partial}{\partial x^\alpha}. \tag{1.82}$$

Since $\hat{g}_{\mu\nu}$ in the background is selected to be the Minkowski metric $\eta_{\mu\nu}$ in Carte-

sian coordinates, it directly implies that the Christoffel symbols $\hat{\Gamma}^\mu_{\nu\rho}$ vanish in the background. Hence, we can define the following perturbations, which serve as sources for the geodesic equations:

$$\delta\hat{\Gamma}^\eta = \hat{\Gamma}^\eta_{\mu\nu}\hat{k}^\mu\hat{k}^\nu = \hat{\Gamma}^\eta_{\eta\eta}(1+\delta\nu)^2 - 2\hat{\Gamma}^\eta_{\alpha\eta}(1+\delta\nu)(n^\alpha+\delta n^\alpha) + \hat{\Gamma}^\eta_{\alpha\beta}(n^\alpha+\delta n^\alpha)(n^\beta+\delta n^\beta),$$
(1.83)

$$\delta\hat{\Gamma}^\alpha = \hat{\Gamma}^\alpha_{\mu\nu}\hat{k}^\mu\hat{k}^\nu = \hat{\Gamma}^\alpha_{\eta\eta}(1+\delta\nu)^2 - 2\hat{\Gamma}^\alpha_{\beta\eta}(1+\delta\nu)(n^\beta+\delta n^\beta) + \hat{\Gamma}^\alpha_{\beta\gamma}(n^\beta+\delta n^\beta)(n^\gamma+\delta n^\gamma).$$
(1.84)

It is crucial to emphasize that $\hat{\Gamma}^\alpha_{\beta\gamma} = 0$ in the background due to our choice of coordinates. Otherwise, $\delta\hat{\Gamma}^\alpha$ would introduce additional contributions proportionate to it.

Subsequently, to derive the formal solution, we integrate the following equations:

$$\delta\nu(x_\lambda) - \delta\nu(x_o) = -\int_0^\lambda d\lambda'\delta\hat{\Gamma}^\eta(x_{\lambda'}), \qquad \delta n^\alpha(x_\lambda) - \delta n^\alpha(x_0) = \int_0^\lambda d\lambda''\delta\hat{\Gamma}^\alpha(x_{\lambda'}),$$
(1.85)

where we set the affine parameter λ equal to zero at the observer position.

1.3.5 Time-Lapse and Spatial Shift at the Observer Position

Because of the metric perturbations and the peculiar velocity $\mathcal{U}^\alpha$, the observer's position $x^\mu(\tau)$ gradually deviates from the background position $\bar{x}^\mu(\tau)$ in the real universe. To quantify this drift, we introduce the time-lapse and spatial shift of the observer today by evaluating the first relation $\delta x^\mu(\tau) = x^\mu(\tau) - \bar{x}^\mu(\tau)$ at the observer position.

The relationship between the vector field u and the associated observer's curve $\mathcal{O}$ is defined by the differential equations governing the integral curve. In coordinate representation, these equations can be expressed as:

$$\frac{dx^\mu(\tau)}{d\tau} = u^\mu(x_\tau), \quad \forall\tau\in[0,\tau_o], \quad x^\mu(0) = 0.$$
(1.86)

These equations can be integrated to yield:

$$x^\mu(\tau_0) = \int_0^{\tau_0} d\tau u^\mu(x_\tau),$$
(1.87)

resulting in the time-lapse and spatial shift at the observer:

$$\delta t(\tau_o) = \int_0^{\tau_o} d\tau\delta u(x_\tau), \qquad \delta x^\alpha = \int_0^{\tau_0} d\tau\frac{1}{a}\mathcal{U}^\alpha(x_\tau).$$
(1.88)

The components of u^μ in cosmic time are represented as:

$$u^\mu = \left(1 + \delta u, \frac{1}{a}\mathcal{U}^\alpha\right)^\mu,$$ (1.89)

and utilizing the solution in the background universe:

$$\hat{t}_o = \int_0^{\tau_o} d\tau = \tau_o = \int_0^\infty \frac{dz}{H(z)(1+z)}.$$ (1.90)

By combining the last equation with the definitions of u in (1.31) and (1.89), the background path can be parametrized by the cosmic time as:

$$\hat{x}_\tau^\mu = (\tau = t = 0)^\mu = \hat{x}_t^\mu.$$ (1.91)

It is essential to acknowledge that (1.88) represents formal solutions, as they necessitate knowledge of the entire trajectory to obtain perturbations.

1.3.6   Observed Redshift

By employing the tetrad basis at the source position, we can determine the emitted light frequency ω_s and compare it with the frequency ω_o observed in the rest frame. As we mentioned before, these frequencies are different because of cosmic expansion. This phenomenon, known as redshift, occurs due to the expanding universe, resulting in a lower observed frequency compared to the original emitted frequency:

$$1 + z = \frac{\omega_s}{\omega_o}.$$ (1.92)

We can use the definition of $\widehat{\Delta\nu}$ from (1.75) for source position to obtain the nonlinear expression for the observed redshift:

$$1 + z = \frac{a_o}{a_s}\left(1 + \widehat{\Delta\nu_s}\right).$$ (1.93)

Now, we are at the moment to define the nonlinear perturbation δz in the observed redshift as:

$$\frac{1 + \delta z}{a_s} = 1 + z.$$ (1.94)

Combining the last two equations yields the nonlinear expression for the perturbation δz in the observed redshift

$$\delta z = a_o\left(1 + \widehat{\Delta\nu_s}\right) - 1,$$ (1.95)

where the nonlinear expression for $\widehat{\Delta\nu_s}$ is written as:

$$\widehat{\Delta\nu_s} = 2\mathcal{A} + \mathcal{U}_\| - \mathcal{B}_\| + (1 + 2\mathcal{A})\left(\delta u + \delta\nu + \delta u \delta\nu\right) + \delta n^\alpha \left(\mathcal{U}_\alpha - \mathcal{B}_\alpha\right) + \mathcal{B}^\alpha \mathcal{U}_\alpha \left(1 + \delta\nu\right) -$$
$$- \delta u \left(\mathcal{B}_\| + \mathcal{B}^\alpha \delta n_\alpha\right) + 2\mathcal{U}^\alpha \left(\mathcal{C}_{\alpha\|} + \mathcal{C}_{\alpha\beta} \delta n^\alpha\right). \quad (1.96)$$

It is important to note that while the right-hand side is evaluated at x_s^μ, there exists a dependence of $\widehat{\Delta\nu_s}$ on the reference position x_o^μ through $\delta\nu$ and δn^α.

1.3.7 Observed Source Position

Now, when we know the observed redshift, we may determine the relationship between $\hat{k}$ and the null light path:

$$\frac{dx^\mu(\lambda)}{d\lambda} = \hat{k}^\mu(x_\lambda), \quad \forall \lambda \in [0, \lambda_s], \quad x^\mu(0) = x_o^\mu. \quad (1.97)$$

Considering background level, integrating the differential equation gives:

$$\bar{\eta}_\lambda - \bar{\eta}_0 = \lambda, \quad \bar{x}_\lambda^\alpha = -\lambda n^\alpha. \quad (1.98)$$

This equation demonstrates the relationship between the affine parameter λ and the time coordinate of the background light path. However, due to the fact that coordinates are defined up to diffeomorphisms, ambiguity arises. To address this [42], one can utilize the observed redshift z to define an unambiguous time coordinate $\bar{\eta}_z$ along the light path in the background:

$$1 + z = \frac{1}{a(\bar{\eta}_z)}, \quad (1.99)$$

and this time coordinate is associated with the affine parameter λ_z defined by:

$$\bar{\eta}_z - \bar{\eta}_o = \lambda_z = -\int_0^z \frac{dz'}{H(z')} = -\hat{r}_z, \quad \hat{x}_z^\alpha = -\lambda_z n^\alpha = \hat{r}_z n^\alpha. \quad (1.100)$$

Here, $\bar{r}_z$ is the comoving distance out to redshift z in the background. Following physical parameterization may be used to denote the reference position of the source in the real universe:

$$\eta_s = \hat{\eta}_s + \delta\eta_s = \hat{\eta}_z + \Delta\eta_s, \quad x_s^\alpha = \hat{x}_s^\alpha + \delta x_s^\alpha = \hat{x}_z^\alpha + \Delta x_s^\alpha, \quad \lambda_s = \lambda_z + \Delta\lambda_s. \quad (1.101)$$

Here, the subscript z denotes evaluation at λ_z, while the subscript s denotes evaluation at λ_s. It is important to note that we use the lowercase delta δ to represent perturbations around the background source position parameterized by λ_s, whereas

we use the uppercase delta Δ to indicate perturbations around the background source position parameterized physically by the observed redshift. These two types of perturbations are generally distinct; for instance, $\delta\eta_s \neq \Delta\eta_s$.

We emphasize that the parameter λ_z, unlike λ_s, has been introduced to parameterize the light path in the background, where the source position $\bar{S}$ is precisely identified by the scalar z. In the case of the real universe, one should account for the perturbation $\Delta\lambda_S$.

With these definitions, we can derive useful nonlinear relations. At the source position, we obtain:

$$\Delta x_s^\mu = \delta x_s^\mu + \hat{x}_s^\mu - \hat{x}_z^\mu = \delta x_s^\mu + \Delta\lambda_s \left(1, -n^\alpha\right)^\mu \tag{1.102}$$

with boundary relation:

$$z\left(x_o; x_o\right) = 0, \Delta\lambda_o = 0, \Delta\eta_o = \delta\eta_0 \neq -, \Delta x_o^\alpha = \delta x_o^\alpha \neq 0. \tag{1.103}$$

Using the relation between δz and $\bar{\eta}_z$ in equations (1.94) and (1.99), we derive the exact relation between $\Delta\eta_s$ and δz:

$$1 + \delta z = \frac{a\left(\hat{\eta}_z + \Delta\eta_s\right)}{a\left(\hat{\eta}_z\right)}. \tag{1.104}$$

In addition to the observed redshift, we can employ the observed angular position n^i, parameterized by the angles θ and ϕ, to fully specify the real location of the source in terms of quantities defined in the rest frame of the observer. These angles are related to n^i by the following expression in Cartesian coordinates:

$$n^i = \left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right)^i. \tag{1.105}$$

The parameterization we adopt for the source position is then:

$$x_s^\mu = (\hat{\eta}_z + \Delta\eta_s, \left(\hat{r}_z + \delta r\right)\sin\left(\theta + \delta\theta\right)\cos\left(\phi + \delta\phi\right)$$
$$\left(\hat{r}_z + \delta r\right)\sin\left(\theta + \delta\theta\right)\sin\left(\phi + \delta\phi\right), \left(\hat{r}_z + \delta r\right)\cos\left(\theta + \delta\theta\right))^\mu, \tag{1.106}$$

where $\Delta\eta_s$, δr, $\delta\theta$, and $\delta\phi$ are the nonlinear perturbations determined by (z, θ, ϕ), it's important to note that technically, the perturbations $(\delta r, \delta\theta, \delta\phi)$ should be denoted as $(\Delta r_s, \Delta\theta_s, \Delta\phi_s)$ for consistency. However, we maintain the current notation convention as it presents no risk of confusion.

We reiterate that z, n^i, θ, and ϕ are observables, and thus their values remain unaffected by our choice of coordinate-based description.

1.3.8 Source Position Distortion

By isolating the perturbative components in (1.97), we find:

$$\frac{d\delta x^{\mu}}{d\lambda} = (\delta\nu, -\delta n^{\alpha})^{\mu}. \tag{1.107}$$

This equation holds true at any point along the light path. Integrating this equation yields the formal solution:

$$\delta x_s^{\mu} = (\delta\eta_s, \delta x_s^{\alpha})^{\mu} = \left(\delta\eta_o + \int_0^{\lambda_s} d\lambda\delta\nu, \delta x_o^{\alpha} - \int_0^{\lambda_s} d\lambda\delta n^{\alpha}\right)^{\mu}. \tag{1.108}$$

Now, utilizing equation (1.88), we derive:

$$\delta x_s^{\mu} = \left(\delta\eta_o + \lambda_s\delta\nu_o - \int_0^{\lambda_s} d\lambda\,(\lambda_s - \lambda)\,\delta\hat{\Gamma}^{\eta}, \delta x_o^{\alpha} - \lambda_s\delta n_o^{\alpha} - \int_0^{\lambda_s} d\lambda\,(\lambda_s - \lambda)\,\delta\hat{\Gamma}^{\alpha}\right)^{\mu}. \tag{1.109}$$

Combining the last expression with (1.103), we obtain a nonlinear expression for the distortion in the source position with respect to $\bar{x}_z^{\mu}$:

$$\Delta\eta_s = \Delta\lambda_s + \delta\eta_o + \lambda_s\delta\nu_o - \int_0^{\lambda_s} d\lambda\,(\lambda_s - \lambda)\,\delta\hat{\Gamma}^{\eta}, \tag{1.110}$$

$$\Delta x_s^{\alpha} = -\Delta\lambda_s n^{\alpha} + \delta x_o^{\alpha} - \lambda_s\delta n_o^{\alpha} - \int_0^{\lambda_s} d\lambda\,(\lambda_s - \lambda)\,\delta\hat{\Gamma}^{\alpha}. \tag{1.111}$$

If we solve the first equation for $\Delta\lambda_s$ and utilize the solution for δn^{α} in equation (1.88), we can derive a more convenient expression for the spatial distortion at the source position:

$$\Delta x_s^{\alpha} = \delta x_o^{\alpha} - n^{\alpha}\,(\Delta\eta_s - \delta\eta_o) - \int_0^{\lambda_s} d\lambda\,(\delta n^{\alpha} - n^{\alpha}\delta\nu). \tag{1.112}$$

Here, we condense the expression by leaving the perturbations δn^{α} and $\delta\nu$ unexpressed. It is important to note that the integration limit λ_s also includes a perturbative part $\Delta\lambda_s$.

1.3.9 Physical Volume, Area Occupied by the Source

Now that we have analyzed the distortion in the photon's path, we can proceed to calculate the physical volume occupied by the observed source galaxies. This

calculation considers small intervals in observed redshift dz and observed angles $(d\theta, d\phi)$. To express this volume in terms of the observed quantities, we may use the approach from [7]. Since the true positions x_s^α of source galaxies are parameterized based on the observed quantities, we can express the physical volume in the rest frame of the observed source galaxies in a covariant manner as:

$$dV = \frac{(1+\delta z)^3}{(1+z)^3}(1+\delta g)\left\{\varepsilon_{\eta\alpha\beta\gamma}(1+\delta u)\frac{\partial\left(\bar{x}_z^\alpha + \Delta x^\alpha\right)}{\partial z}\frac{\partial\left(\bar{x}_z^\beta + \Delta x^\beta\right)}{\partial\theta}\frac{\partial\left(\bar{x}_z^\gamma + \Delta x^\gamma\right)}{\partial\phi}+\right.$$
$$\left.+\varepsilon_{\alpha\mu\nu\rho}\mathcal{U}^\alpha\frac{\partial\left(\bar{x}_z^\mu + \Delta x^\mu\right)}{\partial z}\frac{\partial\left(\bar{x}_z^\nu + \Delta x^\nu\right)}{\partial\theta}\frac{\partial\left(\bar{x}_z^\rho + \Delta x^\rho\right)}{\partial\phi}\right\}dzd\theta d\phi. \quad (1.113)$$

The background contributions to the homogeneous and isotropic universe indeed reduce to the background volume element:

$$d\hat{V} = \frac{\hat{r}_z^2}{H_z\left(1+z\right)^3}dzd\Omega, \quad (1.114)$$

and we define the dimensionless volume fluctuation δV as:

$$dV = \frac{\hat{r}_z^2}{H_z\left(1+z\right)^3}\left(1+\delta V\right)dzd\Omega. \quad (1.115)$$

The definition for the physical area is:

$$dV = dA\left(n_\mu\frac{\partial x^\mu}{\partial z}\right)_s dz. \quad (1.116)$$

So, with this expression, we can rewrite:

$$dA = \sqrt{-g}\varepsilon_{\rho\mu\nu\sigma}u_s^\rho n_s^\mu\frac{\partial x_s^\nu}{\partial\theta}\frac{\partial x_s^\sigma}{\partial\phi}d\theta d\phi, \quad (1.117)$$

where propagation direction at the source position n_s^μ can be expressed as:

$$n_s^\mu = -\frac{k_s^\mu}{\omega_s} + u_s^\mu = \frac{1}{a_s}\left(-\frac{\hat{k}_s^\mu}{1+\widehat{\Delta\nu_s}} + \hat{u}_s^\mu\right), \quad \left(n^\mu n_\mu\right)_s = 1, \quad \left(n^\mu u_\mu\right)_s = 0. \quad (1.118)$$

1.3.10 Luminosity Distance

Galaxy samples are frequently defined by their observed flux or the rest-frame luminosity, which is inferred from the observed flux. The variation in the luminosity distance at the observed redshift z is defined as follows:

$$\mathcal{D}_L(z) = \hat{D}_L(z)\left(1+\delta\mathcal{D}_L\right), \quad \hat{D}_L = (1+z)\hat{r}_s(z), \quad (1.119)$$

where the fluctuation δD_L is dimensionless. Recognizing that the luminosity distance is connected to the angular diameter distance as:

$$\mathcal{D}_A(z)\,\frac{\mathcal{D}_L}{(1+z)^2}, \tag{1.120}$$

we can leverage the calculations of the photon path measured by the observer at the origin to determine the angular diameter distance. Remarkably, the fluctuation in the angular diameter distance is identical to the fluctuations in the luminosity distance.

The relation between the physical area occupied by the source dA and the observed solid angle $d\Omega$ element defines the angular diameter distance $\mathcal{D}_A$:

$$dA = \mathcal{D}_A^2 d\Omega, \tag{1.121}$$

which we can split into the background and the perturbative parts:

$$\mathcal{D}_A = \hat{\mathcal{D}}_A\left(1 + \delta\mathcal{D}_{\mathcal{L}}\right), \quad \hat{\mathcal{D}}_A = \frac{\hat{r}_z}{1+z}. \tag{1.122}$$

Then, solving (1.117) for diameter distance:

$$\mathcal{D}_A^2 = \bar{\mathcal{D}}^2(z)\left[1 + \delta A\right] = \bar{\mathcal{D}}_A^2(z)\left[1 + 2\delta\mathcal{D}_L + \delta\mathcal{D}_L^2\right] = \frac{\hat{\mathcal{D}}_A^2}{\hat{r}_z^2\sin\theta}\left(1 + \delta g\right)\left(1 + \delta z\right)^2 \times$$

$$\times\left[\varepsilon_{\eta\alpha\beta\gamma}\left(1 + \delta u\right)\left(\frac{n^\alpha + \delta n^\alpha}{1 + \widehat{\Delta\nu}} + \mathcal{U}^\alpha\right)\frac{\partial\left(\bar{x}_z^\beta + \Delta x^\beta\right)}{\partial\theta}\frac{\partial\left(\bar{x}_z^\gamma + \Delta x^\gamma\right)}{\partial\phi} + \right.$$

$$+ \varepsilon_{\alpha\eta\beta\gamma}\mathcal{U}^\alpha\left(-\frac{\delta\nu - \widehat{\Delta\nu}}{1 + \widehat{\Delta\nu}} + \delta u\right)\frac{\partial\left(\bar{x}_z^\beta + \Delta x^\beta\right)}{\partial\theta}\frac{\partial\left(\bar{x}_z^\gamma + \Delta x^\gamma\right)}{\partial\phi} + $$

$$\left. + \varepsilon_{\alpha\beta\nu\rho}\mathcal{U}^\alpha\left(\frac{n^\beta + \delta n^\beta}{1 + \widehat{\Delta\nu}} + \mathcal{U}^\beta\right)\frac{\partial\left(\bar{x}_z^\nu + \Delta x^\nu\right)}{\partial\theta}\frac{\partial\left(\bar{x}_z^\rho + \Delta x^\rho\right)}{\partial\phi}\right]. \tag{1.123}$$

Chapter 2

Linear-Order Perturbative Calculations

In this chapter, we will do an exercise to better understand the theoretical ideas we talked about before. Up until now, we have not done many direct calculations and have kept our expressions in a general form. Now, we want to find the specific expression for perturbations in luminosity distance in terms of metric quantities. We will start with basic examples and then, in the next chapter, we will expand this approach to the second order. This exercise will help make the concepts clearer and easier to understand.

2.1 Metric-Related Quantities and Observed Four-Velocity

We can compute the inverse metric using the relation $g_{\mu\rho}g^{\rho\nu} = \delta^\nu_\mu$:

$$g^{\eta\eta}(\eta, \vec{x}) = \frac{1}{a^2(\eta)}\left(-1 + 2\mathcal{A}(\eta, \vec{x})\right), \qquad g_{\eta\alpha}(\eta, \vec{x}) = -\frac{1}{a^2(\eta)}\mathcal{B}^\alpha(\eta, \vec{x}),$$

$$g_{\alpha\beta}(\eta, \vec{x}) = \frac{1}{a^2(\eta)}\left(\delta^{\alpha\beta} - 2\mathcal{C}^{\alpha\beta}(\eta, \vec{x})\right). \tag{2.1}$$

To obtain the determinant of metric, we need to proceed with additional computations:

$$det\left(\exp\{g\}\right) = \exp\{Tr[g]\}. \tag{2.2}$$

As long as $\exp\{L\} = g$, then:

$$det(g) = \exp\{Tr[L]\}, \tag{2.3}$$

$$g = det\left(g\right) \frac{1}{24} \left\{ \left(Tr\left[g\right]\right)^4 - 6Tr\left[g^2\right]\left(Tr\left[g\right]\right)^2 + 3\left(Tr\left[g^2\right]\right)^2 + 8Tr\left[g^3\right]Tr\left[g\right] - 6Tr\left[g^4\right]\right\}.$$
(2.4)

With explicit calculations, we may obtain:

$$
\begin{aligned}
Tr\left[g\right] &= 2a^2\left(1 + \mathcal{C}_\alpha^\alpha - \mathcal{A}\right),\\
Tr\left[g^2\right] &= 4a^4\left(1 + \mathcal{C}_\alpha^\alpha + \mathcal{A}\right),\\
Tr\left[g^3\right] &= 2a^6\left(1 + 3\mathcal{C}_\alpha^\alpha - 3\mathcal{A}\right),\\
Tr\left[g^4\right] &= 4a^8\left(1 + 2\mathcal{C}_\alpha^\alpha + 2\mathcal{A}\right).
\end{aligned}
$$
(2.5)

Now, one can easily obtain:

$$\sqrt{-det\left(g\right)} = a^4\sqrt{1 + 2\mathcal{A} + 2\mathcal{C}_\alpha^\alpha} = a^4\left(1 + \mathcal{A} + \mathcal{C}_\alpha^\alpha\right) = a^4\left(1 + \delta g\right).$$
(2.6)

The Christoffel symbols are:

$$
\begin{aligned}
\hat{\Gamma}^\eta_{\eta\eta} &= \mathcal{A}',\\
\hat{\Gamma}^\eta_{\eta\alpha} &= \mathcal{A}_{,\alpha},\\
\hat{\Gamma}^\eta_{\alpha\beta} &= \mathcal{C}'_{\alpha\beta} + \frac{1}{2}\left(\mathcal{B}_{\alpha,\beta} + \mathcal{B}_{\beta,\alpha}\right),\\
\hat{\Gamma}^\alpha_{\beta\gamma} &= -\mathcal{C}^{;\alpha}_{\beta\gamma} + \mathcal{C}^\alpha_{\beta,\gamma} + \mathcal{C}^\alpha_{\gamma,\beta},\\
\hat{\Gamma}^\alpha_{\beta\eta} &= \mathcal{C}'^\alpha_\beta - \frac{1}{2}\left(\mathcal{B}^\alpha_{,\beta} - \mathcal{B}^\beta_{,\alpha}\right),\\
\hat{\Gamma}^\alpha_{\eta\eta} &= -\mathcal{B}'^\alpha + \mathcal{A}^{;\alpha}.
\end{aligned}
$$
(2.7)

Consequently:
$$\delta\hat{\Gamma}^\eta = \hat{\Gamma}^\eta_{\mu\nu}\hat{k}^\mu\hat{k}^\nu = \mathcal{A}' - 2\mathcal{A}_{,\parallel} + \mathcal{B}_{\parallel,\parallel} + \mathcal{C}'_{\parallel\parallel},$$
(2.8)

$$\delta\hat{\Gamma}^\alpha = \hat{\Gamma}^\alpha_{\mu\nu}\hat{k}^\mu\hat{k}^\nu = \mathcal{A}^{;\alpha} - \mathcal{B}'^\alpha + \mathcal{B}^{;\alpha}_\parallel + \mathcal{B}^\alpha_{,\parallel} - 2\mathcal{C}'^\alpha_\parallel + 2\mathcal{C}^\alpha_{\parallel,\parallel} + \mathcal{C}^{;\alpha}_{\parallel\parallel},$$
(2.9)

where the subscript $\parallel$ should be understood as the convolution of 4-vector with n^α, meaning $B_\parallel = B_\alpha n^\alpha$.

To obtain the perturbed expression of the observer four-velocity, we can use the timelike normalization $u_\mu u^\mu$:

$$\delta u = -\mathcal{A},$$
(2.10)

and therefore, components of u_μ are:

$$u_\eta = -a\left(1 + \mathcal{A}\right), \qquad u_\alpha = a\left(\mathcal{U}_\alpha - \mathcal{B}_\alpha\right) = a\mathcal{V}_\alpha.$$
(2.11)

2.2 Distortion in the Conformal Wavevector

By accounting for (1.88) together with (2.9) or (2.8), it is easy to show that:

$$\delta\nu|_o^s = \left(\mathcal{B}_\| - 2\mathcal{A}\right)\big|_o^z - \int_0^{\bar{r}_z} d\bar{r} \left(\mathcal{A} - \mathcal{B}_\| - \mathcal{C}_{\|\|}\right)', \tag{2.12}$$

$$\delta n_\alpha|_o^s = -\left(\mathcal{B}_\alpha - 2\mathcal{C}_{\alpha\|}\right)\big|_o^z - \int_0^{\bar{r}_z} d\bar{r} \left(\mathcal{A} - \mathcal{B}_\| - \mathcal{C}_{\|\|}\right)_{,\alpha}. \tag{2.13}$$

From the definition and the orthonormality condition, we derive the spacelike tetrad components at linear order as:

$$\delta e_i^\eta = \delta_i^\alpha \left(\mathcal{U}_\alpha - \mathcal{B}_\alpha\right), \quad S_{ij} = \delta_i^\alpha \delta_j^\beta \mathcal{C}_{\alpha\beta}. \tag{2.14}$$

Then, we need to use the expression for an antisymmetric part from [40], and write down the whole expression for $\delta\nu_o$ and $(\delta n_\alpha)_o$ as:

$$\delta\nu_o = -\left(\mathcal{A} + \mathcal{U}_\| - \mathcal{B}_\|\right)_o, \tag{2.15}$$

$$(\delta n_\alpha)_o = -\left(\mathcal{U}_\alpha + \mathcal{C}_{\alpha\|} + C_{[\alpha,\|]} + \varepsilon_{\alpha ij}n^i\Omega^j\right). \tag{2.16}$$

2.3 Observed Redshift

The scale factor at the observer position to the linear order is:

$$a\left(\eta_o + \delta\eta_o\right) = a\left(\eta_o + \delta t_o\right) = 1 + \mathcal{H}_o\delta t_o, \tag{2.17}$$

where $\mathcal{H} = \frac{a'}{a}$ is the conformal Hubble parameter.
Further, we want to use expression (1.88) as explicit form of δt_o:

$$\delta t_o = \int_0^{\bar{t}_0} dt\delta u\left(\bar{x}_t\right) = -\int_0^{\bar{t}_0} dt\mathcal{A}\left(\bar{x}_t\right), \tag{2.18}$$

and similar for δx_o^α from (1.88), one may have:

$$\delta x_o^\alpha = \int_0^{\bar{t}_0} dt\frac{1}{a}\mathcal{U}^\alpha\left(\bar{x}_t\right). \tag{2.19}$$

As we have seen before, combining (1.96) and (1.95):

$$\delta z = a_o\left(1 + \mathcal{A} + \mathcal{U}_\| - \mathcal{B}_\| + \delta\nu\right) - 1 = \mathcal{H}_o\delta t_o + \delta\nu + \mathcal{A} + \mathcal{U}_\| - \mathcal{B}_\|. \tag{2.20}$$

2.4 Distortions in the Source Position

Firstly, it is important to calculate temporal distortion expanding:

$$\Delta\eta_s = \frac{\delta z}{\mathcal{H}_z} = \frac{1}{\mathcal{H}_z}\left(\mathcal{H}_o\delta t_o + \delta\nu + \mathcal{A} + \mathcal{U}_\parallel + \mathcal{B}_\parallel\right). \tag{2.21}$$

For the spatial component, we will use (1.112):

$$\Delta x_s^\alpha = \delta x_o^\alpha - n^\alpha\left(\Delta\eta_s - \delta\eta_o\right) + \int_o^{\bar{r}_z} d\bar{r}\left(\delta n^\alpha - n^\alpha\delta\nu\right). \tag{2.22}$$

Using parametrization given in (1.106), we find:

$$\delta r = -\Delta\lambda_s + \int_0^{\bar{r}_z} d\bar{r}\delta n_\parallel =$$
$$= \delta x_{o\parallel}\delta t_o - \frac{\delta z}{\mathcal{H}_z} + \int_0^{\bar{r}_z} d\bar{r}\left(\mathcal{A} - \mathcal{B}_\parallel - \mathcal{C}_{\parallel\parallel}\right), \tag{2.23}$$

$$\bar{r}_z\delta\theta = \theta_\alpha\delta x_o^\alpha + \int_o^{\bar{r}_z} d\bar{r}\theta_\alpha\delta n^\alpha = \theta_\alpha\delta x_o^\alpha + \bar{r}_z\theta_\alpha\left(\delta n^\alpha + \mathcal{B}^\alpha + 2\mathcal{C}_\parallel^\alpha\right)\big|_o -$$
$$- \int_o^{\bar{r}_z} d\bar{r}\left[\theta_\alpha\left(\mathcal{B}^\alpha + 2\mathcal{C}_\parallel^\alpha\right) + \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}}\right)\frac{\partial}{\partial\theta}\left(\mathcal{A} - \mathcal{B}_\parallel - \mathcal{C}_{\parallel\parallel}\right)\right], \tag{2.24}$$

$$\bar{r}_z\sin\theta\delta\phi = \phi_\alpha\delta x_o^\alpha + \int_o^{\bar{r}_z} d\bar{r}\phi_\alpha\delta n^\alpha = \phi_\alpha\delta x_o^\alpha + \bar{r}_z\phi_\alpha\left(\delta n^\alpha + \mathcal{B}^\alpha + 2\mathcal{C}_\parallel^\alpha\right)\big|_o -$$
$$- \int_o^{\bar{r}_z} d\bar{r}\left[\phi_\alpha\left(\mathcal{B}^\alpha + 2\mathcal{C}_\parallel^\alpha\right) + \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial}{\partial\phi}\left(\mathcal{A} - \mathcal{B}_\parallel - \mathcal{C}_{\parallel\parallel}\right)\right]. \tag{2.25}$$

2.5 Lensing Magnification

The distortion in the solid angle $d\Omega$ at the observed (θ,ϕ) and the (unobserved) source $(\theta+\delta\theta,\phi+\delta\phi)$ is described by the deformation matrix $\mathbb{D}$ (inverse of the magnification matrix), and it is conventionally decomposed as:

$$\mathbb{D} = \frac{\partial\left(\theta+\delta\theta,\phi+\delta\phi\right)}{\partial\left(\theta,\phi\right)} = I - \begin{pmatrix}\kappa & 0 \\ 0 & \kappa\end{pmatrix} - \begin{pmatrix}0 & \omega \\ -\omega & 0\end{pmatrix} - \begin{pmatrix}\gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1\end{pmatrix}. \tag{2.26}$$

Given this definition, we can compute the determinant of the distortion matrix as

$$det\mathbb{D} = 1 - 2\kappa + \kappa^2 - \gamma^2 + \omega^2. \tag{2.27}$$

To the second order in perturbations, the determinant of the deformation matrix is given by:

$$det\mathbb{D} = \frac{\sin(\theta + \delta\theta)}{\sin\theta}\left[1 + \frac{\partial}{\partial\theta}\delta\theta + \frac{\partial}{\partial\phi}\delta\phi + \frac{\partial}{\partial\theta}\delta\theta\frac{\partial}{\partial\phi}\delta\phi - \frac{\partial}{\partial\theta}\delta\phi\frac{\partial}{\partial\phi}\delta\theta\right]. \tag{2.28}$$

To the linear order, one may obtain the following:

$$-2\kappa = \left(\cot\theta + \frac{\partial}{\partial\theta}\right)\delta\theta + \frac{\partial}{\partial\phi}\delta\phi. \tag{2.29}$$

2.6 Luminosity Distance

As it is shown in [7], one may rewrite the expression (1.123) as:

$$\delta\mathcal{D}_L = \delta z - \kappa + \frac{\delta r}{\bar{r}_z} + \frac{1}{2}\left(\mathcal{C}_\alpha^\alpha - \mathcal{C}_{\|\|\|}\right). \tag{2.30}$$

Expanding this relation, using (2.20), (2.23), (2.29), one may show that

$$\delta\mathcal{D}_L = \mathcal{H}_o\delta t_o + \delta\nu + \mathcal{A} + \mathcal{U}_\| - \mathcal{B}_\| + \frac{1}{2}\left[\left(\cot\theta + \frac{\partial}{\partial\theta}\right)\delta\theta + \frac{\partial}{\partial\phi}\delta\phi\right] + \frac{\delta r}{\bar{r}_z} + \frac{1}{2}\left(\mathcal{C}_\alpha^\alpha - \mathcal{C}_{\|\|\|}\right). \tag{2.31}$$

2.7 Newton Conformal Gauge

Here and after, we will use the combination of zero-shear temporal gauge and spatial C-gauge condition. This is commonly referred to in the literature as the Newton Conformal Gauge. In this gauge, the metric to the linear order can be written as:

$$ds^2 = -a^2\left(1 + 2\psi\right)d\eta^2 + a^2\left(1 + 2\phi\right)\delta_{\alpha\beta}dx^\alpha dx^\beta. \tag{2.32}$$

It simply means that we can use the following relation for metric quantities:

$$\mathcal{A} = \psi, \quad \mathcal{B}_\alpha = 0, \quad \mathcal{C}_{\alpha\beta} = \delta_{\alpha\beta}\phi. \tag{2.33}$$

With this definition, 4-velocity is simply:

$$u_\eta = -a\left(1 + \psi\right), \quad u_\alpha = a\mathcal{U}_\alpha. \tag{2.34}$$

Together with distortion in wavevector

$$\delta\nu_o = -\left(\psi + \mathcal{U}_\alpha n^\alpha\right)_o, \tag{2.35}$$

$$\delta\nu = -2\psi + \psi_o - \left(\mathcal{U}_\alpha n^\alpha\right)_o - \int_0^{\bar{r}_z} d\bar{r}\,(\psi - \phi)', \tag{2.36}$$

observed redshift

$$\delta z = \mathcal{H}_o \delta t_o + \left(\mathcal{U}_\alpha n^\alpha - \psi\right)\Big|_o^z - \int_0^{\bar{r}_z} d\bar{r}\,(\psi - \phi)', \tag{2.37}$$

and source position

$$\delta r = \delta x_{o\parallel} + \delta t_o - \frac{\delta z}{\mathcal{H}_z} + \int_0^{\bar{r}_z} (\psi - \phi)\,d\bar{r}, \tag{2.38}$$

$$\delta n_o^\alpha = -\left(\mathcal{U}^\alpha + \mathcal{C}_\parallel^\alpha + C^{[\alpha}{}_{\parallel]} + \varepsilon_{ij}^\alpha n^i \Omega^j\right)_o, \tag{2.39}$$

$$\delta\theta = \frac{\theta_\alpha \delta x_o^\alpha}{\bar{r}_z} + \theta_\alpha \mathcal{U}_o^\alpha - \frac{1}{\bar{r}_z}\int_o^{\bar{r}_z} d\bar{r}\left[\left(\frac{\bar{r}_z - \bar{r}}{\bar{r}}\right)\frac{\partial}{\partial\theta}, (\psi - \phi)\right] \tag{2.40}$$

$$\delta\phi = \frac{\phi_\alpha \delta x_o^\alpha}{\bar{r}_z \sin\theta} + \frac{\phi_\alpha \mathcal{U}_o^\alpha}{\sin\theta} - \frac{1}{\bar{r}_z \sin\theta}\int_o^{\bar{r}_z} d\bar{r}\left[\left(\frac{\bar{r}_z - \bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial}{\partial\phi}(\psi - \phi)\right], \tag{2.41}$$

we may use an identity

$$\left(\cot\theta + \frac{\partial}{\partial\theta}\right)\theta^\alpha + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\phi^\alpha = -2n^\alpha \tag{2.42}$$

to get

$$\kappa = \frac{n_\alpha \delta x_o^\alpha}{\bar{r}_z} + n_\alpha \mathcal{U}^\alpha + \int_o^{\bar{r}_z} d\bar{r}\left(\frac{\bar{r}_z - \bar{r}}{2\bar{r}_z\bar{r}}\right)\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)(\psi - \phi). \tag{2.43}$$

This results into:

$$\delta\mathcal{D}_L = \left(\mathcal{H}_o \delta t_o + \left(\mathcal{U}_\alpha n^\alpha - \psi\right)\Big|_o^z - \int_0^{\bar{r}_z} d\bar{r}\,(\psi - \phi)'\right)\left(1 - \frac{1}{\mathcal{H}_z\bar{r}_z}\right) - n_\alpha \mathcal{U}^\alpha + \frac{\delta t_o}{\hat{r}_z} -$$

$$-\int_o^{\bar{r}_z} d\bar{r}\left(\frac{\bar{r}_z - \bar{r}}{2\bar{r}_z\bar{r}}\right)\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)(\psi - \phi) + \frac{1}{\bar{r}_z}\int_o^{\bar{r}_z}(\psi - \phi) + \phi, \tag{2.44}$$

where:

$$\delta t_o = - \int_0^{\bar{t}_0} \psi \left(\bar{x}_t \right) dt, \tag{2.45}$$

$$\delta x_o^\alpha = \int_0^{\bar{t}_0} \frac{1}{a} \mathcal{U}^\alpha \left(\bar{x}_t \right) dt. \tag{2.46}$$

Chapter 3

Second Order

3.1 Gauge Choice

As we did in the previous chapter, we will again consider a gauge choice. Let's recall scalar-vector-tensor decomposition:

$$\mathcal{A} = \psi, \qquad \mathcal{B}_\alpha = \beta_{,\alpha} + B_\alpha, \qquad \mathcal{C}_{\alpha\beta} = \phi\hat{g}_{\alpha\beta} + \gamma_{,\alpha\beta} + C^{(v)}_{(\alpha,\beta)} + C^{(t)}_{\alpha\beta}. \qquad (3.1)$$

Again, we will choose spatial C-gauge, and for temporal gauge, we will consider zero-shear gauge, meaning $\chi = a\left(\gamma' + \beta\right) = 0$, altogether this means that:

$$\gamma = 0, \qquad C^{(v)}_\alpha = 0, \qquad \beta = 0. \qquad (3.2)$$

We also add flat universe condition and metric elements to the second order:

$$\mathcal{A} = \psi^{(1,2)}, \qquad \mathcal{B}_\alpha = B^{(2)}_\alpha, \qquad \mathcal{C}_{\alpha\beta} = \phi^{(1,2)}\delta_{\alpha\beta} + C^{(2)}_{\alpha\beta}. \qquad (3.3)$$

Here, we can see that to the linear order, it is exactly the Newtonian conformal gauge.

3.2 Intermediate Steps

Now, we would like to find δA to the second order in this gauge. To achieve this, we will use the expression provided in [7]. Firstly, we need to rewrite some intermediate expressions.

3.2.1 Metric-Related Quantities and Four-Velocity

The second-order perturbation in the determinant of the metric in generic form:

$$\delta g = \mathcal{A} + \mathcal{C}_\alpha^\alpha - \frac{1}{2}\mathcal{A}^2 + \frac{1}{2}\mathcal{B}^\alpha\mathcal{B}_\alpha + \mathcal{A}\,\mathcal{C}_\alpha^\alpha + \frac{1}{2}\mathcal{C}_\alpha^\alpha\mathcal{C}_\beta^\beta - \mathcal{C}^{\alpha\beta}\mathcal{C}_{\alpha\beta}.$$

After applying the gauge given in (3.3), this turns to:

$$\delta g = \psi + 3\phi + C_\alpha^\alpha - \frac{1}{2}\psi^2 + 3\psi\phi + \frac{3}{2}\phi^2. \tag{3.4}$$

It is important to note that all terms written linearly in this expression should be understood as second-order expressions, while those presented in quadratic terms are first-order. For example, the first term can be written as $\psi^{(2)}$, while the fifth term as $3\psi^{(1)}\phi^{(1)}$. This distinction may seem insignificant at the moment, but it will become more useful later.

Consequently, $\delta\hat{\Gamma}^\eta$ and $\delta\hat{\Gamma}^\alpha$:

$$\delta\hat{\Gamma}^\eta = \mathcal{A}' - 2\mathcal{A}_{,\|} + \mathcal{B}_{\|,\|} + \mathcal{C}'_{\|\|} + 2\delta\nu\left(\mathcal{A}' - \mathcal{A}_{,\|}\right) - 2\mathcal{A}\left(\mathcal{A}' - 2\mathcal{A}_{,\|} + \mathcal{B}_{\|,\|} + \mathcal{C}'_{\|\|}\right) -$$
$$- 2\delta n^\alpha\left(\mathcal{A}_{,\alpha} - \mathcal{B}_{(\alpha,\|)} - \mathcal{C}'_{\alpha\|}\right) - \mathcal{B}^\alpha\left(\mathcal{A}_{,\alpha} - \mathcal{B}'_\alpha + 2\mathcal{B}_{[\alpha,\|]} - 2\mathcal{C}'_{\alpha\|} + 2\mathcal{C}_{\alpha\|,\|} - \mathcal{C}_{\|\|,\alpha}\right),$$

$$\delta\hat{\Gamma}^\alpha = \mathcal{A}^{,\alpha} - \mathcal{B}'^\alpha - \mathcal{B}_\|{}^{,\alpha} + \mathcal{B}^\alpha{}_{,\|} - 2\mathcal{C}'^\alpha_\| + 2\mathcal{C}^\alpha_{\|,\|} - \mathcal{C}_{\|\|}{}^{,\alpha} + \delta\nu\left(2\mathcal{A}^{,\alpha} - 2\mathcal{B}'^\alpha - \mathcal{B}_\|{}^{,\alpha} + \mathcal{B}^\alpha{}_{,\|} - 2\mathcal{C}'^\alpha_\|\right) -$$
$$- \delta n^\gamma\left(\mathcal{B}_\gamma{}^{,\alpha} - \mathcal{B}^\alpha{}_{,\gamma} + 2\mathcal{C}'^\alpha_\gamma - 4\mathcal{C}^\alpha_{(\|,\gamma)} + 2\mathcal{C}_{\|\gamma}{}^{,\alpha}\right) + \mathcal{B}^\alpha\left(\mathcal{A}' + \mathcal{B}_{\|,\|} + \mathcal{C}'_{\|\|} - 2\mathcal{A}_{,\|}\right) -$$
$$- 2\mathcal{C}^{\alpha\gamma}\left(2\mathcal{C}_{\gamma\|,\|} - \mathcal{C}_{\|\|,\gamma} + \mathcal{A}_{,\gamma} - \mathcal{B}'_\gamma - 2\mathcal{B}_{[\|,\gamma]} - 2\mathcal{C}'_{\|\gamma}\right),$$

Will be used as:

$$\delta\hat{\Gamma}^\eta = \psi' - 2\psi_{,\|} + B_{\|,\|} + \phi' + C'_{\|\|} + 2\delta\nu\left(\psi' - \psi_{,\|}\right) - 2\psi\left(\psi' - 2\psi_{,\|} + \phi'\right) -$$
$$- 2\delta n^\alpha\left(\psi_{,\alpha} - \phi'n_\alpha\right), \quad (3.5)$$

$$\delta\hat{\Gamma}^\alpha = \psi^{,\alpha} - B'^\alpha - B_\|{}^{,\alpha} + B^\alpha{}_{,\|} - 2\phi'n^\alpha - 2C''^\alpha_\| + 2\left(\phi n^\alpha\right)_{,\|} + 2C^\alpha_{\|,\|} - \phi^{,\alpha} - C_{\|\|}{}^{,\alpha} +$$
$$+ 2\delta\nu\left(\psi^{,\alpha} - \phi'n^\alpha\right) - 2\delta n^\gamma\left(\phi'\delta^\alpha_\gamma - \left(\phi n^\alpha\right)_{,\gamma} - \delta^\alpha_\gamma\phi_{,\|} + \left(\phi n_\gamma\right)^{,\alpha}\right) -$$
$$- 2\phi\left(2\left(\phi n^\alpha\right)_{,\|} - \phi^{,\alpha} + \psi^{,\alpha} - 2\phi'n^\alpha\right). \quad (3.6)$$

Perturbed expression of the observer four-velocity without gauge conditions:

$$\delta u = -\mathcal{A} + \frac{3}{2}\mathcal{A}^2 + \frac{1}{2}\mathcal{U}^\alpha\mathcal{U}_\alpha - \mathcal{U}_\alpha\mathcal{B}^\alpha,$$

and after applying the gauge:

$$\delta u = -\psi + \frac{3}{2}\psi^2 + \frac{1}{2}\mathcal{U}^\alpha\mathcal{U}_\alpha. \tag{3.7}$$

3.2.2 Time-Lapse and Spatial Shift

Again, the expression of time-lapse at the observer:

$$
\delta t_o = \delta x_o^\mu (\partial_\mu \delta t)\big|_{\bar{o}} + \int_0^{\bar{t}_o} dt \left(-\mathcal{A} + \frac{1}{2}\mathcal{A}^2 + \frac{1}{2}\mathcal{U}^\alpha \mathcal{U}_\alpha - \mathcal{U}^\alpha \mathcal{B}_\alpha - \frac{1}{a}\mathcal{U}^\alpha \delta t_{,\alpha} \right)\bigg|_{\bar{x}_t}
$$

will be simplified to the form

$$
\delta t_o = \delta x_o^\mu (\partial_\mu \delta t)\big|_{\bar{o}} + \int_0^{\bar{t}_o} dt \left(-\psi + \frac{1}{2}\psi^2 + \frac{1}{2}\mathcal{U}^\alpha \mathcal{U}_\alpha - \frac{1}{a}\mathcal{U}^\alpha \delta t_{,\alpha} \right)\bigg|_{\bar{x}_t}. \tag{3.8}
$$

Analogously, spatial shift:

$$
\delta x_o^\alpha = \delta x_o^\mu (\partial_\mu \delta x^\alpha)\big|_{\bar{o}} + \int_0^{\bar{t}_o} dt \, \frac{1}{a} \left(\mathcal{U}^\alpha + \mathcal{A}\mathcal{U}^\alpha - \mathcal{U}^\beta \delta x^\alpha_{,\beta} \right)\bigg|_{\bar{x}_t}
$$

after applying, the gauge turns into

$$
\delta x_o^\alpha = \delta x_o^\mu (\partial_\mu \delta x^\alpha)\big|_{\bar{o}} + \int_0^{\bar{t}_o} dt \, \frac{1}{a} \left(\mathcal{U}^\alpha + \psi\mathcal{U}^\alpha - \mathcal{U}^\beta \delta x^\alpha_{,\beta} \right)\bigg|_{\bar{x}_t}. \tag{3.9}
$$

Consequently, the second-order expression for the time-lapse in conformal coordinates:

$$
\delta\eta_o = \delta t_o - \frac{1}{2}\mathcal{H}_{\bar{o}}\delta t_o^2
$$

with use of (3.8) can be written as:

$$
\delta\eta_o = \delta x_o^\mu (\partial_\mu \delta t)\big|_{\bar{o}} + \int_0^{\bar{t}_o} dt \left(-\psi + \frac{1}{2}\psi^2 + \frac{1}{2}\mathcal{U}^\alpha \mathcal{U}_\alpha - \frac{1}{a}\mathcal{U}^\alpha \delta t_{,\alpha} \right)\bigg|_{\bar{x}_t} - \frac{1}{2}\mathcal{H}_{\bar{o}}\delta t_o^2. \tag{3.10}
$$

3.2.3 Distortion in the Conformal Wavevector

First of all, let us take a look at the $\delta\nu$ expression:

$$
\delta\nu\big|_o^s = \left[\Delta x_{\bar{r}}^\mu (\partial_\mu \delta\nu)\big|_{\bar{x}_{\bar{r}}} \right]_{\bar{o}}^z + \int_0^{\bar{r}_z} d\bar{r} \left(\delta\hat{\Gamma}^\eta + \delta\nu\delta\nu' - \delta n^\alpha \delta\nu_{,\alpha} \right)\bigg|_{\bar{x}_{\bar{r}}}
$$

that with use of (3.5) will take form of:

$$
\delta\nu\big|_o^s = \left[\Delta x_{\bar{r}}^\mu (\partial_\mu \delta\nu)\big|_{\bar{x}_{\bar{r}}} \right]_{\bar{o}}^z + \int_0^{\bar{r}_z} d\bar{r} \bigg(\delta\nu\delta\nu' - \delta n^\alpha \delta\nu_{,\alpha} + \psi' - 2\psi_{,\parallel} + B_{\parallel,\parallel} + \phi' +
$$
$$
+ C'_{\parallel\parallel} + 2\delta\nu \left(\psi' - \psi_{,\parallel} \right) - 2\psi \left(\psi' - 2\psi_{,\parallel} + \phi' \right) - 2\delta n^\alpha \left(\psi_{,\alpha} - \phi' n_\alpha \right) \bigg)\bigg|_{\bar{x}_{\bar{r}}}. \tag{3.11}
$$

At the observer's position:

$$\delta\nu_o = \left(-\mathcal{A} - \mathcal{U}_\| + \mathcal{B}_\| + \frac{3}{2}\mathcal{A}^2 - \mathcal{C}_{\alpha\|}\left(\mathcal{U}^\alpha + \mathcal{B}^\alpha\right) + \mathcal{A}\left(\mathcal{U}_\| - 2\mathcal{B}_\|\right) - \mathcal{U}^\alpha \mathcal{B}_\alpha + \right.$$

$$\left. +\frac{1}{2}\mathcal{U}_\alpha \mathcal{U}^\alpha + \left(\mathcal{U}^\alpha - \mathcal{B}^\alpha\right)C_{[\alpha,\|]} + \varepsilon_{\alpha i j}\left(\mathcal{U}^\alpha - \mathcal{B}^\alpha\right)n^i \Omega^j \right)_{\bar{o}} + \delta x_o^\mu \left[\partial_\mu\left(-\mathcal{A} - \mathcal{U}_\| + \mathcal{B}_\|\right)\right]\big|_{\bar{o}},$$

the rewritten expression will take the form:

$$\delta\nu_o = \left(-\psi - \mathcal{U}_\| + B_\| + \frac{3}{2}\psi^2 + \mathcal{U}_\|\left(\psi - \phi\right) + \frac{1}{2}\mathcal{U}_\alpha \mathcal{U}^\alpha + \varepsilon_{\alpha i j}\mathcal{U}^\alpha n^i \Omega^j \right)_{\bar{o}} + \delta x_o^\mu \left[\partial_\mu\left(-\psi - \mathcal{U}_\|\right)\right]\big|_{\bar{o}}. \tag{3.12}$$

We will repeat the same with

$$\delta n^\alpha\big|_o^s = \left[\Delta x_{\bar{r}}^\mu (\partial_\mu \delta n^\alpha)\big|_{\bar{x}_{\bar{r}}}\right]_{\bar{o}}^z - \int_0^{\bar{r}_z} d\bar{r}\left(\delta\hat{\Gamma}^\alpha - \delta\nu\delta n'^\alpha + \delta n^\beta \delta n^\alpha_{,\beta}\right)\Bigg|_{\bar{x}_{\bar{r}}}$$

to get:

$$\delta n^\alpha\big|_o^s = \left[\Delta x_{\bar{r}}^\mu (\partial_\mu \delta n^\alpha)\big|_{\bar{x}_{\bar{r}}}\right]_{\bar{o}}^z - \int_0^{\bar{r}_z} d\bar{r}\left(\psi^{,\alpha} - B'^\alpha - B_\|{}^{,\alpha} + B^\alpha_{,\|} - 2\phi' n^\alpha - 2C''^\alpha_\| + \right.$$

$$+2\left(\phi n^\alpha\right)_{,\|} + 2C^\alpha_{\|,\|} - \phi^{,\alpha} - C_{\|\|}{}^{,\alpha} + 2\delta\nu\left(\psi^{,\alpha} - \phi' n^\alpha\right) -$$

$$-2\delta n^\gamma\left(\phi'\delta^\alpha_\gamma - (\phi n^\alpha)_{,\gamma} - \delta^\alpha_\gamma \phi_{,\|} + (\phi n_\gamma)^{,\alpha}\right) - 2\phi\left(2(\phi n^\alpha)_{,\|} - \phi^{,\alpha} + \psi^{,\alpha} - 2\phi' n^\alpha\right) -$$

$$\left. -\delta\nu\delta n'^\alpha + \delta n^\beta \delta n^\alpha_{,\beta}\right)\Bigg|_{\bar{x}_{\bar{r}}}. \tag{3.13}$$

At the observer's position:

$$-\delta n_o^\alpha = \mathcal{U}_o^\alpha + \left(A^\alpha{}_i + \mathcal{C}^\alpha_i + \frac{1}{2}\mathcal{B}_i \mathcal{B}^\alpha - \frac{1}{2}\mathcal{U}_i \mathcal{U}^\alpha - \frac{1}{2}\Omega_i \Omega^\alpha + \frac{1}{2}\delta^\alpha_i \Omega^k \Omega_k - \frac{3}{2}\mathcal{C}^\beta_i \mathcal{C}^\alpha_\beta - \mathcal{C}^\beta_{(i}\varepsilon^{\alpha)}{}_{k\beta}\Omega^k \right.$$

$$-\frac{1}{2}C_{\beta,(i}\mathcal{C}^{\alpha)\beta} + \frac{1}{2}C^\beta{}_{,(i}\varepsilon^{\alpha)}{}_{k\beta}\Omega^k - \frac{1}{2}C_{(i,\beta}\varepsilon^{\alpha)}{}_k{}^\beta \Omega^k + \frac{1}{8}C^\beta{}_{,i}C_\beta{}^{,\alpha} + \frac{1}{8}C_i{}^{,\beta}C^\alpha{}_{,\beta} + \frac{1}{2}\mathcal{C}^\beta_{(i}C^{\alpha)}{}_{,\beta}$$

$$\left. -\frac{1}{4}C^\beta{}_{,(i}C^{\alpha)}{}_{,\beta}\right)n^i + \delta x_o^\mu \left[\partial_\mu\left(\mathcal{U}^\alpha + \mathcal{C}^\alpha_i n^i + C^{[\alpha}{}_{,i]}n^i + \varepsilon^\alpha{}_{ij}n^i \Omega^j\right)\right]\Bigg|_{\bar{o}}$$

the rewritten expression will take the form:

$$-\delta n_o^\alpha = \mathcal{U}_o^\alpha + \left(A^\alpha{}_i + \phi\delta^\alpha_i + C^\alpha_i - \frac{1}{2}\mathcal{U}_i \mathcal{U}^\alpha - \frac{1}{2}\Omega_i \Omega^\alpha + \frac{1}{2}\delta^\alpha_i \Omega^k \Omega_k - \frac{3}{2}\phi^2 \delta^\alpha_i \right)_{\bar{o}}n^i$$

$$+ \delta x_o^\mu \left[\partial_\mu\left(\mathcal{U}^\alpha + \mathcal{C}^\alpha_i n^i + \varepsilon^\alpha{}_{ij}n^i \Omega^j\right)\right]\Bigg|_{\bar{o}}, \tag{3.14}$$

where we used

$$\delta^\beta_{(i}\varepsilon^{\alpha)}{}_{k\beta}\Omega^k n^i = \frac{1}{2}\left(\delta^\beta_i \varepsilon^\alpha{}_{k\beta} + \delta^{\beta\alpha}\varepsilon_{ik\beta}\right)\Omega^k n^i = \frac{1}{2}\left(\varepsilon^\alpha{}_{ki} + \varepsilon_{ik}{}^\alpha\right)\Omega^k n^i = 0. \tag{3.15}$$

3.2.4 Perturbations in the Observed Redshift

In order to express δz in terms of the metric perturbations and the observer velocity, we first need to rewrite the expression

$$\widehat{\Delta\nu}(\bar{x}_z; x_o) = \delta\nu + \mathcal{A} + \mathcal{U}_\parallel - \mathcal{B}_\parallel + \delta\nu\mathcal{A} - \frac{1}{2}\mathcal{A}^2 + \mathcal{A}\mathcal{B}_\parallel + \delta n_\alpha \left(\mathcal{U}^\alpha - \mathcal{B}^\alpha\right) + \frac{1}{2}\mathcal{U}^\alpha\mathcal{U}_\alpha + 2\mathcal{C}_{\alpha\parallel}\mathcal{U}^\alpha$$

that together with (3.3), (3.11) and (3.12) will be written as:

$$\begin{aligned}
\widehat{\Delta\nu}(\bar{x}_z; x_o) = &\left(-\psi - \mathcal{U}_\parallel + B_\parallel + \frac{3}{2}\psi^2 + \mathcal{U}_\parallel(\psi - \phi) + \frac{1}{2}\mathcal{U}_\alpha\mathcal{U}^\alpha + \varepsilon_{\alpha ij}\mathcal{U}^\alpha n^i \Omega^j\right)_{\bar{o}} + \\
&+ \int_0^{\bar{r}_z} d\bar{r}\left(\delta\nu\delta\nu' - \delta n^\alpha \delta\nu_{,\alpha} + \psi' - 2\psi_{,\parallel} + B_{\parallel,\parallel} + \phi' + C'_{\parallel\parallel\parallel} + 2\delta\nu(\psi' - \psi_{,\parallel}) - \right. \\
&\left. - 2\psi(\psi' - 2\psi_{,\parallel} + \phi') - 2\delta n^\alpha(\psi_{,\alpha} - \phi' n_\alpha)\right)\Big|_{\bar{x}_{\bar{r}}} + \psi + \mathcal{U}_\parallel - B_\parallel + \delta\nu\psi - \frac{1}{2}\psi^2 + \\
&+ \delta n_\alpha\mathcal{U}^\alpha + \frac{1}{2}\mathcal{U}^\alpha\mathcal{U}_\alpha + 2\phi\mathcal{U}_\parallel. \quad (3.16)
\end{aligned}$$

That means that we are ready to rewrite

$$\delta z(x_s; x_o) = \mathcal{H}_{\bar{o}}\delta\eta_o + \widehat{\Delta\nu}(\bar{x}_z; x_o) + \frac{1}{2}\left(\mathcal{H}_{\bar{o}}^2 + \mathcal{H}'_{\bar{o}}\right)\delta\eta_o^2 + \mathcal{H}_{\bar{o}}\delta\eta_o\widehat{\Delta\nu}(\bar{x}_z; x_o) + \Delta x_s^\mu(\partial_\mu\widehat{\Delta\nu})\Big|_{\bar{x}_z},$$

$$(3.17)$$

using (3.10) and (3.16), into the:

$$\begin{aligned}
\delta z(x_s; x_o) = &\mathcal{H}_{\bar{o}}\left(\delta x_o^\mu(\partial_\mu\delta t)\Big|_{\bar{o}} - \frac{1}{2}\mathcal{H}_{\bar{o}}\delta t_o^2 + \int_0^{\bar{t}_o} dt\left(-\psi + \frac{1}{2}\psi^2 + \frac{1}{2}\mathcal{U}^\alpha\mathcal{U}_\alpha - \frac{1}{a}\mathcal{U}^\alpha\delta t_{,\alpha}\right)\Big|_{\bar{x}_t}\right) + \\
&+ \left(-\psi - \mathcal{U}_\parallel + B_\parallel + \frac{3}{2}\psi^2 + \mathcal{U}_\parallel(\psi - \phi) + \frac{1}{2}\mathcal{U}_\alpha\mathcal{U}^\alpha + \varepsilon_{\alpha ij}\mathcal{U}^\alpha n^i \Omega^j\right)_{\bar{o}} + \\
&+ \int_0^{\bar{r}_z} d\bar{r}\left(\delta\nu\delta\nu' - \delta n^\alpha \delta\nu_{,\alpha} + \psi' - 2\psi_{,\parallel} + B_{\parallel,\parallel} + \phi' + C'_{\parallel\parallel\parallel} + 2\delta\nu(\psi' - \psi_{,\parallel}) - \right. \\
&\left. - 2\psi(\psi' - 2\psi_{,\parallel} + \phi') - 2\delta n^\alpha(\psi_{,\alpha} - \phi' n_\alpha)\right)\Big|_{\bar{x}_{\bar{r}}} + \psi + \mathcal{U}_\parallel - B_\parallel + \delta\nu\psi - \frac{1}{2}\psi^2 +
\end{aligned}$$

$$+ \delta n_\alpha\mathcal{U}^\alpha + \frac{1}{2}\mathcal{U}^\alpha\mathcal{U}_\alpha + 2\phi\mathcal{U}_\parallel + \frac{1}{2}\left(\mathcal{H}_{\bar{o}}^2 + \mathcal{H}'_{\bar{o}}\right)\delta\eta_o^2 + \mathcal{H}_{\bar{o}}\delta\eta_o\widehat{\Delta\nu}(\bar{x}_z; x_o) + \Delta x_s^\mu(\partial_\mu\widehat{\Delta\nu})\Big|_{\bar{x}_z}.$$

$$(3.18)$$

3.2.5 Distortion in the Source Position

First of all, we will write down the expression for temporal and spatial distortion:

$$\Delta \eta_s = \frac{\delta z}{\mathcal{H}_z} - \frac{1}{2}\frac{\mathcal{H}_z^2 + \mathcal{H}_z'}{\mathcal{H}_z}\left(\frac{\delta z}{\mathcal{H}_z}\right)^2, \tag{3.19}$$

$$\Delta x_s^\alpha = \int_0^{\bar{r}_z} d\bar{r}\left[\delta n^\alpha - n^\alpha \delta\nu + \delta\nu\left(\Delta x'^\alpha + n^\alpha \Delta\eta'\right) - \delta n^\beta\left(\Delta x^\alpha{}_{,\beta} + n^\alpha \Delta\eta_{,\beta}\right)\right]\Big|_{\bar{x}_{\bar{r}}}$$
$$+ \delta x_o^\alpha - n^\alpha(\Delta\eta_s - \delta\eta_o) + \left[\Delta x_{\bar{r}}^\mu \,\partial_\mu\left(\Delta x^\alpha + n^\alpha \Delta\eta\right)\big|_{\bar{x}_{\bar{r}}}\right]_{\bar{o}}^z, \tag{3.20}$$

which we will leave in this form. But we rather compute the radial and transverse distortion:

$$\delta r = n_\alpha \Delta x_s^\alpha + \frac{\bar{r}_z}{2}\left(\delta\theta^2 + \sin^2\theta\,\delta\phi^2\right),$$
$$\bar{r}_z\delta\theta = \theta_\alpha \Delta x_s^\alpha - \delta r\delta\theta + \frac{\bar{r}_z}{2}\cos\theta\sin\theta\,\delta\phi^2, \tag{3.21}$$
$$\bar{r}_z\sin\theta\delta\phi = \phi_\alpha \Delta x_s^\alpha - \sin\theta\,\delta r\delta\phi - \bar{r}_z\cos\theta\,\delta\theta\delta\phi.$$

With the use of (3.19), (3.20), (3.18):

$$\delta r = -\bar{r}_z\left[\phi + C_{\|\|} - \frac{1}{2}\mathcal{U}_\|^2 - \frac{1}{2}\Omega_\|^2 + \frac{1}{2}\Omega^k\Omega_k - \frac{3}{2}\phi^2 - \psi + B_\| + \frac{3}{2}\psi^2 + \mathcal{U}_\|\left(\psi - \phi\right) + \right.$$

$$\left. + \frac{1}{2}\mathcal{U}_\alpha\mathcal{U}^\alpha + \varepsilon_{\alpha ij}\mathcal{U}^\alpha n^i\Omega^j\right]\Big|_{\bar{o}} + n_\alpha\delta x_o^\mu(\partial_\mu\delta x^\alpha)\big|_{\bar{o}} + n_\alpha\int_0^{\bar{t}_o}dt\,\frac{1}{a}\left(\mathcal{U}^\alpha + \psi\mathcal{U}^\alpha - \mathcal{U}^\beta\delta x^\alpha{}_{,\beta}\right)\Big|_{\bar{x}_t} -$$

$$- \frac{1}{\mathcal{H}_z}\left[\left(-\psi - \mathcal{U}_\| + B_\| + \frac{3}{2}\psi^2 + \mathcal{U}_\|\left(\psi - \phi\right) + \frac{1}{2}\mathcal{U}_\alpha\mathcal{U}^\alpha + \varepsilon_{\alpha ij}\mathcal{U}^\alpha n^i\Omega^j\right)_{\bar{o}} + \right.$$

$$\left. + \int_0^{\bar{r}_z}d\bar{r}\left[\delta\nu\delta\nu' - \delta n^\alpha\delta\nu_{,\alpha} + \psi' - 2\psi_{,\|} + B_{\|,\|} + \phi' + C'_{\|\|} + 2\delta\nu\left(\psi' - \psi_{,\|}\right) - \right.\right.$$

$$\left.\left. - 2\psi\left(\psi' - 2\psi_{,\|} + \phi'\right) - 2\delta n^\alpha\left(\psi_{,\alpha} - \phi'n_\alpha\right)\right]\Big|_{\bar{x}_{\bar{r}}} + \right.$$

$$+ \psi + \mathcal{U}_\| - B_\| + \delta\nu\psi - \frac{1}{2}\psi^2 + \delta n_\alpha\mathcal{U}^\alpha + \frac{1}{2}\mathcal{U}^\alpha\mathcal{U}_\alpha + 2\phi\mathcal{U}_\| +$$

$$+ \frac{1}{2}\left(\mathcal{H}_{\bar{o}}^2 + \mathcal{H}_{\bar{o}}'\right)\delta\eta_o^2 + \mathcal{H}_{\bar{o}}\delta\eta_o\widehat{\Delta\nu} + \Delta x_s^\mu(\partial_\mu\widehat{\Delta\nu})\big|_{\bar{x}_z}\right] + \frac{1}{2}\frac{\mathcal{H}_z^2 + \mathcal{H}_z'}{\mathcal{H}_z}\left(\frac{\delta z}{\mathcal{H}_z}\right)^2 -$$

$$- \left(\frac{\mathcal{H}_{\bar{o}}}{\mathcal{H}_z} - 1\right)\left[\delta x_o^\mu(\partial_\mu\delta t)\big|_{\bar{o}} + \int_0^{\bar{t}_o}dt\left(-\psi + \frac{1}{2}\psi^2 + \frac{1}{2}\mathcal{U}^\alpha\mathcal{U}_\alpha - \frac{1}{a}\mathcal{U}^\alpha\delta t_{,\alpha}\right)\Big|_{\bar{x}_t}\right] +$$

$$+ \frac{1}{2}\mathcal{H}_{\bar{o}}\delta t_o^2 + n_\alpha\left[\Delta x_{\bar{r}}^\mu\,\partial_\mu\left(\Delta x^\alpha + n^\alpha\Delta\eta\right)\big|_{\bar{x}_{\bar{r}}}\right]_{\bar{o}}^z + n_\alpha\int_0^{\bar{r}_z}d\bar{r}\left[-\delta n^\beta\left(\Delta x^\alpha{}_{,\beta} + n^\alpha\Delta\eta_{,\beta}\right)\right.$$

$$
\begin{aligned}
&+ \delta\nu\left(\Delta x'^{\alpha} + n^{\alpha}\Delta\eta'\right) - (\bar{r}_z - \bar{r})\bigg(-\delta\nu\delta n'^{\alpha} + \delta n^{\beta}\delta n^{\alpha}{}_{,\beta} + \psi^{,\alpha} - B'^{\alpha} - B_{\parallel}{}^{;\alpha} + B^{\alpha}{}_{,\parallel} - \\
&\qquad - 2\phi' n^{\alpha} - 2C'^{\alpha}_{\parallel} + 2\left(\phi n^{\alpha}\right)_{,\parallel} + 2C^{\alpha}_{\parallel,\parallel} - \phi^{,\alpha} - C_{\parallel\parallel}{}^{,\alpha} + 2\delta\nu\left(\psi^{,\alpha} - \phi' n^{\alpha}\right) - \\
&\quad - 2\delta n^{\gamma}\left(\phi'\delta^{\alpha}_{\gamma} - \left(\phi n^{\alpha}\right)_{,\gamma} - \delta^{\alpha}_{\gamma}\phi_{,\parallel} + \left(\phi n_{\gamma}\right)^{;\alpha}\right) - 2\phi\left(2\left(\phi n^{\alpha}\right)_{,\parallel} - \phi^{,\alpha} + \psi^{,\alpha} - 2\phi' n^{\alpha}\right) + \\
&+ n^{\alpha}\bigg(\delta\nu\delta\nu' - \delta n^{\alpha}\delta\nu_{,\alpha} + \psi' - 2\psi_{,\parallel} + B_{\parallel,\parallel} + \phi' + C'_{\parallel\parallel} + 2\delta\nu\left(\psi' - \psi_{,\parallel}\right) - 2\psi\left(\psi' - 2\psi_{,\parallel} + \phi'\right) - \\
&\qquad\qquad - 2\delta n^{\alpha}\left(\psi_{,\alpha} - \phi' n_{\alpha}\right)\bigg)\bigg)\bigg]\bigg|_{\bar{x}\bar{r}} + \frac{\bar{r}_z}{2}\left(\delta\theta^2 + \sin^2\theta\,\delta\phi^2\right), \quad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
\delta\theta = \frac{1}{\bar{r}_z}\,\theta_{\alpha}\bigg\{&-\bar{r}_z\bigg[\mathcal{U}^{\alpha} + n^i\bigg(A^{\alpha}{}_i + \phi\delta^{\alpha}_i + C^{\alpha}_i - \frac{1}{2}\mathcal{U}_i\mathcal{U}^{\alpha} - \frac{1}{2}\Omega_i\Omega^{\alpha} + \frac{1}{2}\delta^{\alpha}_i\Omega^k\Omega_k - \frac{3}{2}\phi^2\delta^{\alpha}_i\bigg)\bigg]\bigg|_{\bar{o}} \\
&+ \delta x^{\mu}_o(\partial_{\mu}\delta x^{\alpha})\big|_{\bar{o}} + \left[\Delta x^{\mu}_{\bar{r}}\,\partial_{\mu}\left(\Delta x^{\alpha} + n^{\alpha}\Delta\eta\right)\big|_{\bar{x}\bar{r}}\right]^z_{\bar{o}} \\
&+ \int_0^{\bar{t}_o} dt\,\frac{1}{a}\left(\mathcal{U}^{\alpha} + \psi\mathcal{U}^{\alpha} - \mathcal{U}^{\beta}\delta x^{\alpha}{}_{,\beta}\right)\bigg|_{\bar{x}_t} + \int_0^{\bar{r}_z} d\bar{r}\bigg[-\delta n^{\beta}\left(\Delta x^{\alpha}{}_{,\beta} + n^{\alpha}\Delta\eta_{,\beta}\right) \\
&+ \delta\nu\left(\Delta x'^{\alpha} + n^{\alpha}\Delta\eta'\right) - (\bar{r}_z - \bar{r})\bigg(-\delta\nu\delta n'^{\alpha} + \delta n^{\beta}\delta n^{\alpha}{}_{,\beta} + \psi^{,\alpha} - B'^{\alpha} - B_{\parallel}{}^{;\alpha} + B^{\alpha}{}_{,\parallel} - \\
&\qquad - 2\phi' n^{\alpha} - 2C'^{\alpha}_{\parallel} + 2\left(\phi n^{\alpha}\right)_{,\parallel} + 2C^{\alpha}_{\parallel,\parallel} - \phi^{,\alpha} - C_{\parallel\parallel}{}^{,\alpha} + 2\delta\nu\left(\psi^{,\alpha} - \phi' n^{\alpha}\right) - \\
&-2\delta n^{\gamma}\left(\phi'\delta^{\alpha}_{\gamma} - \left(\phi n^{\alpha}\right)_{,\gamma} - \delta^{\alpha}_{\gamma}\phi_{,\parallel} + \left(\phi n_{\gamma}\right)^{;\alpha}\right) - 2\phi\left(2\left(\phi n^{\alpha}\right)_{,\parallel} - \phi^{,\alpha} + \psi^{,\alpha} - 2\phi' n^{\alpha}\right)\bigg)\bigg]\bigg|_{\bar{x}\bar{r}}\bigg\} - \\
&\qquad\qquad\qquad - \frac{\delta r}{\bar{r}_z}\delta\theta + \frac{1}{2}\cos\theta\sin\theta\,\delta\phi^2. \quad (3.23)
\end{aligned}
$$

The expression for $\delta\phi$ is analogous to that of $\delta\theta$ up to replacing θ_{α} with $\phi_{\alpha}/\sin\theta$ and $-\frac{\delta r}{\bar{r}_z}\delta\theta + \frac{1}{2}\cos\theta\sin\theta\,\delta\phi^2$ with $-\frac{\delta r}{\bar{r}_z}\delta\phi - \bar{r}_z\cot\theta\,\delta\theta\delta\phi$.

3.2.6   Distortion Matrix Perturbation

Our goal in this section is to write down the expression for distortion in the physical area and to do this, first of all, we need to recover distortion matrix perturbation as:

$$
\delta\mathbb{D} = \left(\cot\theta + \frac{\partial}{\partial\theta}\right)\delta\theta + \frac{\partial}{\partial\phi}\delta\phi + \frac{\partial}{\partial\theta}\delta\theta\frac{\partial}{\partial\phi}\delta\phi - \frac{\partial}{\partial\phi}\delta\theta\frac{\partial}{\partial\theta}\delta\phi + \cot\theta\,\delta\theta\left(\frac{\partial}{\partial\theta}\delta\theta + \frac{\partial}{\partial\phi}\delta\phi\right) - \frac{1}{2}\delta\theta^2.
$$
$$(3.24)$$

Now we can substitute previously obtained expressions for $\delta\theta$ and $\delta\phi$:

$$\delta\mathbb{D} =$$

$$= \left(\cot\theta + \frac{\partial}{\partial\theta}\right)\left(\frac{1}{\bar{r}_z}\,\theta_\alpha\Bigg\{-\bar{r}_z\left[\mathcal{U}^\alpha + n^i\left(A^\alpha{}_i + \phi\delta_i^\alpha + C_i^\alpha - \frac{1}{2}\mathcal{U}_i\mathcal{U}^\alpha - \frac{1}{2}\Omega_i\Omega^\alpha + \frac{1}{2}\delta_i^\alpha\Omega^k\Omega_k - \frac{3}{2}\phi^2\delta_i^\alpha\right)\right]\Bigg|_{\bar{o}}\right.$$

$$+\left.\delta x_o^\mu(\partial_\mu\delta x^\alpha)\big|_{\bar{o}} + \left[\Delta x_{\bar{r}}^\mu\,\partial_\mu\left(\Delta x^\alpha + n^\alpha\Delta\eta\right)\big|_{\bar{x}_{\bar{r}}}\right]_{\bar{o}}^z + \int_0^{\bar{t}_o} dt\,\frac{1}{a}\left(\mathcal{U}^\alpha + \psi\mathcal{U}^\alpha - \mathcal{U}^\beta\delta x^\alpha{}_{,\beta}\right)\bigg|_{\bar{x}_t} +\right.$$

$$+\int_0^{\bar{r}_z} d\bar{r}\left[-\delta n^\beta\left(\Delta x^\alpha{}_{,\beta} + n^\alpha\Delta\eta_{,\beta}\right) + \delta\nu\left(\Delta x'^\alpha + n^\alpha\Delta\eta'\right) - \left(\bar{r}_z - \bar{r}\right)\left(-\delta\nu\delta n'^\alpha +\right.\right.$$

$$+\,\delta n^\beta\delta n^\alpha{}_{,\beta} + \psi^{;\alpha} - B'^\alpha - B_\parallel{}^{,\alpha} + B^\alpha{}_{,\parallel} - 2\phi'n^\alpha - 2C'^\alpha_\parallel + 2\left(\phi n^\alpha\right)_{,\parallel} + 2C^\alpha_{\parallel,\parallel} -$$

$$-\,\phi^{;\alpha} - C_{\parallel\parallel}{}^{,\alpha} + 2\delta\nu\left(\psi^{;\alpha} - \phi'n^\alpha\right) - 2\delta n^\gamma\left(\phi'\delta_\gamma^\alpha - \left(\phi n^\alpha\right)_{,\gamma} - \delta_\gamma^\alpha\phi_{,\parallel} + \left(\phi n_\gamma\right)^{;\alpha}\right) -$$

$$-\left.\left.2\phi\left(2\left(\phi n^\alpha\right)_{,\parallel} - \phi^{;\alpha} + \psi^{;\alpha} - 2\phi'n^\alpha\right)\right)\right]\Bigg|_{\bar{x}_{\bar{r}}}\Bigg\} - \frac{\delta r}{\bar{r}_z}\delta\theta + \frac{1}{2}\cos\theta\sin\theta\,\delta\phi^2\right) +$$

$$+\frac{\partial}{\partial\phi}\left(\frac{1}{\bar{r}_z\sin\theta}\phi_\alpha\Bigg\{-\bar{r}_z\left[\mathcal{U}^\alpha + n^i\left(A^\alpha{}_i + \phi\delta_i^\alpha + C_i^\alpha - \frac{1}{2}\mathcal{U}_i\mathcal{U}^\alpha - \frac{1}{2}\Omega_i\Omega^\alpha + \frac{1}{2}\delta_i^\alpha\Omega^k\Omega_k - \frac{3}{2}\phi^2\delta_i^\alpha\right)\right]\Bigg|_{\bar{o}}\right.$$

$$+\,\delta x_o^\mu(\partial_\mu\delta x^\alpha)\big|_{\bar{o}} + \left[\Delta x_{\bar{r}}^\mu\,\partial_\mu\left(\Delta x^\alpha + n^\alpha\Delta\eta\right)\big|_{\bar{x}_{\bar{r}}}\right]_{\bar{o}}^z$$

$$+\int_0^{\bar{t}_o} dt\,\frac{1}{a}\left(\mathcal{U}^\alpha + \psi\mathcal{U}^\alpha - \mathcal{U}^\beta\delta x^\alpha{}_{,\beta}\right)\bigg|_{\bar{x}_t} + \int_0^{\bar{r}_z} d\bar{r}\left[-\delta n^\beta\left(\Delta x^\alpha{}_{,\beta} + n^\alpha\Delta\eta_{,\beta}\right)\right.$$

$$+\,\delta\nu\left(\Delta x'^\alpha + n^\alpha\Delta\eta'\right) - \left(\bar{r}_z - \bar{r}\right)\left(-\delta\nu\delta n'^\alpha + \delta n^\beta\delta n^\alpha{}_{,\beta} + \psi^{;\alpha} - B'^\alpha - B_\parallel{}^{,\alpha} + B^\alpha{}_{,\parallel} -\right.$$

$$-\,2\phi'n^\alpha - 2C'^\alpha_\parallel + 2\left(\phi n^\alpha\right)_{,\parallel} + 2C^\alpha_{\parallel,\parallel} - \phi^{;\alpha} - C_{\parallel\parallel}{}^{,\alpha} + 2\delta\nu\left(\psi^{;\alpha} - \phi'n^\alpha\right) -$$

$$-2\delta n^\gamma\left(\phi'\delta_\gamma^\alpha - \left(\phi n^\alpha\right)_{,\gamma} - \delta_\gamma^\alpha\phi_{,\parallel} + \left(\phi n_\gamma\right)^{;\alpha}\right) -2\phi\left(2\left(\phi n^\alpha\right)_{,\parallel} - \phi^{;\alpha} + \psi^{;\alpha} - 2\phi'n^\alpha\right)\right)\Bigg]\Bigg|_{\bar{x}_{\bar{r}}}\Bigg\} -$$

$$-\frac{\delta r}{\bar{r}_z}\delta\phi - \bar{r}_z\cot\theta\,\delta\theta\delta\phi\right) +$$

$$+\frac{\partial}{\partial\theta}\delta\theta\frac{\partial}{\partial\phi}\delta\phi - \frac{\partial}{\partial\phi}\delta\theta\frac{\partial}{\partial\theta}\delta\phi + \cot\theta\,\delta\theta\left(\frac{\partial}{\partial\theta}\delta\theta + \frac{\partial}{\partial\phi}\delta\phi\right) - \frac{1}{2}\delta\theta^2\;. \quad (3.25)$$

Let us rewrite it using:

$$\left(\cot\theta + \frac{\partial}{\partial\theta}\right)\left[\theta_\alpha\frac{1}{\bar{r}_z}F^\alpha + A\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left[\phi_\alpha\frac{1}{\bar{r}_z}F^\alpha + B\right] =$$

$$= \left[\left(\cot\theta + \frac{\partial}{\partial\theta}\right)\theta_\alpha + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\phi_\alpha\right]\cdot\frac{1}{\bar{r}_z}F^\alpha + \left[\theta_\alpha\frac{\partial}{\partial\theta} + \frac{\phi_\alpha}{\sin\theta}\frac{\partial}{\partial\phi}\right]\frac{1}{\bar{r}_z}F^\alpha +$$

$$+ \left(\cot\theta + \frac{\partial}{\partial\theta}\right)A + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}B =$$

$$= -\frac{2}{\bar{r}_z}n_\alpha F^\alpha + \left[\theta_\alpha\frac{\partial}{\partial\theta} + \frac{\phi_\alpha}{\sin\theta}\frac{\partial}{\partial\phi}\right]\frac{1}{\bar{r}_z}F^\alpha + \left(\cot\theta + \frac{\partial}{\partial\theta}\right)A + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}B, \quad (3.26)$$

where the identity with the unit vectors has been used:

$$\left(\cot\theta + \frac{\partial}{\partial\theta}\right)\theta^\alpha + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\phi^\alpha = -2n^\alpha. \tag{3.27}$$

So now, the expression for the distortion matrix:

$$\delta\mathbb{D} = \left[2\mathcal{U}_\| + 2\phi + 2C_{\|\|} - \mathcal{U}_\|^2 - \Omega_\|^2 + \Omega^k\Omega_k - 3\phi^2\right]\bigg|_{\bar{\mathrm{o}}} - \frac{2n^\alpha}{\bar{r}_z}\bigg\{\delta x_o^\mu(\partial_\mu \delta x^\alpha)\big|_{\bar{\mathrm{o}}} +$$

$$+ \left[\Delta x_{\bar{r}}^\mu\,\partial_\mu\left(\Delta x^\alpha + n^\alpha\Delta\eta\right)\big|_{\bar{x}_{\bar{r}}}\right]_{\bar{\mathrm{o}}}^z + \int_0^{\bar{t}_{\mathrm{o}}} dt\,\frac{1}{a}\left(\mathcal{U}^\alpha + \psi\mathcal{U}^\alpha - \mathcal{U}^\beta\delta x^\alpha_{,\beta}\right)\bigg|_{\bar{x}_t} +$$

$$+ \int_0^{\bar{r}_z} d\bar{r}\left[-\delta n^\beta\left(\Delta x^\alpha_{,\beta} + n^\alpha\Delta\eta_{,\beta}\right) + \delta\nu\left(\Delta x'^\alpha + n^\alpha\Delta\eta'\right) - (\bar{r}_z - \bar{r})\left(-\delta\nu\delta n'^\alpha +\right.\right.$$

$$+ \delta n^\beta\delta n^\alpha_{,\beta} + \psi'^\alpha - B'^\alpha - B_\|^{,\alpha} + B^\alpha_{,\|} - 2\phi'n^\alpha - 2C''^\alpha_\| + 2\left(\phi n^\alpha\right)_{,\|} + 2C^\alpha_{\|,\|} -$$

$$- \phi^{,\alpha} - C_{\|\|}{}^{,\alpha} + 2\delta\nu\left(\psi^{,\alpha} - \phi'n^\alpha\right) - 2\delta n^\gamma\left(\phi'\delta^\alpha_\gamma - \left(\phi n^\alpha\right)_{,\gamma} - \delta^\alpha_\gamma\phi_{,\|} + \left(\phi n_\gamma\right)^{,\alpha}\right) -$$

$$\left.\left.-2\phi\left(2\left(\phi n^\alpha\right)_{,\|} - \phi^{,\alpha} + \psi^{,\alpha} - 2\phi'n^\alpha\right)\right)\right]\bigg|_{\bar{x}_{\bar{r}}}\bigg\} + \frac{1}{\bar{r}_z}\left(\theta_\alpha\frac{\partial}{\partial\theta} + \frac{\phi_\alpha}{\sin\theta}\frac{\partial}{\partial\phi}\right)\bigg\{\delta x_o^\mu(\partial_\mu\delta x^\alpha)\big|_{\bar{\mathrm{o}}} +$$

$$+ \left[\Delta x_{\bar{r}}^\mu\,\partial_\mu\left(\Delta x^\alpha + n^\alpha\Delta\eta\right)\big|_{\bar{x}_{\bar{r}}}\right]_{\bar{\mathrm{o}}}^z + \int_0^{\bar{t}_{\mathrm{o}}} dt\,\frac{1}{a}\left(\mathcal{U}^\alpha + \psi\mathcal{U}^\alpha - \mathcal{U}^\beta\delta x^\alpha_{,\beta}\right)\bigg|_{\bar{x}_t} +$$

$$+ \int_0^{\bar{r}_z} d\bar{r}\left[-\delta n^\beta\left(\Delta x^\alpha_{,\beta} + n^\alpha\Delta\eta_{,\beta}\right) + \delta\nu\left(\Delta x'^\alpha + n^\alpha\Delta\eta'\right) - (\bar{r}_z - \bar{r})\left(-\delta\nu\delta n'^\alpha +\right.\right.$$

$$+ \delta n^\beta\delta n^\alpha_{,\beta} + \psi'^\alpha - B'^\alpha - B_\|^{,\alpha} + B^\alpha_{,\|} - 2\phi'n^\alpha - 2C''^\alpha_\| + 2\left(\phi n^\alpha\right)_{,\|} + 2C^\alpha_{\|,\|} -$$

$$- \phi^{,\alpha} - C_{\|\|}{}^{,\alpha} + 2\delta\nu\left(\psi^{,\alpha} - \phi'n^\alpha\right) - 2\delta n^\gamma\left(\phi'\delta^\alpha_\gamma - \left(\phi n^\alpha\right)_{,\gamma} - \delta^\alpha_\gamma\phi_{,\|} + \left(\phi n_\gamma\right)^{,\alpha}\right) -$$

$$\left.\left.-2\phi\left(2\left(\phi n^\alpha\right)_{,\|} - \phi^{,\alpha} + \psi^{,\alpha} - 2\phi'n^\alpha\right)\right)\right]\bigg|_{\bar{x}_{\bar{r}}}\bigg\} + \left(\cot\theta + \frac{\partial}{\partial\theta}\right)\left(-\frac{\delta r}{\bar{r}_z}\delta\theta + \frac{1}{2}\cos\theta\sin\theta\,\delta\phi^2\right) +$$

$$+ \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(-\frac{\delta r}{\bar{r}_z}\delta\phi - \bar{r}_z\cot\theta\,\delta\theta\delta\phi\right) + \frac{\partial}{\partial\theta}\delta\theta\frac{\partial}{\partial\phi}\delta\phi - \frac{\partial}{\partial\phi}\delta\theta\frac{\partial}{\partial\theta}\delta\phi +$$

$$+ \cot\theta\,\delta\theta\left(\frac{\partial}{\partial\theta}\delta\theta + \frac{\partial}{\partial\phi}\delta\phi\right) - \frac{1}{2}\delta\theta^2\,. \tag{3.28}$$

Chapter 4

Distortion in the Luminosity Distance

According to equation (1.123), the fluctuations in the physical area occupied by the source and the angular diameter distance are identical. Combined with relation (1.26), this implies that the perturbation in the angular diameter distance is equal to the perturbation in the luminosity distance.

4.1 Distortion in the Physical Area

Now, we are at the point where we are ready to substitute everything in expression for the physical area. In general form, it can be written as:

$$
\delta A = 2\delta z + \delta g - \widehat{\Delta\nu} + 2\frac{\delta r}{\bar{r}_z} + \delta\mathbb{D} + \delta n_\parallel + \delta u + \mathcal{U}_\parallel - \mathcal{A}\left(-2\kappa + 2\frac{\delta r}{\bar{r}_z} - \mathcal{C}_{\parallel\parallel\parallel}\right) + \mathcal{U}_\parallel\left(\mathcal{B}_\parallel - \mathcal{U}_\parallel\right) -
$$
$$
-\frac{1}{\bar{r}_z}\left(\mathcal{U}_\theta\frac{\partial}{\partial\theta} + \frac{\mathcal{U}_\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\Delta\eta - 2\kappa\left(2\frac{\delta r}{\bar{r}_z} - \mathcal{C}_{\parallel\parallel\parallel}\right) - 2\frac{\delta r}{\bar{r}_z}\mathcal{C}_{\parallel\parallel\parallel} + \frac{\delta r^2}{\bar{r}_z^2} - \frac{1}{2}\delta\theta^2 - \frac{1}{2}(\sin\theta\,\delta\phi)^2
$$
$$
+ \widehat{\Delta\nu}\left(\mathcal{U}_\parallel + \mathcal{C}_{\parallel\parallel\parallel}\right) + \delta\theta\left(\mathcal{U}_\theta + \delta n_\theta\right) - \frac{1}{\bar{r}_z}\left[\left(\mathcal{U}_\theta + \delta n_\theta\right)\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\left(\mathcal{U}_\phi + \delta n_\phi\right)\frac{\partial}{\partial\phi}\right]\delta r
$$
$$
+ \frac{1}{\bar{r}_z}\delta\theta\frac{\partial}{\partial\theta}\delta r + \sin\theta\,\delta\phi\left(\mathcal{U}_\phi + \delta n_\phi\right) + \frac{1}{\bar{r}_z}\delta\phi\frac{\partial}{\partial\phi}\delta r + \delta z^2 + 2\,\delta z\delta g
$$
$$
+(2\,\delta z + \delta g)\left(-\mathcal{A} + 2\frac{\delta r}{\bar{r}_z} - 2\kappa - \mathcal{C}_{\parallel\parallel\parallel}\right) + \Delta x^\mu\partial_\mu\left(2\,\delta z + \delta g - \mathcal{A} + \mathcal{U}_\parallel + \delta n_\parallel - \widehat{\Delta\nu}\right).
$$

$$
(4.1)
$$

So, the result is

$$\delta A = 2\Bigg[\mathcal{H}_{\bar{o}}\left(\delta x_o^\mu (\partial_\mu \delta t)\big|_{\bar{o}} - \frac{1}{2}\mathcal{H}_{\bar{o}}\delta t_o^2 + \int_0^{\bar{t}_o} dt \left(-\psi + \frac{1}{2}\psi^2 + \frac{1}{2}\mathcal{U}^\alpha \mathcal{U}_\alpha - \frac{1}{a}\mathcal{U}^\alpha \delta t_{,\alpha} \right)\bigg|_{\bar{x}_t} \right) +$$

$$+ \frac{1}{2}\left(\mathcal{H}_{\bar{o}}^2 + \mathcal{H}_{\bar{o}}' \right)\delta\eta_o^2 + \mathcal{H}_{\bar{o}}\delta\eta_o\widehat{\Delta\nu}(\bar{x}_z; x_o) + \Delta x_s^\mu(\partial_\mu\widehat{\Delta\nu})\big|_{\bar{x}_z} \Bigg] + \psi + 3\phi + C_\alpha^\alpha + \frac{1}{2}\psi^2 +$$

$$+ 3\psi\phi + \frac{3}{2}\phi^2 - \left(-\psi + \mathcal{U}_\| + B_\| + \frac{3}{2}\psi^2 + \mathcal{U}_\|(\psi - \phi) + \frac{1}{2}\mathcal{U}_\alpha\mathcal{U}^\alpha + \varepsilon_{\alpha ij}\mathcal{U}^\alpha n^i \Omega^j \right)_{\bar{o}} +$$

$$+ \int_0^{\bar{r}_z} d\bar{r}\left(\delta\nu\delta\nu' - \delta n^\alpha \delta\nu_{,\alpha} + \psi' - 2\psi_{,\|} + B_{\|,\|} + \phi' + C_{\|\|}' + 2\delta\nu\left(\psi' - \psi_{,\|} \right) - \right.$$

$$\left. - 2\psi\left(\psi' - 2\psi_{,\|} + \phi' \right) - 2\delta n^\alpha \left(\psi_{,\alpha} - \phi' n_\alpha \right) \right)\bigg|_{\bar{x}_{\bar{r}}} + 2\mathcal{U}_\| - B_\| + \delta\nu\psi +$$

$$+ \delta n_\alpha \mathcal{U}^\alpha + \mathcal{U}^\alpha \mathcal{U}_\alpha + 2\phi\mathcal{U}_\| + \frac{2}{\bar{r}_z}\Bigg[-\frac{1}{\mathcal{H}_z}\Bigg[\left(-\psi - \mathcal{U}_\| + B_\| + \frac{3}{2}\psi^2 + \mathcal{U}_\|(\psi - \phi) + \right.$$

$$\left. + \frac{1}{2}\mathcal{U}_\alpha\mathcal{U}^\alpha + \varepsilon_{\alpha ij}\mathcal{U}^\alpha n^i \Omega^j \right)_{\bar{o}} + \int_0^{\bar{r}_z} d\bar{r}\Big[\delta\nu\delta\nu' - \delta n^\alpha \delta\nu_{,\alpha} + \psi' - 2\psi_{,\|} + B_{\|,\|} + \phi' +$$

$$+ C_{\|\|}' + 2\delta\nu\left(\psi' - \psi_{,\|} \right) - 2\psi\left(\psi' - 2\psi_{,\|} + \phi' \right) - 2\delta n^\alpha \left(\psi_{,\alpha} - \phi' n_\alpha \right) \Big]\bigg|_{\bar{x}_{\bar{r}}} +$$

$$+ \psi + \mathcal{U}_\| - B_\| + \delta\nu\psi - \frac{1}{2}\psi^2 + \delta n_\alpha \mathcal{U}^\alpha + \frac{1}{2}\mathcal{U}^\alpha \mathcal{U}_\alpha + 2\phi\mathcal{U}_\| + \frac{1}{2}\left(\mathcal{H}_{\bar{o}}^2 + \mathcal{H}_{\bar{o}}' \right)\delta\eta_o^2 +$$

$$+ \mathcal{H}_{\bar{o}}\delta\eta_o\widehat{\Delta\nu} + \Delta x_s^\mu(\partial_\mu\widehat{\Delta\nu})\big|_{\bar{x}_z} \Bigg] + \frac{1}{2}\frac{\mathcal{H}_z^2 + \mathcal{H}_z'}{\mathcal{H}_z}\left(\frac{\delta z}{\mathcal{H}_z} \right)^2 -$$

$$- \left(\frac{\mathcal{H}_{\bar{o}}}{\mathcal{H}_z} - 1 \right)\Bigg[\delta x_o^\mu (\partial_\mu \delta t)\big|_{\bar{o}} + \int_0^{\bar{t}_o} dt \left(-\psi + \frac{1}{2}\psi^2 + \frac{1}{2}\mathcal{U}^\alpha \mathcal{U}_\alpha - \frac{1}{a}\mathcal{U}^\alpha \delta t_{,\alpha} \right)\bigg|_{\bar{x}_t} \Bigg] +$$

$$+ \frac{1}{2}\mathcal{H}_{\bar{o}}\delta t_o^2 - \int_0^{\bar{r}_z} d\bar{r}\Bigg[(\bar{r}_z - \bar{r})\left(\delta\nu\delta\nu' - \delta n^\alpha \delta\nu_{,\alpha} + \psi' - 2\psi_{,\|} + B_{\|,\|} + \phi' + C_{\|\|}' + 2\delta\nu\left(\psi' - \psi_{,\|} \right) - \right.$$

$$\left. - 2\psi\left(\psi' - 2\psi_{,\|} + \phi' \right) - 2\delta n^\alpha \left(\psi_{,\alpha} - \phi' n_\alpha \right) \right)\Bigg]\bigg|_{\bar{x}_{\bar{r}}} + \frac{\bar{r}_z}{2}\left(\delta\theta^2 + \sin^2\theta\, \delta\phi^2 \right)\Bigg] +$$

$$
+ \frac{1}{\bar{r}_z} \left(\theta_\alpha \frac{\partial}{\partial \theta} + \frac{\phi_\alpha}{\sin \theta} \frac{\partial}{\partial \phi} \right) \left\{ \delta x_o^\mu (\partial_\mu \delta x^\alpha) \big|_{\bar{o}} + \left[\Delta x_{\bar{r}}^\mu \, \partial_\mu \left(\Delta x^\alpha + n^\alpha \Delta \eta \right) \big|_{\bar{x}_{\bar{r}}} \right]_{\bar{o}}^z + \right.
$$

$$
+ \int_0^{\bar{t}_o} dt \, \frac{1}{a} \left(\mathcal{U}^\alpha + \psi \mathcal{U}^\alpha - \mathcal{U}^\beta \delta x^\alpha_{,\beta} \right) \Big|_{\bar{x}_t} + \int_0^{\bar{r}_z} d\bar{r} \left[- \delta n^\beta \left(\Delta x^\alpha_{,\beta} + n^\alpha \Delta \eta_{,\beta} \right) + \right.
$$

$$
+ \delta\nu \left(\Delta x'^\alpha + n^\alpha \Delta \eta' \right) - (\bar{r}_z - \bar{r}) \left(- \delta\nu \delta n'^\alpha + \delta n^\beta \delta n^\alpha_{,\beta} + \psi^{,\alpha} - B'^\alpha - B_\|{}^{,\alpha} + \right.
$$

$$
+ B^\alpha_{,\|} - 2\phi' n^\alpha - 2C''^\alpha_\| + 2 \left(\phi n^\alpha \right)_{,\|} + 2C^\alpha_{\|,\|} - \phi^{,\alpha} - C_{\|\|\|}{}^{,\alpha} + 2\delta\nu \left(\psi^{,\alpha} - \phi' n^\alpha \right) -
$$

$$
- 2\delta n^\gamma \left(\phi' \delta^\alpha_\gamma - (\phi n^\alpha)_{,\gamma} - \delta^\alpha_\gamma \phi_{,\|} + (\phi n_\gamma)^{,\alpha} \right) - 2\phi \left(2 \left(\phi n^\alpha \right)_{,\|} - \phi^{,\alpha} + \psi^{,\alpha} - 2\phi' n^\alpha \right) \right) \Big] \Big|_{\bar{x}_{\bar{r}}} \right\} +
$$

$$
+ \left(\cot \theta + \frac{\partial}{\partial \theta} \right) \left(- \frac{\delta r}{\bar{r}_z} \delta \theta + \frac{1}{2} \cos \theta \sin \theta \, \delta \phi^2 \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \left(- \frac{\delta r}{\bar{r}_z} \delta \phi - \bar{r}_z \cot \theta \, \delta\theta \delta\phi \right) +
$$

$$
+ \frac{\partial}{\partial \theta} \delta\theta \frac{\partial}{\partial \phi} \delta\phi - \frac{\partial}{\partial \phi} \delta\theta \frac{\partial}{\partial \theta} \delta\phi + \cot \theta \, \delta\theta \left(\frac{\partial}{\partial \theta} \delta\theta + \frac{\partial}{\partial \phi} \delta\phi \right) - \frac{1}{2} \delta\theta^2 -
$$

$$
- n_\alpha \int_0^{\bar{r}_z} d\bar{r} \left[\psi^{,\alpha} - B'^\alpha - B_\|{}^{,\alpha} + B^\alpha_{,\|} - 2\phi' n^\alpha - 2C''^\alpha_\| + 2 \left(\phi n^\alpha \right)_{,\|} + 2C^\alpha_{\|,\|} - \phi^{,\alpha} - C_{\|\|\|}{}^{,\alpha} + 2\delta\nu \left(\psi^{,\alpha} - \phi' n^\alpha \right) - \right.
$$

$$
- 2\delta n^\gamma \left(\phi' \delta^\alpha_\gamma - (\phi n^\alpha)_{,\gamma} - \delta^\alpha_\gamma \phi_{,\|} + (\phi n_\gamma)^{,\alpha} \right) - 2\phi \left(2 \left(\phi n^\alpha \right)_{,\|} - \phi^{,\alpha} + \psi^{,\alpha} - 2\phi' n^\alpha \right) -
$$

$$
- \delta\nu \delta n'^\alpha + \delta n^\beta \delta n^\alpha_{,\beta} \Big] \Big|_{\bar{x}_{\bar{r}}} - \left(- \mathcal{U}_\| + \phi + C_{\|\|\|} - \frac{1}{2} \mathcal{U}_\|^2 - \frac{1}{2} \Omega_\|^2 + \frac{1}{2} \Omega^k \Omega_k - \frac{3}{2} \phi^2 \right)_{\bar{o}} - \psi \left(- 2\kappa + 2 \frac{\delta r}{\bar{r}_z} - \phi \right) - \mathcal{U}_\|^2 -
$$

$$
- \frac{1}{\bar{r}_z} \left(\mathcal{U}_\theta \frac{\partial}{\partial \theta} + \frac{\mathcal{U}_\phi}{\sin \theta} \frac{\partial}{\partial \phi} \right) \Delta\eta - 2\kappa \left(2 \frac{\delta r}{\bar{r}_z} - \phi \right) - 2 \frac{\delta r}{\bar{r}_z} \phi + \frac{\delta r^2}{\bar{r}_z^2} - \frac{1}{2} \delta\theta^2 - \frac{1}{2} (\sin \theta \, \delta\phi)^2 +
$$

$$
+ \widehat{\Delta\nu} \left(\mathcal{U}_\| + \phi \right) + \delta\theta \left(\mathcal{U}_\theta + \delta n_\theta \right) - \frac{1}{\bar{r}_z} \left[\left(\mathcal{U}_\theta + \delta n_\theta \right) \frac{\partial}{\partial \theta} + \frac{1}{\sin \theta} \left(\mathcal{U}_\phi + \delta n_\phi \right) \frac{\partial}{\partial \phi} \right] \delta r +
$$

$$
+ \frac{1}{\bar{r}_z} \delta\theta \frac{\partial}{\partial \theta} \delta r + \sin \theta \, \delta\phi \left(\mathcal{U}_\phi + \delta n_\phi \right) + \frac{1}{\bar{r}_z} \delta\phi \frac{\partial}{\partial \phi} \delta r + \delta z^2 + 2 \, \delta z \delta g +
$$

$$
+ (2 \, \delta z + \delta g) \left(-\psi + 2 \frac{\delta r}{\bar{r}_z} - 2\kappa - \phi \right) + \Delta x^\mu \partial_\mu \left(2 \, \delta z + \delta g - \psi + \mathcal{U}_\| + \delta n_\| - \widehat{\Delta\nu} \right).
$$

$$
\tag{4.2}
$$

4.1.1 First-Order Quantities and Other Simplifications

We are ready to introduce some additional simplifications.

First, let us set $\Omega_i = 0$, as it is an arbitrarily chosen quantity. Secondly, for the first-order quantities, we can use the Einstein equations to the first order, resulting in a well-known expression. Therefore, for linear expressions, we will use:

$$
\phi = -\psi, \qquad \mathcal{B}_\alpha = 0, \qquad \mathcal{C}_{\alpha\beta} = \delta_{\alpha\beta} \phi = -\delta_{\alpha\beta} \psi. \tag{4.3}
$$

Additionally, there is another "simplification". We initially wrote $\mathcal{U}^\alpha$, but if it happens to be a first-order quantity, it is scalar only, so $\mathcal{U}^\alpha = -\partial^\alpha U$. In the Newtonian gauge, U is equivalent to v (the velocity potential) because in this gauge,

$\beta = 0$. In the previous chapter, we have shown that to the linear order:

$$\delta g = -2\psi, \tag{4.4}$$

$$\delta t_o = -\int_0^{\bar{t}_o} dt\ \psi, \qquad \delta x_o^\alpha = \int_0^{\bar{t}_o} dt\ \frac{1}{a}\mathcal{U}^\alpha\ , \tag{4.5}$$

$$\delta \eta_o = \delta t_o = -\int_0^{\bar{t}_o} dt\ \psi, \tag{4.6}$$

$$\delta \nu_o = -\left(\psi + \mathcal{U}_\|\right)_{\bar{o}} \qquad (\delta n_\alpha)_o = -\left(\mathcal{U}_\alpha - \psi n_\alpha\right)_{\bar{o}}, \tag{4.7}$$

$$\delta\nu = -2\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\psi'\right) + \left(\psi - \mathcal{U}_\|\right)_{\bar{o}},$$

$$\delta n_\alpha = 2\left(n_\alpha\psi - \int_0^{\bar{r}_z} d\bar{r}\psi_{,\alpha}\right) - \left(\mathcal{U}_\alpha + \psi n_\alpha\right)_{\bar{o}}, \tag{4.8}$$

$$\widehat{\Delta\nu} = \left(\mathcal{U}_\| - \psi\right)\big|_o^z - 2\int_0^{\bar{r}_z} d\bar{r}\psi', \tag{4.9}$$

$$\delta z = \left(\mathcal{U}_\| - \psi\right)\big|_o^z - 2\int_0^{\bar{r}_z} d\bar{r}\psi' - \mathcal{H}_{\bar{o}}\int_0^{\bar{t}_o}\psi\,dt, \tag{4.10}$$

$$\Delta\eta_s = \frac{1}{\mathcal{H}_z}\left(\left(\mathcal{U}_\| - \psi\right)\big|_o^z - 2\int_0^{\bar{r}_z} d\bar{r}\psi' - \mathcal{H}_{\bar{o}}\int_0^{\bar{t}_o}\psi\,dt\right), \tag{4.11}$$

$$\Delta x_s^\alpha = \int_0^{\bar{t}_o}\frac{1}{a}\mathcal{U}^\alpha\,dt - n^\alpha\frac{1}{\mathcal{H}_z}\left(\left(\mathcal{U}_\| - \psi\right)\big|_o^z - 2\int_0^{\bar{r}_z} d\bar{r}\psi' - \left(\mathcal{H}_{\bar{o}} - \mathcal{H}_z\right)\int_0^{\bar{t}_o}\psi\,dt\right) +$$

$$+ \int_0^{\bar{r}_z} d\bar{r}\left(2n^\alpha\psi - 2\left[\left(\theta^\alpha\frac{\bar{r}_z - \bar{r}}{\bar{r}}\frac{\partial\psi}{\partial\theta} + \phi^\alpha\frac{\bar{r}_z - \bar{r}}{\bar{r}\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]\right) + \bar{r}_z\left(\mathcal{U}_\| n^\alpha - \mathcal{U}^\alpha\right)_{\bar{o}}, \tag{4.12}$$

$$\delta r = \int_0^{\bar{t}_o}\frac{1}{a}\mathcal{U}_\|\,dt - \frac{1}{\mathcal{H}_z}\left(\left(\mathcal{U}_\| - \psi\right)\big|_o^z - 2\int_0^{\bar{r}_z} d\bar{r}\psi' - \left(\mathcal{H}_{\bar{o}} - \mathcal{H}_z\right)\int_0^{\bar{t}_o}\psi\,dt\right) + 2\int_0^{\bar{r}_z} d\bar{r}\psi, \tag{4.13}$$

$$\delta\theta = \frac{1}{\bar{r}_z}\left(\int_0^{\bar{t}_o}\frac{1}{a}\theta_\alpha\mathcal{U}^\alpha\,dt - \bar{r}_z\left(\theta_\alpha\mathcal{U}^\alpha\right)_{\bar{o}} - 2\int_0^{\bar{r}_z} d\bar{r}\ \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}}\right)\frac{\partial\psi}{\partial\theta}\right), \tag{4.14}$$

$$\delta\phi = \frac{1}{\bar{r}_z \sin\theta}\left(\int_0^{\bar{t}_o}\frac{1}{a}\phi_\alpha \mathcal{U}^\alpha\, dt - \bar{r}_z\left(\phi_\alpha \mathcal{U}^\alpha\right)_{\bar{o}} - 2\int_0^{\bar{r}_z} d\bar{r}\ \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\psi}{\partial\phi}\right), \quad (4.15)$$

and again rewriting the result for δA:

$$\delta A = 2\left[\mathcal{H}_{\bar{o}}\left(\delta x_o^\mu\left(\partial_\mu\left(\int_0^t dt\,\psi\right)\right)\right)\Big|_{\bar{o}} - \frac{1}{2}\mathcal{H}_{\bar{o}}\left(\int_0^{t_{\bar{o}}} dt\,\psi\right)^2 + \int_0^{\bar{t}_o} dt\left(-\psi + \frac{1}{2}\psi^2 + \frac{1}{2}\mathcal{U}^\alpha\mathcal{U}_\alpha\right.\right.$$

$$\left.- \frac{1}{a}\mathcal{U}^\alpha\left(\int_0^t dt\,\psi\right)_{,\alpha}\right)\Big|_{\bar{x}_t}\right) + \frac{1}{2}\left(\mathcal{H}_{\bar{o}}^2 + \mathcal{H}_{\bar{o}}'\right)\left(\int_0^{\bar{t}_o} dt\,\psi\right)^2 - \mathcal{H}_{\bar{o}}\left(\int_0^{\bar{t}_o} dt\,\psi\right)\left(\left(\mathcal{U}_\| - \psi\right)\big|_o^z - 2\int_0^{\bar{r}_z} d\bar{r}\,\psi'\right) +$$

$$+ \Delta x_s^\mu\left(\partial_\mu\left(\left(\mathcal{U}_\| - \psi\right)\big|_o^z - 2\int_0^{\bar{r}_z} d\bar{r}\,\psi'\right)\right)\Big|_{\bar{x}_z}\right] + \psi + 3\phi + C_\alpha^\alpha + 4\psi^2 + \left(\psi - \phi - C_{\|\|}-\right.$$

$$\left.- B_\| - 2\mathcal{U}_\|\psi - \frac{1}{2}\mathcal{U}_\alpha\mathcal{U}^\alpha + \frac{1}{2}\mathcal{U}_\|^2\right)_{\bar{o}} + \int_0^{\bar{r}_z} d\bar{r}\left(2\left\{2\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\,\psi'\right) - \left(\psi - \mathcal{U}_\|\right)_{\bar{o}}\right\}\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\,\psi'\right)' +\right.$$

$$\left.+ 2\left\{2\left(n^\alpha\psi - \int_0^{\bar{r}_z} d\bar{r}\,\psi'^{,\alpha}\right) - \left(\mathcal{U}^\alpha + \psi n^\alpha\right)_{\bar{o}}\right\}\left\{\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\,\psi'\right\}_{,\alpha} + \psi' + \phi' - 2\psi_{,\|} + B_{\|,\|} + C'_{\|\|\|} +\right.$$

$$+ 2\left(-2\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\,\psi'\right) + \left(\psi - \mathcal{U}_\|\right)_{\bar{o}}\right)\left(\psi' - \psi_{,\|}\right) + 4\psi\,\psi_{,\|} -$$

$$- 2\left(2\left(n^\alpha\psi - \int_0^{\bar{r}_z} d\bar{r}\,\psi'^{,\alpha}\right) - \left(\mathcal{U}^\alpha + \psi n^\alpha\right)_{\bar{o}}\right)\left(\psi_{,\alpha} + \psi' n_\alpha\right)\right)\Big|_{\bar{x}_{\bar{r}}} + 2\mathcal{U}_\| - B_\| +$$

$$+ \left(-2\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\,\psi'\right) + \left(\psi - \mathcal{U}_\|\right)_{\bar{o}}\right)\psi + \left(2\left(n_\alpha\psi - \int_0^{\bar{r}_z} d\bar{r}\,\psi_{,\alpha}\right) - \left(\mathcal{U}_\alpha + \psi n_\alpha\right)_{\bar{o}}\right)\mathcal{U}^\alpha +$$

$$+ \mathcal{U}^\alpha\mathcal{U}_\alpha - 3\mathcal{U}_\|^2 - 2\psi\mathcal{U}_\| + \frac{2}{\bar{r}_z}\left[-\frac{1}{\mathcal{H}_z}\left[\left(-\psi - \mathcal{U}_\| + B_\| + \frac{3}{2}\psi^2 + 2\mathcal{U}_\|\psi +\right.\right.\right.$$

$$\left.+ \frac{1}{2}\mathcal{U}_\alpha\mathcal{U}^\alpha\right)_{\bar{o}} + \int_0^{\bar{r}_z} d\bar{r}\left[2\left\{2\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\,\psi'\right) - \left(\psi - \mathcal{U}_\|\right)_{\bar{o}}\right\}\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\,\psi'\right)' +\right.$$

$$+ 2\left\{2\left(n^\alpha\psi - \int_0^{\bar{r}_z} d\bar{r}\,\psi'^{,\alpha}\right) - \left(\mathcal{U}^\alpha + \psi n^\alpha\right)_{\bar{o}}\right\}\left\{\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\,\psi'\right\}_{,\alpha} + \psi' + \phi' - 2\psi_{,\|} + B_{\|,\|} +$$

$$+ C'_{\|\|\|} + 2\left(-2\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\,\psi'\right) + \left(\psi - \mathcal{U}_\|\right)_{\bar{o}}\right)\left(\psi' - \psi_{,\|}\right) + 4\psi\,\psi_{,\|}$$

$$\left.-2\left(2\left(n^\alpha\psi - \int_0^{\bar{r}_z} d\bar{r}\,\psi'^{,\alpha}\right) - \left(\mathcal{U}^\alpha + \psi n^\alpha\right)_{\bar{o}}\right)\left(\psi_{,\alpha} + \psi' n_\alpha\right)\right]\right]\Big|_{\bar{x}_{\bar{r}}} + \psi + \mathcal{U}_\| - B_\| - \frac{1}{2}\psi^2 + \frac{1}{2}\mathcal{U}^\alpha\mathcal{U}_\alpha - 2\psi\mathcal{U}_\| +$$

$$+ \left(-2\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\,\psi'\right) + \left(\psi - \mathcal{U}_\|\right)_{\bar{o}}\right)\psi + \left(2\left(n_\alpha\psi - \int_0^{\bar{r}_z} d\bar{r}\,\psi_{,\alpha}\right) - \left(\mathcal{U}_\alpha + \psi n_\alpha\right)_{\bar{o}}\right)\mathcal{U}^\alpha +$$

$$+ \frac{1}{2}\left(\mathcal{H}_{\bar{o}}^2 + \mathcal{H}_{\bar{o}}'\right)\left(\int_0^{\bar{t}_o} dt\,\psi\right)^2 - \mathcal{H}_{\bar{o}}\left(\int_0^{\bar{t}_o} dt\,\psi\right)\left((\mathcal{U}_\| - \psi)\,|_o^z - 2\int_0^{\bar{r}_z} d\bar{r}\psi'\right) +$$

$$+\Delta x_s^\mu\left(\partial_\mu\left((\mathcal{U}_\| - \psi)\,|_o^z - 2\int_0^{\bar{r}_z} d\bar{r}\psi'\right)\right)\Big|_{\bar{x}_z}\Bigg] + \frac{1}{2}\frac{\mathcal{H}_z^2 + \mathcal{H}_z'}{\mathcal{H}_z^3}\left((\mathcal{U}_\| - \psi)\,|_o^z - 2\int_0^{\bar{r}_z} d\bar{r}\psi' - \mathcal{H}_{\bar{o}}\int_0^{\bar{t}_o} \psi\,dt\right)^2 -$$

$$-\left(\frac{\mathcal{H}_{\bar{o}}}{\mathcal{H}_z} - 1\right)\Bigg[\delta x_o^\mu\left(\partial_\mu\left(\int_0^t dt\,\psi\right)\right)\Big|_{\bar{o}} + \int_0^{\bar{t}_o} dt\left(-\psi + \frac{1}{2}\psi^2 + \frac{1}{2}\mathcal{U}^\alpha\mathcal{U}_\alpha - \frac{1}{a}\mathcal{U}^\alpha\left(\int_0^t dt\,\psi\right)_{,\alpha}\right)\Big|_{\bar{x}_t}\Bigg] +$$

$$+\frac{1}{2}\mathcal{H}_{\bar{o}}\left(\int_0^{\bar{t}_o} dt\,\psi\right)^2 - \int_0^{\bar{r}_z} d\bar{r}\Bigg[(\bar{r}_z - \bar{r})\left(2\left\{2\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\psi'\right) - (\psi - \mathcal{U}_\|)_{\bar{o}}\right\}\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\psi'\right)' +$$

$$+2\left\{2\left(n^\alpha\psi - \int_0^{\bar{r}_z} d\bar{r}\psi'^{,\alpha}\right) - (\mathcal{U}^\alpha + \psi n^\alpha)_{\bar{o}}\right\}\left\{\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\psi'\right\}_{,\alpha} + \psi' + \phi' - 2\psi_{,\|} + B_{\|,\|} + C_{\|\|\|}' +$$

$$+2\left(-2\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\psi'\right) + (\psi - \mathcal{U}_\|)_{\bar{o}}\right)(\psi' - \psi_{,\|}) + 4\psi\,\psi_{,\|} - 2\left(2\left(n^\alpha\psi - \int_0^{\bar{r}_z} d\bar{r}\psi'^{,\alpha}\right) -$$

$$-(\mathcal{U}^\alpha + \psi n^\alpha)_{\bar{o}}\right)(\psi_{,\alpha} + \psi' n_\alpha)\right)\Bigg]\Big|_{\bar{x}_{\bar{r}}} + \frac{1}{2\bar{r}_z}\Bigg[\left(\int_0^{\bar{t}_o} \frac{1}{a}\mathcal{U}_\theta\,dt - \bar{r}_z\,(\mathcal{U}_\theta)_{\bar{o}} - 2\int_0^{\bar{r}_z} d\bar{r}\left(\frac{\bar{r}_z - \bar{r}}{\bar{r}}\right)\frac{\partial\psi}{\partial\theta}\right)^2 +$$

$$+\left(\int_0^{\bar{t}_o} \frac{1}{a}\mathcal{U}_\phi\,dt - \bar{r}_z\,(\mathcal{U}_\phi)_{\bar{o}} - 2\int_0^{\bar{r}_z} d\bar{r}\left(\frac{\bar{r}_z - \bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\psi}{\partial\phi}\right)^2\Bigg]\Bigg] + \frac{1}{\bar{r}_z}\left(\theta_\alpha\frac{\partial}{\partial\theta} + \frac{\phi_\alpha}{\sin\theta}\frac{\partial}{\partial\phi}\right)\left\{\delta x_o^\mu(\partial_\mu\delta x^\alpha)|_{\bar{o}} +$$

$$+\Bigg[\Delta x_{\bar{r}}^\mu\,\partial_\mu\Bigg[\int_0^{\bar{t}_o} \frac{1}{a}\mathcal{U}^\alpha\,dt + \int_0^{\bar{r}_z} d\bar{r}\left(2n^\alpha\psi - 2\left[\left(\theta^\alpha\frac{\bar{r}_z - \bar{r}}{\bar{r}}\frac{\partial\psi}{\partial\theta} + \phi^\alpha\frac{\bar{r}_z - \bar{r}}{\bar{r}\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]\right) +$$

$$+\bar{r}_z\,(\mathcal{U}_\| n^\alpha - \mathcal{U}^\alpha)_{\bar{o}}\Bigg]\Big|_{\bar{x}_{\bar{r}}}\Bigg]_{\bar{o}}^z + \int_0^{\bar{t}_o} dt\,\frac{1}{a}\left(\mathcal{U}^\alpha + \psi\mathcal{U}^\alpha - \mathcal{U}^\beta\delta x^\alpha_{,\beta}\right)\Big|_{\bar{x}_t} + \int_0^{\bar{r}_z} d\bar{r}\Bigg[-\left\{2\left(n^\beta\psi - \int_0^{\bar{r}_z} d\bar{r}\psi'^{,\beta}\right) -$$

$$-(\mathcal{U}^\beta + \psi n^\beta)_{\bar{o}}\right\}\left(\left[\int_0^{\bar{t}_o} \frac{1}{a}\mathcal{U}^\alpha\,dt + \int_0^{\bar{r}_z} d\bar{r}\left(2n^\alpha\psi - 2\left[\left(\theta^\alpha\frac{\bar{r}_z - \bar{r}}{\bar{r}}\frac{\partial\psi}{\partial\theta} + \phi^\alpha\frac{\bar{r}_z - \bar{r}}{\bar{r}\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]\right)\right) +$$

$$+\bar{r}_z\,(\mathcal{U}_\| n^\alpha - \mathcal{U}^\alpha)_{\bar{o}}\Bigg]_{,\beta} + n^\alpha_{,\beta}\left[\frac{1}{\mathcal{H}_z}\left((\mathcal{U}_\| - \psi)\,|_o^z - 2\int_0^{\bar{r}_z} d\bar{r}\psi' - \mathcal{H}_{\bar{o}}\int_0^{\bar{t}_o} \psi\,dt\right)\right]\right) +$$

$$+\left\{-2\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\psi'\right) + (\psi - \mathcal{U}_\|)_{\bar{o}}\right\}\Bigg[\int_0^{\bar{t}_o} \frac{1}{a}\mathcal{U}^\alpha\,dt + \int_0^{\bar{r}_z} d\bar{r}\left(2n^\alpha\psi - 2\left[\theta^\alpha\frac{\bar{r}_z - \bar{r}}{\bar{r}}\frac{\partial\psi}{\partial\theta} +\right.\right.$$

$$+\phi^\alpha\frac{\bar{r}_z - \bar{r}}{\bar{r}\sin\theta}\frac{\partial\psi}{\partial\phi}\Bigg]\right) + \bar{r}_z\,(\mathcal{U}_\| n^\alpha - \mathcal{U}^\alpha)_{\bar{o}}\Bigg]' - (\bar{r}_z - \bar{r})\left(\psi^{,\alpha} - \phi^{,\alpha} - 2\phi' n^\alpha + 2\,(\phi n^\alpha)_{,\|} - B'^\alpha - B_\|^{,\alpha} + B^\alpha_{,\|} +\right.$$

$$+4\psi\left(\psi^{,\alpha} - (\psi n^\alpha)_{,\|} + \psi' n^\alpha\right) - 2C_\|'^\alpha + 2C^\alpha_{\|,\|} - C_{\|\|\|}{}^{,\alpha} + 2\left\{(\psi - \mathcal{U}_\|)_{\bar{o}} - 2\left(\psi + \int_0^{\bar{r}_z} d\bar{r}\psi'\right)\right\}\left(\psi^{,\alpha} +\right.$$

$$+\psi' n^\alpha\right) + 2\left\{(\mathcal{U}^\gamma + \psi n^\gamma)_{\bar{o}} - 2\left(n^\gamma\psi - \int_0^{\bar{r}_z} d\bar{r}\psi'^{,\gamma}\right)\right\}\left(\delta^\alpha_\gamma\,(\psi_{,\|} - \psi') + (\psi n^\alpha)_{,\gamma} + (\psi n_\gamma)^{,\alpha}\right) +$$

$$+\left\{4\left(\psi+\int_0^{\bar{r}_z}d\bar{r}\psi'\right)-2\left(\psi-\mathcal{U}_\parallel\right)_{\bar{\mathrm{o}}}\right\}\left(n^\alpha\psi'-\int_0^{\bar{r}_z}d\bar{r}\psi'^{,\alpha}\right)+2\left(2\psi-\psi_{\bar{\mathrm{o}}}\right)\left\{n^\alpha\psi-\int_0^{\bar{r}_z}d\bar{r}\psi^\alpha\right\}_{,\parallel}-$$

$$-2\left(\mathcal{U}_{\bar{\mathrm{o}}}^\beta+2\int_0^{\bar{r}_z}d\bar{r}\psi'^{,\beta}\right)\left\{n^\alpha\psi-\int_0^{\bar{r}_z}d\bar{r}\psi^\alpha\right\}_{,\beta}\right)\Big]\Big|_{\bar{x}_{\bar{r}}}\right\}+\frac{1}{\bar{r}_z^2}\left(\cot\theta+\frac{\partial}{\partial\theta}\right)\left[-\left(\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\parallel\,dt-\right.\right.$$

$$-\frac{1}{\mathcal{H}_z}\left(\left.\left(\mathcal{U}_\parallel-\psi\right)\right|_o^z-2\int_0^{\bar{r}_z}d\bar{r}\psi'-\left(\mathcal{H}_{\bar{\mathrm{o}}}-\mathcal{H}_z\right)\int_0^{\bar{t}_{\mathrm{o}}}\psi\,dt\right)+$$

$$+2\int_0^{\bar{r}_z}d\bar{r}\psi\right)\left(\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\theta\,dt-\bar{r}_z\left(\mathcal{U}_\theta\right)_{\bar{\mathrm{o}}}-2\int_0^{\bar{r}_z}d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}}\right)\frac{\partial\psi}{\partial\theta}\right)+\frac{\cot\theta}{2}\left(\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\phi\,dt-\right.$$

$$-\bar{r}_z\left(\mathcal{U}_\phi\right)_{\bar{\mathrm{o}}}-2\int_0^{\bar{r}_z}d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\psi}{\partial\phi}\right)^2\Big]-\frac{1}{\bar{r}_z\sin^2\theta}\frac{\partial}{\partial\phi}\left[\frac{1}{\bar{r}_z}\left(\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\parallel\,dt-\right.\right.$$

$$-\frac{1}{\mathcal{H}_z}\left(\left.\left(\mathcal{U}_\parallel-\psi\right)\right|_o^z-2\int_0^{\bar{r}_z}d\bar{r}\psi'-\left(\mathcal{H}_{\bar{\mathrm{o}}}-\mathcal{H}_z\right)\int_0^{\bar{t}_{\mathrm{o}}}\psi\,dt\right)+2\int_0^{\bar{r}_z}d\bar{r}\psi\right)\left(\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\phi\,dt-\right.$$

$$-\bar{r}_z\left(\mathcal{U}_\phi\right)_{\bar{\mathrm{o}}}-2\int_0^{\bar{r}_z}d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\psi}{\partial\phi}\right)+\cot\theta\left(\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\theta\,dt-\bar{r}_z\left(\mathcal{U}_\theta\right)_{\bar{\mathrm{o}}}-\right.$$

$$-2\int_0^{\bar{r}_z}d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}}\right)\frac{\partial\psi}{\partial\theta}\right)\left(\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\phi\,dt-\bar{r}_z\left(\mathcal{U}_\phi\right)_{\bar{\mathrm{o}}}-2\int_0^{\bar{r}_z}d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\psi}{\partial\phi}\right)\Big]-$$

$$-\frac{1}{\bar{r}_z^2}\frac{\partial}{\partial\phi}\left[\left(\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\theta\,dt-2\int_0^{\bar{r}_z}d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}}\right)\frac{\partial\psi}{\partial\theta}\right)\right]\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\left(\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\phi\,dt-\right.\right.$$

$$-2\int_0^{\bar{r}_z}d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\psi}{\partial\phi}\right)\Big]+\frac{1}{\bar{r}_z^2\sin\theta}\frac{\partial}{\partial\phi}\left[\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\phi\,dt-\right.$$

$$-2\int_0^{\bar{r}_z}d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\psi}{\partial\phi}\right]\frac{\partial}{\partial\theta}\left[\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\theta\,dt-2\int_0^{\bar{r}_z}d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}}\right)\frac{\partial\psi}{\partial\theta}\right]+$$

$$+\frac{\cot\theta}{\bar{r}_z^2}\left(\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\theta\,dt-\bar{r}_z\left(\mathcal{U}_\theta\right)_{\bar{\mathrm{o}}}-2\int_0^{\bar{r}_z}d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}}\right)\frac{\partial\psi}{\partial\theta}\right)\left(\frac{\partial}{\partial\theta}\left[\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\theta\,dt-\right.\right.$$

$$-2\int_0^{\bar{r}_z}d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}}\right)\frac{\partial\psi}{\partial\theta}\right]+\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left[\int_0^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_\phi\,dt-2\int_0^{\bar{r}_z}d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\psi}{\partial\phi}\right]\right)-$$

$$-n_\alpha\int_0^{\bar{r}_z}d\bar{r}\Big[\psi^{,\alpha}-\phi^{,\alpha}-2\phi'n^\alpha+2\left(\phi n^\alpha\right)_{,\parallel}-B'^\alpha-B_\parallel^{,\alpha}+B_{,\parallel}^\alpha+4\psi\left(\psi^{,\alpha}-\left(\psi n^\alpha\right)_{,\parallel}+\psi'n^\alpha\right)-$$

$$-2C'^\alpha_\parallel+2C^\alpha_{\parallel,\parallel}-C_{\parallel\parallel}{}^{,\alpha}+2\left\{\left(\psi-\mathcal{U}_\parallel\right)_{\bar{\mathrm{o}}}-2\left(\psi+\int_0^{\bar{r}_z}d\bar{r}\psi'\right)\right\}\left(\psi^{,\alpha}+\psi'n^\alpha\right)+$$

$$+2\left\{\left(\mathcal{U}^\gamma+\psi n^\gamma\right)_{\bar{\mathrm{o}}}-2\left(n^\gamma\psi-\int_0^{\bar{r}_z}d\bar{r}\psi^{,\gamma}\right)\right\}\left(\delta_\gamma^\alpha\left(\psi_{,\parallel}-\psi'\right)+\left(\psi n^\alpha\right)_{,\gamma}+\left(\psi n_\gamma\right)^{,\alpha}\right)+$$

$$+\left\{4\left(\psi+\int_0^{\bar{r}_z}d\bar{r}\psi'\right)-2\left(\psi-\mathcal{U}_\parallel\right)_{\bar{\mathrm{o}}}\right\}\left(n^\alpha\psi'-\int_0^{\bar{r}_z}d\bar{r}\psi'^{,\alpha}\right)+2\left(2\psi-\psi_{\bar{\mathrm{o}}}\right)\left\{n^\alpha\psi-\int_0^{\bar{r}_z}d\bar{r}\psi^\alpha\right\}_{,\parallel}-$$

$$- 2 \left(\mathcal{U}_{\bar{\mathrm{o}}}^{\beta} + 2 \int_{0}^{\bar{r}_z} d\bar{r}\psi^{,\beta}\right) \left\{n^{\alpha}\psi - \int_{0}^{\bar{r}_z} d\bar{r}\psi^{\alpha}\right\}_{,\beta}\Bigg]\Bigg|_{\bar{x}_{\bar{r}}} - \frac{1}{\bar{r}_z}\left(\mathcal{U}_{\theta}\frac{\partial}{\partial\theta} + \frac{\mathcal{U}_{\phi}}{\sin\theta}\frac{\partial}{\partial\phi}\right)\Delta\eta -$$

$$-4\left\{\frac{1}{\bar{r}_z}\int_{0}^{\bar{t}_{\mathrm{o}}}dt\,\frac{1}{a}\mathcal{U}_{\|}+\mathcal{U}_{\|}+\int_{o}^{\bar{r}_z}d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}_z\bar{r}}\right)\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\psi\right\}\left\{\frac{1}{\bar{r}_z}\int_{0}^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_{\|}\,dt-\right.$$

$$\left. -\frac{1}{\bar{r}_z\mathcal{H}_z}\left((\mathcal{U}_{\|}-\psi)\,|_{o}^{z}-2\int_{0}^{\bar{r}_z}d\bar{r}\psi'-(\mathcal{H}_{\bar{\mathrm{o}}}-\mathcal{H}_z)\int_{0}^{\bar{t}_{\mathrm{o}}}\psi\,dt\right)+\frac{2}{\bar{r}_z}\int_{0}^{\bar{r}_z}d\bar{r}\psi\right\}+$$

$$+\frac{1}{\bar{r}_z^2}\left[\left(\int_{0}^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_{\|}\,dt-\frac{1}{\mathcal{H}_z}\left((\mathcal{U}_{\|}-\psi)\,|_{o}^{z}-2\int_{0}^{\bar{r}_z}d\bar{r}\psi'-(\mathcal{H}_{\bar{\mathrm{o}}}-\mathcal{H}_z)\int_{0}^{\bar{t}_{\mathrm{o}}}\psi\,dt\right)+2\int_{0}^{\bar{r}_z}d\bar{r}\psi\right)^2-\right.$$

$$-\left(\int_{0}^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_{\theta}\,dt-\bar{r}_z\,(\mathcal{U}_{\theta})_{\bar{\mathrm{o}}}-2\int_{0}^{\bar{r}_z}d\bar{r}\,\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}}\right)\frac{\partial\psi}{\partial\theta}\right)^2-$$

$$\left. -\frac{1}{2}\left(\int_{0}^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_{\phi}\,dt-\bar{r}_z\,(\mathcal{U}_{\phi})_{\bar{\mathrm{o}}}-2\int_{0}^{\bar{r}_z}d\bar{r}\,\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\psi}{\partial\phi}\right)^2\right]-$$

$$-\frac{1}{\bar{r}_z}\left[\left(\left(\mathcal{U}_{\theta}-(\mathcal{U}_{\theta})_{\bar{\mathrm{o}}}-2\theta^{\alpha}\int_{0}^{\bar{r}_z}d\bar{r}\psi_{,\alpha}\right)\right)\frac{\partial}{\partial\theta}+\frac{1}{\sin\theta}\left(\left(\mathcal{U}_{\phi}-(\mathcal{U}_{\phi})_{\bar{\mathrm{o}}}-2\phi^{\alpha}\int_{0}^{\bar{r}_z}d\bar{r}\psi_{,\alpha}\right)\right)\frac{\partial}{\partial\phi}\right]$$

$$\left\{\int_{0}^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_{\|}\,dt-\frac{1}{\mathcal{H}_z}\left((\mathcal{U}_{\|}-\psi)\,|_{o}^{z}-2\int_{0}^{\bar{r}_z}d\bar{r}\psi'-(\mathcal{H}_{\bar{\mathrm{o}}}-\mathcal{H}_z)\int_{0}^{\bar{t}_{\mathrm{o}}}\psi\,dt\right)+2\int_{0}^{\bar{r}_z}d\bar{r}\psi\right\}+$$

$$+\frac{1}{\bar{r}_z}\left(\int_{0}^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_{\theta}\,dt-\bar{r}_z\,(\mathcal{U}_{\theta})_{\bar{\mathrm{o}}}-2\int_{0}^{\bar{r}_z}d\bar{r}\,\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}}\right)\frac{\partial\psi}{\partial\theta}\right)\left(\mathcal{U}_{\theta}-(\mathcal{U}_{\theta})_{\bar{\mathrm{o}}}-2\theta^{\alpha}\int_{0}^{\bar{r}_z}d\bar{r}\psi_{,\alpha}\right)+$$

$$+\frac{1}{\bar{r}_z}\left(\int_{0}^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_{\phi}\,dt-\bar{r}_z\,(\mathcal{U}_{\phi})_{\bar{\mathrm{o}}}-2\int_{0}^{\bar{r}_z}d\bar{r}\,\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\psi}{\partial\phi}\right)\left(\mathcal{U}_{\phi}-(\mathcal{U}_{\phi})_{\bar{\mathrm{o}}}-2\phi^{\alpha}\int_{0}^{\bar{r}_z}d\bar{r}\psi_{,\alpha}\right)+$$

$$+\frac{1}{\bar{r}_z^2}\left[\left(\int_{0}^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_{\theta}\,dt-\bar{r}_z\,(\mathcal{U}_{\theta})_{\bar{\mathrm{o}}}-2\int_{0}^{\bar{r}_z}d\bar{r}\,\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}}\right)\frac{\partial\psi}{\partial\theta}\right)\frac{\partial}{\partial\theta}+\right.$$

$$\left. +\frac{1}{\sin\theta}\left(\int_{0}^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_{\phi}\,dt-\bar{r}_z\,(\mathcal{U}_{\phi})_{\bar{\mathrm{o}}}-2\int_{0}^{\bar{r}_z}d\bar{r}\,\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\psi}{\partial\phi}\right)\frac{\partial}{\partial\phi}\right]\left\{\int_{0}^{\bar{t}_{\mathrm{o}}}\frac{1}{a}\mathcal{U}_{\|}\,dt-\right.$$

$$\left. -\frac{1}{\mathcal{H}_z}\left((\mathcal{U}_{\|}-\psi)\,|_{o}^{z}-2\int_{0}^{\bar{r}_z}d\bar{r}\psi'-(\mathcal{H}_{\bar{\mathrm{o}}}-\mathcal{H}_z)\int_{0}^{\bar{t}_{\mathrm{o}}}\psi\,dt\right)+2\int_{0}^{\bar{r}_z}d\bar{r}\psi\right\}+$$

$$+\left((\mathcal{U}_{\|}-\psi)_{\bar{\mathrm{o}}}+2\int_{0}^{\bar{r}_z}d\bar{r}\psi'+\mathcal{H}_{\bar{\mathrm{o}}}\int_{0}^{\bar{t}_{\mathrm{o}}}\psi\,dt\right)^2+(\mathcal{U}_{\|}-\psi)\,\mathcal{H}_{\bar{\mathrm{o}}}\int_{0}^{\bar{t}_{\mathrm{o}}}\psi\,dt+$$

$$+\left(7\psi+\mathcal{U}_{\|}\right)\left((\mathcal{U}_{\|}-\psi)_{\bar{\mathrm{o}}}+2\int_{0}^{\bar{r}_z}d\bar{r}\psi'+\mathcal{H}_{\bar{\mathrm{o}}}\int_{0}^{\bar{t}_{\mathrm{o}}}\psi\,dt\right)+$$

$$+4\left(\mathcal{U}_{\|}-2\psi-(\mathcal{U}_{\|}-\psi)_{\bar{\mathrm{o}}}-2\int_{0}^{\bar{r}_z}d\bar{r}\psi'-\mathcal{H}_{\bar{\mathrm{o}}}\int_{0}^{\bar{t}_{\mathrm{o}}}\psi\,dt\right)\left\{\frac{2}{\bar{r}_z}\int_{0}^{\bar{r}_z}d\bar{r}\psi-\right.$$

$$-\frac{1}{\bar{r}_z \mathcal{H}_z}\left[(\mathcal{U}_\parallel - \psi)\,|_o^z - 2\int_0^{\bar{r}_z} d\bar{r}\psi' - (\mathcal{H}_{\bar{o}} - \mathcal{H}_z)\int_0^{\bar{t}_o}\psi\,dt\right] -$$

$$-\int_o^{\bar{r}_z} d\bar{r}\left(\frac{\bar{r}_z - \bar{r}}{\bar{r}_z \bar{r}}\right)\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\psi\Bigg\}$$

$$+ 2\Delta x^\mu \partial_\mu\left(\mathcal{U}_\parallel - \psi - \int_0^{\bar{r}_z} d\bar{r}\psi' - \mathcal{H}_{\bar{o}}\int_0^{\bar{t}_o}\psi\,dt - n^\alpha\int_0^{\bar{r}_z} d\bar{r}\psi_{,\alpha}\right). \quad (4.16)$$

4.1.2 Einstein-de Sitter Universe

On first learning of Edwin Hubble's discovery of a linear relation between the red-shift of the galaxies and their distance, Einstein set the cosmological constant to zero in the Friedmann equations, resulting in a model of the expanding universe known as the Friedmann–Einstein universe. Later, Einstein and de Sitter proposed an even simpler cosmic model by assuming a vanishing spatial curvature as well as a vanishing cosmological constant. In modern parlance, the Einstein–de Sitter universe can be described as a cosmological model for a flat matter-only FLRW metric universe.

We will take the Einstein-de Sitter limit by implementing a simple set of rules:

$$a = \frac{\eta^2}{\eta_{\bar{o}}^2}, \quad (4.17)$$

$$\mathcal{H} = \frac{a'}{a} = \frac{2}{\eta}, \quad (4.18)$$

$$\mathcal{U}_\alpha^{(1)} = -\partial_\alpha U = \frac{1}{5}\eta\partial_\alpha \mathcal{R}(x), \quad (4.19)$$

$$\psi^{(1)} = -\frac{3\mathcal{R}(x)}{5}. \quad (4.20)$$

This leads to a large amount of possible simplifications, for example,

$$\psi^{1'} = 0, \qquad \int_0^{t_{\bar{o}}} \psi^{(1)}dt = -\frac{\eta_{\bar{o}}}{5}\mathcal{R}_{\bar{o}}. \quad (4.21)$$

Using this limit, the expression for δA is written as:

$$\delta A = 2\left[\frac{2}{\eta_{\bar{o}}}\left(\delta x_o^{\mu}\left(\partial_{\mu}\left(-\frac{\eta^3}{5\eta_{\bar{o}}^2}\mathcal{R}\right)\right)\right)\Big|_{\bar{o}} - \frac{\eta_{\bar{o}}}{25}\mathcal{R}_{\bar{o}}^2 + \int_0^{\bar{t}_o} dt\left(-\psi + \frac{9}{50}\mathcal{R}^2 + \frac{3\eta^2}{50}\left(\partial_{\alpha}\mathcal{R}\right)^2\right)\Big|_{\bar{x}_t}\right) +$$

$$+ \frac{\mathcal{R}_{\bar{o}}^2}{25} + \frac{2}{25}\mathcal{R}_{\bar{o}}\left(3\mathcal{R} + \eta\left(\partial_{\|}\mathcal{R}\right)\right)\Big|_o^z + \Delta x_s^{\mu}\left(\partial_{\mu}\left(\frac{3}{5}\mathcal{R} + \frac{\eta}{5}\left(\partial_{\|}\mathcal{R}\right)\right)\Big|_o^z\right)\Big|_{\bar{x}_z}\right]$$

$$+ \psi + 3\phi + C_{\alpha}^{\alpha} + \frac{36}{25}\mathcal{R}^2 + \left(\psi - \phi - C_{\|\|} - B_{\|} + \frac{6\eta}{25}\mathcal{R}\partial_{\|}\mathcal{R} + \frac{\eta^2}{50}\left(\left(\partial_{\|}\mathcal{R}\right)^2 - \left(\partial_{\alpha}\mathcal{R}\right)^2\right)\right)_{\bar{o}}$$

$$+ \int_0^{\bar{r}_z} d\bar{r}\left(\psi' + \phi' - 2\psi_{,\|} + B_{\|,\|} + C_{\|\|\|}' + \frac{72}{25}\mathcal{R}\partial_{\|}\mathcal{R} - \frac{6}{25}\left(\partial_{\|}\mathcal{R}\right)\left(3\mathcal{R} + \eta\left(\partial_{\|}\mathcal{R}\right)\right)\right)_{\bar{o}}\Big|_{\bar{x}_{\bar{r}}} +$$

$$+ 2\mathcal{U}_{\|} - B_{\|} - \frac{3}{25}\mathcal{R}\left(6\mathcal{R} - \left(3\mathcal{R} + \eta\left(\partial_{\|}\mathcal{R}\right)\right)_{\bar{o}}\right) + \frac{\eta\left(\partial^{\alpha}\mathcal{R}\right)}{25}\left(6\left(\int_0^{\bar{r}_z} d\bar{r}\left(\partial_{\alpha}\mathcal{R}\right) - n_{\alpha}\mathcal{R}\right) +$$

$$+ \left(3n_{\alpha}\mathcal{R} - \eta\left(\partial_{\alpha}\mathcal{R}\right)\right)_{\bar{o}}\right) + \frac{\eta^2}{25}\left(\left(\partial_{\alpha}\mathcal{R}\right)^2 - 3\left(\partial_{\|}\mathcal{R}\right)^2\right) + \frac{6\eta}{25}\mathcal{R}\left(\partial_{\|}\mathcal{R}\right) +$$

$$+ \frac{2}{\bar{r}_z}\left[-\frac{\eta}{2}\left[\left(-\psi - \mathcal{U}_{\|} + B_{\|} + \frac{27}{50}\mathcal{R}^2 - \frac{6\eta}{25}\mathcal{R}\left(\partial_{\|}\mathcal{R}\right) + \frac{\eta^2}{50}\left(\partial_{\alpha}\mathcal{R}\right)^2\right)_{\bar{o}} +$$

$$+ \int_0^{\bar{r}_z} d\bar{r}\left[\psi' + \phi' - 2\psi_{,\|} + B_{\|,\|} + C_{\|\|\|}' + \frac{72}{25}\mathcal{R}\partial_{\|}\mathcal{R} - \frac{6}{25}\left(\partial_{\|}\mathcal{R}\right)\left(3\mathcal{R} + \eta\left(\partial_{\|}\mathcal{R}\right)\right)\right]_{\bar{o}}\Big|_{\bar{x}_{\bar{r}}} +$$

$$+ \psi + \mathcal{U}_{\|} - B_{\|} - \frac{9}{50}\mathcal{R}^2 + \frac{\eta^2}{50}\left(\partial_{\alpha}\mathcal{R}\right)^2 + \frac{6\eta}{25}\mathcal{R}\left(\partial_{\|}\mathcal{R}\right) +$$

$$- \frac{3}{25}\mathcal{R}\left(6\mathcal{R} - \left(3\mathcal{R} + \eta\left(\partial_{\|}\mathcal{R}\right)\right)_{\bar{o}}\right) + \frac{\eta\left(\partial^{\alpha}\mathcal{R}\right)}{25}\left(6\left(\int_0^{\bar{r}_z} d\bar{r}\left(\partial_{\alpha}\mathcal{R}\right) - n_{\alpha}\mathcal{R}\right) + \left(3n_{\alpha}\mathcal{R} - \eta\left(\partial_{\alpha}\mathcal{R}\right)\right)_{\bar{o}}\right) +$$

$$+ \frac{\mathcal{R}_{\bar{o}}^2}{25} + \frac{2}{25}\mathcal{R}_{\bar{o}}\left(3\mathcal{R} + \eta\left(\partial_{\|}\mathcal{R}\right)\right)\Big|_o^z + \Delta x_s^{\mu}\left(\partial_{\mu}\left(\frac{3}{5}\mathcal{R} + \frac{\eta}{5}\left(\partial_{\|}\mathcal{R}\right)\right)\Big|_o^z\right)\Big|_{\bar{x}_z}\right] +$$

$$+ \frac{\eta}{8}\left(\frac{1}{5}\left(\eta\left(\partial_{\|}\mathcal{R}\right) + 3\mathcal{R}\right)\Big|_o^z + \frac{2}{5}\mathcal{R}_{\bar{o}}\right)^2 -$$

$$- \left(\frac{\eta}{\eta_{\bar{o}}} - 1\right)\left[\delta x_o^{\mu}\left(\partial_{\mu}\left(-\frac{\eta^3}{5\eta_{\bar{o}}^2}\mathcal{R}\right)\right)\Big|_{\bar{o}} + \int_0^{\bar{t}_o} dt\left(-\psi + \frac{9}{50}\mathcal{R}^2 + \frac{3\eta^2}{50}\left(\partial_{\alpha}\mathcal{R}\right)^2\right)\Big|_{\bar{x}_t}\right] +$$

$$+ \frac{\eta_{\bar{o}}}{25}\mathcal{R}_{\bar{o}}^2 - \int_0^{\bar{r}_z} d\bar{r}\left[\left(\bar{r}_z - \bar{r}\right)\left(\psi' + \phi' - 2\psi_{,\|} + B_{\|,\|} + C_{\|\|\|}' + \frac{72}{25}\mathcal{R}\partial_{\|}\mathcal{R} - \frac{6}{25}\left(\partial_{\|}\mathcal{R}\right)\left(3\mathcal{R} + \eta\left(\partial_{\|}\mathcal{R}\right)\right)\right)_{\bar{o}}\right]\Big|_{\bar{x}_{\bar{r}}} +$$

$$+ \frac{1}{2\bar{r}_z}\left[\left(\frac{\eta_{\bar{o}}^2}{10}\left(\partial_{\theta}\mathcal{R}\right)_{\bar{o}} - \frac{\bar{r}_z}{5}\left(\eta\partial_{\theta}\mathcal{R}\right)_{\bar{o}} + \frac{6}{5}\int_0^{\bar{r}_z} d\bar{r}\left(\frac{\bar{r}_z - \bar{r}}{\bar{r}}\right)\frac{\partial\mathcal{R}}{\partial\theta}\right)^2 +$$

$$+ \left(\frac{\eta_{\bar{o}}^2}{10}\left(\partial_{\phi}\mathcal{R}\right)_{\bar{o}} - \frac{\bar{r}_z}{5}\left(\eta\partial_{\phi}\mathcal{R}\right)_{\bar{o}} + \frac{6}{5}\int_0^{\bar{r}_z} d\bar{r}\left(\frac{\bar{r}_z - \bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\mathcal{R}}{\partial\phi}\right)^2\right] +$$

$$+ \frac{1}{\bar{r}_z}\left(\theta_{\alpha}\frac{\partial}{\partial\theta} + \frac{\phi_{\alpha}}{\sin\theta}\frac{\partial}{\partial\phi}\right)\left\{\delta x_o^{\mu}\left(\partial_{\mu}\frac{\eta_{\bar{o}}^2}{10}\partial^{\alpha}\mathcal{R}\right)\Big|_{\bar{o}} +$$

$$+\left[\Delta x_{\bar{r}}^{\mu}\,\partial_{\mu}\left[\frac{\eta_{\bar{o}}^2}{10}\,(\partial^{\alpha}\mathcal{R})_{\bar{o}}+\int_{0}^{\bar{r}_z}d\bar{r}\frac{6}{5}\left(\left[\left(\theta^{\alpha}\frac{\bar{r}_z-\bar{r}}{\bar{r}}\frac{\partial\mathcal{R}}{\partial\theta}+\phi^{\alpha}\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\frac{\partial\mathcal{R}}{\partial\phi}\right)\right]-n^{\alpha}\mathcal{R}\right)+\right.\right.$$

$$\left.\left.+\frac{\bar{r}_z\eta_{\bar{o}}}{5}\left(\left(\partial_{\|}\mathcal{R}\right)n^{\alpha}-(\partial^{\alpha}\mathcal{R})\right)_{\bar{o}}\right]\Big|_{\bar{x}_{\bar{r}}}^{z}\right]_{\bar{o}}+\int_{0}^{\bar{t}_o}\mathcal{U}_{\alpha}d\eta-\frac{3\eta_{\bar{o}}^2}{50}\left(\mathcal{R}\partial_{\alpha}\mathcal{R}\right)_{\bar{o}}-\frac{\eta_{\bar{o}}^3}{30}\left((\partial^{\beta}\mathcal{R})\,(\partial^{\alpha}\mathcal{R})_{,\beta}\right)_{\bar{o}}+$$

$$+\int_{0}^{\bar{r}_z}d\bar{r}\left[-\left\{\frac{6}{5}\left(\int_{0}^{\bar{r}_z}d\bar{r}\,(\partial^{\beta}\mathcal{R})-n^{\beta}\mathcal{R}\right)-\frac{1}{5}\left(\eta\,(\partial^{\beta}\mathcal{R})-3n^{\beta}\mathcal{R}\right)_{\bar{o}}\right\}\left(\left[\frac{\eta_{\bar{o}}^2}{10}\,(\partial^{\alpha}\mathcal{R})_{\bar{o}}+\right.\right.\right.$$

$$\left.+\int_{0}^{\bar{r}_z}d\bar{r}\frac{6}{5}\left(\left[\left(\theta^{\alpha}\frac{\bar{r}_z-\bar{r}}{\bar{r}}\frac{\partial\mathcal{R}}{\partial\theta}+\phi^{\alpha}\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\frac{\partial\mathcal{R}}{\partial\phi}\right)\right]-n^{\alpha}\mathcal{R}\right)+\right.$$

$$\left.\left.+\frac{\bar{r}_z\eta_{\bar{o}}}{5}\left(\left(\partial_{\|}\mathcal{R}\right)n^{\alpha}-(\partial^{\alpha}\mathcal{R})\right)_{\bar{o}}\right]_{,\beta}+n_{,\beta}^{\alpha}\left[\frac{\eta}{2}\left(\left(\frac{\eta}{5}\,(\partial_{\|}\mathcal{R})+\frac{3}{5}\mathcal{R}\right)\Big|_{o}^{z}+\frac{2}{5}\mathcal{R}_{\bar{o}}\right)\right]\right)+$$

$$+\left\{\frac{6}{5}\mathcal{R}-\frac{1}{5}\left(3\mathcal{R}+\eta\,(\partial_{\|}\mathcal{R})\right)_{\bar{o}}\right\}\left[\int_{0}^{\bar{r}_z}d\bar{r}\frac{6}{5}\left(\left[\left(\theta^{\alpha}\frac{\bar{r}_z-\bar{r}}{\bar{r}}\frac{\partial\mathcal{R}}{\partial\theta}+\phi^{\alpha}\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\frac{\partial\mathcal{R}}{\partial\phi}\right)\right]-n^{\alpha}\mathcal{R}\right)+\right.$$

$$\left.+\frac{\bar{r}_z\eta_{\bar{o}}}{5}\left(\left(\partial_{\|}\mathcal{R}\right)n^{\alpha}-(\partial^{\alpha}\mathcal{R})\right)_{\bar{o}}\right]'-(\bar{r}_z-\bar{r})\left(\psi^{,\alpha}-\phi^{,\alpha}-2\phi'n^{\alpha}+2\,(\phi n^{\alpha})_{,\|}-B'^{\alpha}-B_{\|}^{,\alpha}+B_{,\|}^{\alpha}+\right.$$

$$+\frac{36}{25}\mathcal{R}\left(\partial^{\alpha}\mathcal{R}-(\mathcal{R}n^{\alpha})_{,\|}\right)-2C_{\|}'^{\alpha}+2C_{\|,\|}^{\alpha}-C_{\|\|}^{,\alpha}+\frac{6}{25}\partial^{\alpha}\mathcal{R}\left\{6\mathcal{R}-\left(\eta\,(\partial_{\|}\mathcal{R})+3\mathcal{R}\right)_{\bar{o}}\right\}-$$

$$-\frac{6}{5}\left\{\left(\frac{\eta}{5}\,(\partial^{\gamma}\mathcal{R})-\frac{3}{5}\mathcal{R}n^{\gamma}\right)_{\bar{o}}-\frac{6}{5}\left(\int_{0}^{\bar{r}_z}d\bar{r}\partial^{\gamma}\mathcal{R}-\mathcal{R}n^{\gamma}\right)\right\}\left(\delta_{\gamma}^{\alpha}\,(\partial_{\|}\mathcal{R})+(\mathcal{R}n^{\alpha})_{,\gamma}+(\mathcal{R}n_{\gamma})^{,\alpha}\right)+$$

$$+\frac{18}{25}\left(2\mathcal{R}-\mathcal{R}_{\bar{o}}\right)\left\{\int_{0}^{\bar{r}_z}d\bar{r}\,(\partial^{\alpha}\mathcal{R})-\mathcal{R}n^{\alpha}\right\}_{,\|}-$$

$$-\frac{6}{25}\left(\eta_{\bar{o}}\,(\partial^{\beta}\mathcal{R})_{\bar{o}}-6\int_{0}^{\bar{r}_z}d\bar{r}\,(\partial^{\beta}\mathcal{R})\right)\left\{\int_{0}^{\bar{r}_z}d\bar{r}\,(\partial^{\alpha}\mathcal{R})-n^{\alpha}\mathcal{R}\right\}_{,\beta}\right)\Big|_{\bar{x}_{\bar{r}}}\right]\Bigg\}+$$

$$+\frac{1}{\bar{r}_z^2}\left(\cot\theta+\frac{\partial}{\partial\theta}\right)\left[-\left(\frac{\eta_{\bar{o}}^2}{10}\,(\partial_{\|}\mathcal{R})_{\bar{o}}-\frac{\eta}{2}\left(\left(\frac{\eta}{5}\partial_{\|}\mathcal{R}+\frac{3}{5}\mathcal{R}\right)\Big|_{o}^{z}+\frac{\eta-\eta_{\bar{o}}}{\eta}\frac{2\mathcal{R}_{\bar{o}}}{5}\right)-\right.$$

$$-\frac{6}{5}\int_{0}^{\bar{r}_z}d\bar{r}\mathcal{R}\right)\left(\frac{\eta_{\bar{o}}^2}{10}\,(\partial_{\theta}\mathcal{R})_{\bar{o}}-\frac{\bar{r}_z}{5}\,(\eta\partial_{\theta}\mathcal{R})_{\bar{o}}+\frac{6}{5}\int_{0}^{\bar{r}_z}d\bar{r}\,\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}}\right)\frac{\partial\mathcal{R}}{\partial\theta}\right)+$$

$$\left.+\frac{\cot\theta}{2}\left(\frac{\eta_{\bar{o}}^2}{10}\,(\partial_{\phi}\mathcal{R})_{\bar{o}}-\frac{\bar{r}_z}{5}\,(\eta\partial_{\phi}\mathcal{R})_{\bar{o}}+\frac{6}{5}\int_{0}^{\bar{r}_z}d\bar{r}\,\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\mathcal{R}}{\partial\phi}\right)^2\right]-$$

$$\frac{1}{\bar{r}_z\sin^2\theta}\frac{\partial}{\partial\phi}\left[\frac{1}{\bar{r}_z}\left(\frac{\eta_{\bar{o}}^2}{10}\,(\partial_{\|}\mathcal{R})_{\bar{o}}-\frac{\eta}{2}\left(\left(\frac{\eta}{5}\partial_{\|}\mathcal{R}+\frac{3}{5}\mathcal{R}\right)\Big|_{o}^{z}+\frac{\eta-\eta_{\bar{o}}}{\eta}\frac{2\mathcal{R}_{\bar{o}}}{5}\right)-\frac{6}{5}\int_{0}^{\bar{r}_z}d\bar{r}\mathcal{R}\right)\times$$

$$\times\left(\frac{\eta_{\bar{o}}^2}{10}\,(\partial_{\phi}\mathcal{R})_{\bar{o}}-\frac{\bar{r}_z}{5}\,(\eta\partial_{\phi}\mathcal{R})_{\bar{o}}+\frac{6}{5}\int_{0}^{\bar{r}_z}d\bar{r}\,\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\mathcal{R}}{\partial\phi}\right)+\cot\theta\left(\frac{\eta_{\bar{o}}^2}{10}\,(\partial_{\theta}\mathcal{R})_{\bar{o}}-\frac{\bar{r}_z}{5}\,(\eta\partial_{\theta}\mathcal{R})_{\bar{o}}+\right.$$

$$\left.+\frac{6}{5}\int_{0}^{\bar{r}_z}d\bar{r}\,\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}}\right)\frac{\partial\mathcal{R}}{\partial\theta}\right)\left(\frac{\eta_{\bar{o}}^2}{10}\,(\partial_{\phi}\mathcal{R})_{\bar{o}}-\frac{\bar{r}_z}{5}\,(\eta\partial_{\phi}\mathcal{R})_{\bar{o}}+\frac{6}{5}\int_{0}^{\bar{r}_z}d\bar{r}\,\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}\sin\theta}\right)\frac{\partial\mathcal{R}}{\partial\phi}\right)\right]-$$

$$-\frac{1}{\bar{r}_z^2}\frac{\partial}{\partial\phi}\left[\frac{\eta_{\bar{o}}^2}{10}\,(\partial_{\theta}\mathcal{R})_{\bar{o}}+\frac{6}{5}\int_{0}^{\bar{r}_z}d\bar{r}\,\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}}\right)\frac{\partial\mathcal{R}}{\partial\theta}\right]\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\left(\frac{\eta_{\bar{o}}^2}{10}\,(\partial_{\phi}\mathcal{R})_{\bar{o}}+\right.\right.$$

$$+ \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \, \left(\frac{\bar{r}_z - \bar{r}}{\bar{r} \sin\theta} \right) \frac{\partial \mathcal{R}}{\partial \phi} \Big) \Big] + \frac{1}{\bar{r}_z^2 \sin\theta} \frac{\partial}{\partial \phi} \Big[\frac{\eta_{\bar{o}}^2}{10} (\partial_\phi \mathcal{R})_{\bar{o}} +$$

$$+ \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \, \left(\frac{\bar{r}_z - \bar{r}}{\bar{r} \sin\theta} \right) \frac{\partial \mathcal{R}}{\partial \phi} \Big] \frac{\partial}{\partial \theta} \Big[\frac{\eta_{\bar{o}}^2}{10} (\partial_\theta \mathcal{R})_{\bar{o}} + \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \, \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}} \right) \frac{\partial \mathcal{R}}{\partial \theta} \Big] +$$

$$+ \frac{\cot\theta}{\bar{r}_z^2} \left(\frac{\eta_{\bar{o}}^2}{10} (\partial_\theta \mathcal{R})_{\bar{o}} - \frac{\bar{r}_z}{5} (\eta \partial_\theta \mathcal{R})_{\bar{o}} + \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \, \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}} \right) \frac{\partial \mathcal{R}}{\partial \theta} \right) \left(\frac{\partial}{\partial \theta} \Big[\frac{\eta_{\bar{o}}^2}{10} (\partial_\theta \mathcal{R})_{\bar{o}} +$$

$$+ \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \, \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}} \right) \frac{\partial \mathcal{R}}{\partial \theta} \Big] + \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \Big[\frac{\eta_{\bar{o}}^2}{10} (\partial_\phi \mathcal{R})_{\bar{o}} + \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \, \left(\frac{\bar{r}_z - \bar{r}}{\bar{r} \sin\theta} \right) \frac{\partial \mathcal{R}}{\partial \phi} \Big] \right) -$$

$$- n_\alpha \int_0^{\bar{r}_z} d\bar{r} \Big(\psi^{,\alpha} - \phi^{,\alpha} - 2\phi' n^\alpha + 2 (\phi n^\alpha)_{,\parallel} - B'^\alpha - B_{\parallel}{}^{,\alpha} + B^\alpha{}_{,\parallel} +$$

$$+ \frac{36}{25} \mathcal{R} \left(\partial^\alpha \mathcal{R} - (\mathcal{R} n^\alpha)_{,\parallel} \right) - 2C'^\alpha_\parallel + 2C^\alpha_{\parallel,\parallel} - C_{\parallel\parallel}{}^{,\alpha} + \frac{6}{25} \partial^\alpha \mathcal{R} \Big\{ 6\mathcal{R} - \Big(\eta \left(\partial_\parallel \mathcal{R} \right) + 3\mathcal{R} \Big)_{\bar{o}} \Big\} -$$

$$- \frac{6}{5} \Big\{ \left(\frac{\eta}{5} (\partial^\gamma \mathcal{R}) - \frac{3}{5} \mathcal{R} n^\gamma \right)_{\bar{o}} - \frac{6}{5} \left(\int_0^{\bar{r}_z} d\bar{r} \partial^\gamma \mathcal{R} - \mathcal{R} n^\gamma \right) \Big\} \left(\delta^\alpha_\gamma \left(\partial_{,\parallel} \mathcal{R} \right) + (\mathcal{R} n^\alpha)_{,\gamma} + (\mathcal{R} n_\gamma)^{;\alpha} \right) +$$

$$+ \frac{18}{25} \left(2\mathcal{R} - \mathcal{R}_{\bar{o}} \right) \Big\{ \int_0^{\bar{r}_z} d\bar{r} \left(\partial^\alpha \mathcal{R} \right) - \mathcal{R} n^\alpha \Big\}_{,\parallel} -$$

$$- \frac{6}{25} \left(\eta_{\bar{o}} \left(\partial^\beta \mathcal{R} \right)_{\bar{o}} - 6 \int_0^{\bar{r}_z} d\bar{r} \left(\partial^\beta \mathcal{R} \right) \right) \Big\{ \int_0^{\bar{r}_z} d\bar{r} \left(\partial^\alpha \mathcal{R} \right) - n^\alpha \mathcal{R} \Big\}_{,\beta} \Big) \Big|_{\bar{x}_{\bar{r}}} - \frac{\eta}{5\bar{r}_z} \left(\partial_\theta \mathcal{R} \frac{\partial}{\partial \theta} + \frac{\partial_\phi \mathcal{R}}{\sin\theta} \frac{\partial}{\partial \phi} \right) \Delta\eta -$$

$$- 4 \Big\{ \frac{\eta_{\bar{o}}^2}{10\bar{r}_z} \left(\partial_\parallel \mathcal{R} \right)_{\bar{o}} + \frac{\eta}{5} \left(\partial_\parallel \mathcal{R} \right) - \frac{3}{5} \int_o^{\bar{r}_z} d\bar{r} \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}_z \bar{r}} \right) \left(\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right) \mathcal{R} \Big\} \times$$

$$\times \Big\{ \frac{1}{\bar{r}_z} \left(\frac{\eta_{\bar{o}}^2}{10} (\partial_\parallel \mathcal{R})_{\bar{o}} - \frac{\eta}{2} \left(\left(\frac{\eta}{5} \partial_\parallel \mathcal{R} + \frac{3}{5} \mathcal{R} \right) \Big|_o^z + \frac{\eta - \eta_{\bar{o}}}{\eta} \frac{2\mathcal{R}_{\bar{o}}}{5} \right) - \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \mathcal{R} \right) \Big\} +$$

$$+ \frac{1}{\bar{r}_z^2} \Big[\left(\frac{\eta_{\bar{o}}^2}{10} (\partial_\parallel \mathcal{R})_{\bar{o}} - \frac{\eta}{2} \left(\left(\frac{\eta}{5} \partial_\parallel \mathcal{R} + \frac{3}{5} \mathcal{R} \right) \Big|_o^z + \frac{\eta - \eta_{\bar{o}}}{\eta} \frac{2\mathcal{R}_{\bar{o}}}{5} \right) - \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \mathcal{R} \right)^2 -$$

$$- \left(\frac{\eta_{\bar{o}}^2}{10} (\partial_\theta \mathcal{R})_{\bar{o}} - \frac{\bar{r}_z}{5} (\eta \partial_\theta \mathcal{R})_{\bar{o}} + \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \, \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}} \right) \frac{\partial \mathcal{R}}{\partial \theta} \right)^2 -$$

$$- \frac{1}{2} \left(\frac{\eta_{\bar{o}}^2}{10} (\partial_\phi \mathcal{R})_{\bar{o}} - \frac{\bar{r}_z}{5} (\eta \partial_\phi \mathcal{R})_{\bar{o}} + \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \, \left(\frac{\bar{r}_z - \bar{r}}{\bar{r} \sin\theta} \right) \frac{\partial \mathcal{R}}{\partial \phi} \right)^2 \Big] -$$

$$- \frac{1}{\bar{r}_z} \Big[\left(\left(\frac{\eta}{5} \partial_\theta \mathcal{R} \right) \Big|_{\bar{o}}^z + \frac{6}{5} \theta^\alpha \int_0^{\bar{r}_z} d\bar{r} \partial_\alpha \mathcal{R} \right) \frac{\partial}{\partial \theta} + \frac{1}{\sin\theta} \left(\left(\frac{\eta}{5} \partial_\phi \mathcal{R} \right) \Big|_{\bar{o}}^z + \frac{6}{5} \phi^\alpha \mathcal{R} \int_0^{\bar{r}_z} d\bar{r} \partial_\alpha \right) \frac{\partial}{\partial \phi} \Big]$$

$$\Big\{ \frac{\eta_{\bar{o}}^2}{10} (\partial_\parallel \mathcal{R})_{\bar{o}} - \frac{\eta}{2} \left(\left(\frac{\eta}{5} \partial_\parallel \mathcal{R} + \frac{3}{5} \mathcal{R} \right) \Big|_o^z + \frac{\eta - \eta_{\bar{o}}}{\eta} \frac{2\mathcal{R}_{\bar{o}}}{5} \right) - \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \mathcal{R} \Big\} +$$

$$+ \frac{1}{\bar{r}_z} \left(\frac{\eta_{\bar{o}}^2}{10} (\partial_\theta \mathcal{R})_{\bar{o}} - \frac{\bar{r}_z}{5} (\eta \partial_\theta \mathcal{R})_{\bar{o}} + \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \, \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}} \right) \frac{\partial \mathcal{R}}{\partial \theta} \right) \left(\left(\frac{\eta}{5} \partial_\theta \mathcal{R} \right) \Big|_{\bar{o}}^z + \frac{6}{5} \theta^\alpha \int_0^{\bar{r}_z} d\bar{r} \partial_\alpha \mathcal{R} \right) +$$

$$+ \frac{1}{\bar{r}_z} \left(\frac{\eta_{\bar{o}}^2}{10} (\partial_\phi \mathcal{R})_{\bar{o}} - \frac{\bar{r}_z}{5} (\eta \partial_\phi \mathcal{R})_{\bar{o}} + \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \, \left(\frac{\bar{r}_z - \bar{r}}{\bar{r} \sin\theta} \right) \frac{\partial \mathcal{R}}{\partial \phi} \right) \left(\left(\frac{\eta}{5} \partial_\phi \mathcal{R} \right) \Big|_{\bar{o}}^z + \frac{6}{5} \phi^\alpha \mathcal{R} \int_0^{\bar{r}_z} d\bar{r} \partial_\alpha \mathcal{R} \right) +$$

$$+ \frac{1}{\bar{r}_z^2} \left[\left(\frac{\eta_{\bar{o}}^2}{10} \left(\partial_\theta \mathcal{R} \right)_{\bar{o}} - \frac{\bar{r}_z}{5} \left(\eta \partial_\theta \mathcal{R} \right)_{\bar{o}} + \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \, \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}} \right) \frac{\partial \mathcal{R}}{\partial \theta} \right) \frac{\partial}{\partial \theta} + $$

$$+ \frac{1}{\sin\theta} \left(\frac{\eta_{\bar{o}}^2}{10} \left(\partial_\phi \mathcal{R} \right)_{\bar{o}} - \frac{\bar{r}_z}{5} \left(\eta \partial_\phi \mathcal{R} \right)_{\bar{o}} + \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \, \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}\sin\theta} \right) \frac{\partial \mathcal{R}}{\partial \phi} \right) \frac{\partial}{\partial \phi} \right] \left\{ \frac{\eta_{\bar{o}}^2}{10} \left(\partial_\parallel \mathcal{R} \right)_{\bar{o}} - \right.$$

$$- \frac{\eta}{2} \left(\left(\frac{\eta}{5} \partial_\parallel \mathcal{R} + \frac{3}{5} \mathcal{R} \right) \Big|_o^z + \frac{\eta - \eta_{\bar{o}}}{\eta} \frac{2\mathcal{R}_{\bar{o}}}{5} \right) - \frac{6}{5} \int_0^{\bar{r}_z} d\bar{r} \mathcal{R} \Bigg\} + $$

$$+ \left(\frac{1}{5} \left(\eta \partial_\parallel \mathcal{R} - 3\mathcal{R} \right)_{\bar{o}} + \frac{2}{5} \mathcal{R}_{\bar{o}} \right)^2 + \left(\eta \partial_\parallel \mathcal{R} + 3\mathcal{R} \right) \frac{2}{25} \mathcal{R}_{\bar{o}} + $$

$$+ \frac{1}{25} \left(\eta \partial_\parallel \mathcal{R} - 21\mathcal{R} \right) \left(\left(\eta \partial_\parallel \mathcal{R} - 3\mathcal{R} \right)_{\bar{o}} + 2\mathcal{R}_{\bar{o}} \right) + $$

$$+ 4 \left(\frac{\eta}{5} \partial_\parallel \mathcal{R} + \frac{6}{5} \mathcal{R} - \frac{1}{5} \left(\eta \partial_\parallel \mathcal{R} + 3\mathcal{R} \right)_{\bar{o}} - \frac{2}{5} \mathcal{R}_{\bar{o}} \right) \left\{ -\frac{6}{5\bar{r}_z} \int_0^{\bar{r}_z} d\bar{r} \mathcal{R} - \frac{\eta}{2\bar{r}_z} \left[\frac{1}{5} \left(\eta \partial_\parallel \mathcal{R} + 3\mathcal{R} \right) \Big|_o^z - \right. \right.$$

$$\left. - \frac{\eta - \eta_{\bar{o}}}{\eta} \frac{2\mathcal{R}_{\bar{o}}}{5} \right] + \frac{3}{5} \int_o^{\bar{r}_z} d\bar{r} \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}_z \bar{r}} \right) \left(\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \mathcal{R} \right\}$$

$$+ 2\Delta x^\mu \partial_\mu \left(\frac{\eta}{5} \partial_\parallel \mathcal{R} + \frac{3}{5} \mathcal{R} + \frac{3}{5} n^\alpha \int_0^{\bar{r}_z} d\bar{r} \partial_\alpha \mathcal{R} \right). \quad (4.22)$$

4.2 Monopole Calculation

It is evident that expression (4.22) exhibits angular dependence. This arises because metric perturbations generally depend on coordinates. However, due to the random nature of these perturbations, this formula does not provide specific information about our Universe. What is truly interesting are the statistical quantities, such as the luminosity distance monopole.

4.2.1 What is a Monopole?

The observed luminosity distance anisotropy could be decomposed in terms of spherical harmonics $Y_{lm}(\hat{n})$ as:

$$\delta A(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n}), \quad (4.23)$$

and the multipole coefficients are:

$$a_{lm} = \int d^2 \hat{n} \, Y_{lm}^*(\hat{n}) \, \delta A(\hat{n}). \quad (4.24)$$

The monopole is the contribution that arises from the a_{00} coefficient. Since Y_{00} is just a constant, we will understand the monopole as the angular average $\langle \delta A \rangle_\Omega$:

$$\langle \delta A \rangle_\Omega = \int \frac{d\hat{n}^2}{4\pi} \delta A. \tag{4.25}$$

The total luminosity distance can be written as:

$$\mathcal{D}_L = \bar{\mathcal{D}}_L \left[1 + \delta A \right], \tag{4.26}$$

where the angular average of this equation leads to

$$\langle \mathcal{D}_L \rangle_\Omega = \bar{\mathcal{D}}_L \left[1 + \langle \delta A \rangle_\Omega \right]. \tag{4.27}$$

This averaging simply gives $\bar{\mathcal{D}}_L$ at the linear order in perturbation theory because, from our theory of initial condition, the mean of every fluctuation is zero. That is the reason why we need second-order calculations, as they generate a nonvanishing prediction for the expectation value of the luminosity distance fluctuations.

4.2.2 Contributions to the Monopole

Let us examine some of the terms in (4.22) to qualitatively describe their contribution to the monopole.

First, consider the terms proportional to $\mathcal{R}^2$. Using the Fourier transformation, we can express the following:

$$\mathcal{R}(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \mathcal{R}(\mathbf{k}). \tag{4.28}$$

This implies that the product $\mathcal{R}^2$ can be written as:

$$\int d^3\mathbf{k} \int d^3\mathbf{k}'\, e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \mathcal{R}(\mathbf{k}) \mathcal{R}(\mathbf{k}'). \tag{4.29}$$

If we take an ensemble average, this results in the following relation:

$$\int d^3\mathbf{k} \int d^3\mathbf{k}'\, e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \langle \mathcal{R}(\mathbf{k}) \mathcal{R}(\mathbf{k}') \rangle_\Omega = \int d^3\mathbf{k} \int d^3\mathbf{k}'\, e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \delta_D(\mathbf{k}+\mathbf{k}') P_\mathcal{R}(k), \tag{4.30}$$

where P denotes the power spectrum. Typically, the power spectrum is parameterized as a power law, and to get the scale-invariant prediction from standard inflation theory, it should be proportional to k^{-3}, leading to

$$\int d^3\mathbf{k}\, P_\mathcal{R}(k) \propto \int_0^\infty k^2 \frac{1}{k^3}\, dk = \int_0^\infty \frac{dk}{k}, \tag{4.31}$$

which is clearly divergent. Therefore, these terms require special treatment, which will be discussed later.

A similar approach can be applied when calculating the $\mathcal{R}_{\bar{o}}\mathcal{R}_{\bar{o}}$ terms. However, the terms proportional to $\mathcal{R}\mathcal{R}_{\bar{o}}$ are more intricate. Considering $x_{\bar{o}} = 0$, we can write:

$$\int d^3\mathbf{k} d^3\mathbf{k}' \, e^{i\mathbf{k}\cdot\mathbf{x}} \langle \mathcal{R}(\mathbf{k}) \mathcal{R}(\mathbf{k}') \rangle \approx \int d^3\mathbf{k} d^3\mathbf{k}' \, (1 + i\mathbf{k}\cdot\mathbf{x} + ...) \langle \mathcal{R}(\mathbf{k}) \mathcal{R}(\mathbf{k}') \rangle . \quad (4.32)$$

Of course, the second term contains an additional factor of k, but the first term remains divergent.

The next step will be to take a look at those terms that have the line of sight derivatives $RR_{\bar{o}} \left(\partial_\parallel \mathcal{R} \right)_{\bar{o}}$. Beginning with the same transformation:

$$\int \left\{ \int d^3\mathbf{k} \int d^3\mathbf{k}' \mathcal{R}(\mathbf{k}) \, n^\alpha \frac{\partial}{\partial x^\alpha} \left(e^{i\mathbf{k}'\cdot\mathbf{x}} \mathcal{R}(\mathbf{k}') \right)_{\bar{o}} \right\} d\Omega =$$
$$= \int \left\{ \int d^3\mathbf{k} \int d^3\mathbf{k}' \mathcal{R}(\mathbf{k}) \, n^\alpha k'_\alpha \mathcal{R}(\mathbf{k}') \right\} d\Omega, \quad (4.33)$$

one may rewrite the dot product $n^\alpha k'_\alpha$ using Legendre polynomials as:

$$n^\alpha k'_\alpha = |k'| \, \hat{n} \cdot \hat{k}' = |k'| \, P_1 \left(\hat{n} \cdot \hat{k}' \right), \quad (4.34)$$

and using well-known relation between Legendre polynomials and spherical harmonics:

$$n^\alpha k'_\alpha = |k'| \frac{4\pi}{3} \sum_m Y_{1m}(\hat{n}) Y^*_{1m} \left(\hat{k}' \right). \quad (4.35)$$

The only thing that remains in integration over angles is:

$$\sum_m \int d\Omega Y_{1m} \propto \sum_m \int d\Omega Y_{1m} Y^*_{00} = 0 \quad (4.36)$$

due to the orthogonality of spherical harmonics with different l. It is easy to see that if we integrated over $d\Omega Y^*_{1m'}$, there will be a non-zero contribution. So, one can say that such terms contribute only to dipole.

While terms proportional to $R_{\bar{o}} \left(R_{,\parallel} \right)_{\bar{o}}$ contribute only to the dipole and therefore drop out in the monopole computation, terms proportional to $\left(R_{,\parallel} \right)_{\bar{o}} \left(R_{,\parallel} \right)_{\bar{o}}$ have a non-vanishing contribution to the monopole, as it is expected from the composition

of two angular momenta with $l = 1$. Explicitly:

$$\int \left\{ \int d^3\mathbf{k} \int d^3\mathbf{k}' n^\beta \frac{\partial}{\partial x^\beta} \left(e^{i\mathbf{k}\mathbf{x}} \mathcal{R}(\mathbf{k}) \right)_{\bar{o}} n^\alpha \frac{\partial}{\partial x^\alpha} \left(e^{i\mathbf{k}'\mathbf{x}} \mathcal{R}(\mathbf{k}') \right)_{\bar{o}} \right\} d\Omega =$$

$$= \int \left\{ \int d^3\mathbf{k} \int d^3\mathbf{k}' n^\beta k_\beta \mathcal{R}(\mathbf{k}) n^\alpha k'_\alpha \mathcal{R}(\mathbf{k}') \right\} d\Omega \propto$$

$$\propto \sum_m \sum_{m'} \int Y_{1m} Y_{1m'} d\Omega. \quad (4.37)$$

The final integral is recognized as a Wigner 3-j symbol. Integrals of the form:

$$\int Y_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3} \, d\Omega \quad (4.38)$$

will yield a non-zero contribution if the following conditions are satisfied:

$$\begin{cases} m_1 + m_2 + m_3 = 0, \\ |l_1 - l_2| \leq l_3 \leq l_1 + l_2. \end{cases} \quad (4.39)$$

Thus, for the specific case where $l_1 = 1$ and $l_2 = 1$, the possible values of l_3 are 0, 1, and 2. This implies that such terms contribute to the monopole, dipole, and quadrupole.

Also, one may show how to simplify $\mathcal{R}_{\bar{o}} \partial_\| \mathcal{R}$. To do this, we again refer to:

$$\int \left\{ \int d^3\mathbf{k} \int d^3\mathbf{k}' \mathcal{R}(\mathbf{k}) n^\alpha \frac{\partial}{\partial x^\alpha} \left(e^{i\mathbf{k}'\mathbf{x}} \mathcal{R}(\mathbf{k}') \right) \right\} d\Omega =$$

$$= \int \left\{ \int d^3\mathbf{k} \int d^3\mathbf{k}' \mathcal{R}(\mathbf{k}) n^\alpha k'_\alpha \mathcal{R}(\mathbf{k}') e^{i\mathbf{k}'\mathbf{x}} \right\} d\Omega \quad (4.40)$$

To proceed further, one may use decomposition of $e^{i\mathbf{k}\mathbf{x}}$ as:

$$e^{i\mathbf{k}\mathbf{x}} = \frac{4}{\pi} \sum_{l,m} i^l j_l(|k| |x|) Y_{lm}(\hat{n}) Y_{lm}^*\left(\hat{k}\right), \quad (4.41)$$

so the only thing that is left in angular integration is:

$$\sum_{l,m} \sum_{m'} \int d\Omega Y_{lm} Y_{1,m'}. \quad (4.42)$$

From the orthogonality condition one may say that only terms with $l = 1$, will survive. The very similar logic can be applied to $\mathcal{R} \left(\partial_\| \mathcal{R} \right)_{\bar{o}}$.

4.2.3 Further Problems and Complications

Direct computation of the angular average for (4.22) is a rather complex task, both analytically and numerically. The most challenging aspects in proceeding further can be highlighted by the following factors:

- **Terms that can cause divergences.** As we demonstrated in the section above, the terms proportional to $\mathcal{R}\mathcal{R}$, $\mathcal{R}_{\bar{o}}\mathcal{R}_{\bar{o}}$, and $\mathcal{R}_{\bar{o}}\mathcal{R}$ result in divergent integrals.

- **Linear second-order terms.** The contribution of linear second-order terms in (4.22) is not clear. It also can be somehow written in terms of $\mathcal{R}$, and its decomposition will have contributions proportional to terms that have been listed above. However, it is necessary to solve the Einstein equations in the second order of perturbations to know the exact prefactors.

Despite these complications, this problem has the potential to be solved. As noted above, solving Einstein's equations and considering linear second-order terms can fix the issue of divergent terms. This, in turn, will make it possible to directly obtain values for the shift in the Luminosity Distance from the second-order relativistic perturbation theory.

Chapter 5

Conclusions and Outlook

This work investigates the shift in Luminosity Distance using second-order relativistic perturbation theory. Analytical expressions for second-order distortions in the physical area of an Einstein-de Sitter Universe were derived.

Starting from the generic form of the perturbed FLRW metric (1.29), we derived the observed redshift, source position, physical volume, and area occupied by the source, relating them to luminosity distance. Initially, we performed calculations using the first order of perturbation theory to verify the results against those previously obtained in the literature.

We then considered the zero-shear temporal gauge combined with the spatial C-gauge as our gauge choice. In this gauge, metric-related quantities, time-lapse, spatial shift, wavevector distortion, observed redshift, and source position distortion were presented. Using these intermediate steps, we obtained the distortion matrix perturbation and the perturbation in the physical area.

For the perturbation in the considered gauge, we utilized first-order quantities derived earlier in this thesis. This led us to second-order expressions in terms of only metric quantities, which had not been presented in the literature before.

Next, we selected the flat matter-only FLRW metric limit, known as the Einstein-de Sitter Universe, to derive an expression ready for further statistical analysis. We identified several issues that persist in this area of study and highlighted potential directions for future research.

The obtained equation can be used for the direct calculation of the monopole contribution to the luminosity distance. Achieving this result and its subsequent

analysis may shed light on the current description of our Universe. In the future, it might allow the exclusion of the controversial idea of "dark energy" from our current model of the Universe's expansion. This prospect is incredibly exciting and deserves a separate investigation.

Bibliography

[1] S. Perlmutter et al. "Discovery of a supernova explosion at half the age of the Universe". In: *Nature* 391.6662 (Jan. 1998), pp. 51–54. ISSN: 1476-4687. DOI: 10.1038/34124. URL: http://dx.doi.org/10.1038/34124.

[2] Adam G. Riess et al. "Type Ia Supernova Discoveries atzgt; 1 from theHubble Space Telescope: Evidence for Past Deceleration and Constraints on Dark Energy Evolution". In: *The Astrophysical Journal* 607.2 (June 2004), pp. 665–687. ISSN: 1538-4357. DOI: 10.1086/383612. URL: http://dx.doi.org/10.1086/383612.

[3] L. Smolin. *The Trouble with Physics: The Rise of String Theory, the Fall of a Science, and what Comes Next.* A Mariner Book. Houghton Mifflin, 2006. ISBN: 9780618551057. URL: https://books.google.ch/books?id=UVzdfHF_phIC.

[4] J. B. Hartle and S. W. Hawking. "Wave function of the Universe". In: *Phys. Rev. D* 28 (12 Dec. 1983), pp. 2960–2975. DOI: 10.1103/PhysRevD.28.2960. URL: https://link.aps.org/doi/10.1103/PhysRevD.28.2960.

[5] Stephen W. Hawking and Thomas Hertog. "A smooth exit from eternal inflation?" In: *Journal of High Energy Physics* 2018.4 (2018), p. 147. DOI: 10.1007/JHEP04(2018)147. URL: https://doi.org/10.1007/JHEP04(2018)147.

[6] Valery A. Rubakov and Dmitry S. Gorbunov. *Introduction to the Theory of the Early Universe: Hot big bang theory.* Singapore: World Scientific, 2017. ISBN: 978-981-320-987-9, 978-981-320-988-6, 978-981-322-005-8. DOI: 10.1142/10447.

[7] Matteo Magi and Jaiyul Yoo. "Second-order gauge-invariant formalism for the cosmological observables: complete verification of their gauge-invariance". In: *Journal of Cosmology and Astroparticle Physics* 2022.09 (Sept. 2022), p. 071. DOI: 10.1088/1475-7516/2022/09/071. URL: https://dx.doi.org/10.1088/1475-7516/2022/09/071.

[8] Edward W. Kolb et al. "Effect of inhomogeneities on the expansion rate of the universe". In: *Physical Review D* 71.2 (Jan. 2005). ISSN: 1550-2368. DOI: 10.1103/physrevd.71.023524. URL: http://dx.doi.org/10.1103/PhysRevD.71.023524.

[9] ALESSIO NOTARI. "LATE TIME FAILURE OF FRIEDMANN EQUATION". In: *Modern Physics Letters A* 21.40 (Dec. 2006), pp. 2997–3007. ISSN: 1793-6632. DOI: 10.1142/s0217732306021852. URL: http://dx.doi.org/10.1142/S0217732306021852.

[10] Edward W Kolb, Sabino Matarrese, and Antonio Riotto. "On cosmic acceleration without dark energy". In: *New Journal of Physics* 8.12 (Dec. 2006), pp. 322–322. ISSN: 1367-2630. DOI: 10.1088/1367-2630/8/12/322. URL: http://dx.doi.org/10.1088/1367-2630/8/12/322.

[11] Enrico Barausse, Sabino Matarrese, and Antonio Riotto. "Effect of inhomogeneities on the luminosity distance-redshift relation: Is dark energy necessary in a perturbed universe?" In: *Physical Review D* 71.6 (Mar. 2005). ISSN: 1550-2368. DOI: 10.1103/physrevd.71.063537. URL: http://dx.doi.org/10.1103/PhysRevD.71.063537.

[12] Shishir Sankhyayan et al. *Identification of Superclusters and their Properties in the Sloan Digital Sky Survey Using WHL Cluster Catalog.* 2023. arXiv: 2309.06251 [astro-ph.CO]. URL: https://arxiv.org/abs/2309.06251.

[13] R. Brent Tully et al. "The Laniakea supercluster of galaxies". In: *Nature* 513.7516 (Sept. 2014), pp. 71–73. ISSN: 1476-4687. DOI: 10.1038/nature13674. URL: http://dx.doi.org/10.1038/nature13674.

[14] Elizabeth Gibney. "Earth's new address: 'Solar System, Milky Way, Laniakea'". In: *Nature* (Sept. 2014). ISSN: 1476-4687. DOI: 10.1038/nature.2014.15819. URL: https://doi.org/10.1038/nature.2014.15819.

[15] Dominique Proust et al. "The Shapley Supercluster: the Largest Matter Concentration in the Local Universe". In: *The Messenger* 124 (June 2006), p. 30.

[16] *Sloan Digital Sky Survey.* https://www.sdss.org/. Accessed: (Use the date of access).

[17] Chris Stoughton et al. "Sloan Digital Sky Survey: Early Data Release". In: *Astron. J.* 123.1 (Jan. 2002), pp. 485–548. DOI: 10.1086/324741.

[18] V. M. Slipher. "The radial velocity of the Andromeda Nebula". In: *Lowell Observatory Bulletin* 2.8 (Jan. 1913), pp. 56–57.

[19] V. M. Slipher. "Radial velocity observations of spiral nebulae". In: *The Observatory* 40 (Aug. 1917), pp. 304–306.

[20] Edwin Hubble. "A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae". In: *Proceedings of the National Academy of Science* 15.3 (Mar. 1929), pp. 168–173. DOI: 10.1073/pnas.15.3.168.

[21] M. L. Humason, N. U. Mayall, and A. R. Sandage. "Redshifts and magnitudes of extragalactic nebulae." In: *Astron. J.* 61 (Jan. 1956), pp. 97–162. DOI: 10.1086/107297.

[22] Allan Sandage. "Current Problems in the Extragalactic Distance Scale." In: *Astrophys. J.* 127 (May 1958), p. 513. DOI: 10.1086/146483.

[23] SIDNEY VAN DEN BERGH. "Extra-galactic Distance Scale". In: *Nature* 225.5232 (Feb. 1970), pp. 503–505. ISSN: 1476-4687. DOI: 10.1038/225503a0. URL: https://doi.org/10.1038/225503a0.

[24] G. de Vaucouleurs and W. L. Peters. "The Extragalactic Distance Scale Derived from "Sosie" Galaxies. II. Solar Motion and Hubble Constant". In: *Astrophys. J.* 303 (Apr. 1986), p. 19. DOI: 10.1086/164048.

[25] Adam G. Riess et al. *Crowded No More: The Accuracy of the Hubble Constant Tested with High Resolution Observations of Cepheids by JWST*. 2023. arXiv: 2307.15806 [astro-ph.CO]. URL: https://arxiv.org/abs/2307.15806.

[26] Adam G. Riess et al. "A Comprehensive Measurement of the Local Value of the Hubble Constant with 1 km s1 Mpc1 Uncertainty from the Hubble Space Telescope and the SH0ES Team". In: *The Astrophysical Journal Letters* 934.1 (July 2022), p. L7. DOI: 10.3847/2041-8213/ac5c5b. URL: https://dx.doi.org/10.3847/2041-8213/ac5c5b.

[27] J. A. Wheeler. "Geometrodynamics and the issue of final state". In: *Les Houches Summer Shcool of Theoretical Physics: Relativity, Groups and Topology*. 1964, pp. 317–522.

[28] W.A. Zajc. "The Fluid Nature of Quark-Gluon Plasma". In: *Nuclear Physics A* 805.1–4 (June 2008), pp. 283c–294c. ISSN: 0375-9474. DOI: 10.1016/j.nuclphysa.2008.02.285. URL: http://dx.doi.org/10.1016/j.nuclphysa.2008.02.285.

[29] T. Prusti et al. "TheGaiamission". In: *Astronomy amp; Astrophysics* 595 (Nov. 2016), A1. ISSN: 1432-0746. DOI: 10.1051/0004-6361/201629272. URL: http://dx.doi.org/10.1051/0004-6361/201629272.

[30] D. W. Evans et al. "Gaia Data Release 2: Photometric content and validation". In: *Astronomy amp; Astrophysics* 616 (Aug. 2018), A4. ISSN: 1432-0746. DOI: 10.1051/0004-6361/201832756. URL: http://dx.doi.org/10.1051/0004-6361/201832756.

[31] J. D. Fernie. "THE PERIOD-LUMINOSITY RELATION: A HISTORICAL REVIEW". In: *Publications of the Astronomical Society of the Pacific* 81.483 (Dec. 1969), p. 707. DOI: 10.1086/128847. URL: https://dx.doi.org/10.1086/128847.

[32] Adam G. Riess et al. "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant". In: *The Astronomical Journal* 116.3 (Sept. 1998), pp. 1009–1038. ISSN: 0004-6256. DOI: 10.1086/300499. URL: http://dx.doi.org/10.1086/300499.

[33] S. Perlmutter et al. "Measurements of and from 42 High-Redshift Supernovae". In: *The Astrophysical Journal* 517.2 (June 1999), pp. 565–586. ISSN: 1538-4357. DOI: 10.1086/307221. URL: http://dx.doi.org/10.1086/307221.

[34] Jacques Colin et al. "Evidence for anisotropy of cosmic acceleration". In: *Astronomy amp; Astrophysics* 631 (Nov. 2019), p. L13. ISSN: 1432-0746. DOI: 10.1051/0004-6361/201936373. URL: http://dx.doi.org/10.1051/0004-6361/201936373.

[35] Karim A Malik and David R Matravers. "A concise introduction to perturbation theory in cosmology". In: *Classical and Quantum Gravity* 25.19 (Sept. 2008), p. 193001. ISSN: 1361-6382. DOI: 10.1088/0264-9381/25/19/193001. URL: http://dx.doi.org/10.1088/0264-9381/25/19/193001.

[36] V. F. Mukhanov, L. Raul W. Abramo, and Robert H. Brandenberger. "Backreaction Problem for Cosmological Perturbations". In: *Physical Review Letters* 78.9 (Mar. 1997), pp. 1624–1627. ISSN: 1079-7114. DOI: 10.1103/physrevlett.78.1624. URL: http://dx.doi.org/10.1103/PhysRevLett.78.1624.

[37] Hyerim Noh and Jai-chan Hwang. "Second-order perturbations of the Friedmann world model". In: *Physical Review D* 69.10 (May 2004). ISSN: 1550-2368. DOI: 10.1103/physrevd.69.104011. URL: http://dx.doi.org/10.1103/PhysRevD.69.104011.

[38] James M. Bardeen, ed. *Cosmological perturbations from quantum fluctuations to large scale structure*. Jan. 1988.

[39] Ermis Mitsou and Jaiyul Yoo. *Tetrad Formalism for Exact Cosmological Observables*. Springer International Publishing, 2020. ISBN: 9783030500399. DOI: 10.1007/978-3-030-50039-9. URL: http://dx.doi.org/10.1007/978-3-030-50039-9.

[40]   Jaiyul Yoo et al. "Gauge-invariant formalism of cosmological weak lensing". In: *Journal of Cosmology and Astroparticle Physics* 2018.04 (Apr. 2018), pp. 029–029. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2018/04/029. URL: http://dx.doi.org/10.1088/1475-7516/2018/04/029.

[41]   R.M. Wald. *General Relativity*. University of Chicago Press, 2010. ISBN: 9780226870373. URL: https://books.google.ch/books?id=9S-hzg6-moYC.

[42]   Jaiyul Yoo and Matias Zaldarriaga. "Beyond the linear-order relativistic effect in galaxy clustering: Second-order gauge-invariant formalism". In: *Physical Review D* 90.2 (July 2014). ISSN: 1550-2368. DOI: 10.1103/physrevd.90.023513. URL: http://dx.doi.org/10.1103/PhysRevD.90.023513.