Exercise 1. ***Free radiation field***

The radiation field can be written as

$$\vec{A}(\vec{r},t) = \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_\lambda \sqrt{\frac{2\pi\hbar c^2}{\omega_k}} \left(a_{k,\lambda}\vec{\varepsilon}_{k,\lambda}\, e^{i\vec{k}\cdot\vec{r}-i\omega_k t} + a^*_{k,\lambda}\vec{\varepsilon}^{\,*}_{k,\lambda}\, e^{-i\vec{k}\cdot\vec{r}+i\omega_k t} \right)$$

1. Compute the Hamiltonian $H = \frac{1}{8\pi}\int d^3\vec{r}\,(\vec{E}^2 + \vec{B}^2)$
2. Compute the Poynting vector $\vec{P} = \frac{1}{4\pi c}\int d^3\vec{r}\,\vec{E}\times\vec{B}$

Exercise 2. ***Polarization vectors***

Assuming the wave vector points along the z-axis, $\vec{k} = (0,0,k)$, common choices for the polarization vectors $\vec{\varepsilon}(\vec{k},\lambda) = \vec{\varepsilon}_{k,\lambda}$ are:
circular polarization: $\vec{\varepsilon}_{k,+} = (-1,-i,0)/\sqrt{2}$ $\quad\vec{\varepsilon}_{k,-} = (1,-i,0)/\sqrt{2}$
linear polarization: $\vec{\varepsilon}_{k,1} = (1,0,0)$; $\quad\vec{\varepsilon}_{k,2} = (0,1,0)$;

1. Show
$$\vec{\varepsilon}^{\,*}(\vec{k},\lambda)\cdot\vec{\varepsilon}(\vec{k},\lambda') = \delta_{\lambda\lambda'}$$
2. Show
$$\sum_\lambda \varepsilon^*_i(\vec{k},\lambda)\varepsilon_j(\vec{k},\lambda) = \delta_{ij} - \frac{k_i\,k_j}{k^2}$$

Exercise 3. ***Commutation relations***

The Lagrangian density of the free radiation field is given by $\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}$ where $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field strength tensor.

1. Show that the canonical momentum to $\vec{A}$ is given by $\vec{\pi} = -\frac{1}{4\pi c}\vec{E}$
2. After quantisation of the radiation field, both $\vec{A}$ and $\vec{\pi}$ are operators. Show that the commutation relations
$$[\hat{a}(\vec{k},\lambda), \hat{a}^\dagger(\vec{k}',\lambda')] = (2\pi)^3\delta(\vec{k}-\vec{k}')\,\delta_{\lambda\lambda'}$$
$$[\hat{a}(\vec{k},\lambda), \hat{a}(\vec{k}',\lambda')] = [\hat{a}^\dagger(\vec{k},\lambda), \hat{a}^\dagger(\vec{k}',\lambda')] = 0$$
result in equal-time commutation relations
$$[\hat{A}_i(\vec{x},t), \hat{\pi}_j(\vec{y},t)] = i\hbar \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right)$$
How would these relations simplify if there were three polarizations of the radiation field? Interpret the results.