FOURIER SERIES

Trigonometric polynomials

A trigonometric polynomial is a function $f: \mathbb{R} \to \mathbb{C}$ of the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \qquad a_i, b_i \in \mathbb{C}$$

↑ real representation

Is there a way to compute the a_i and b_i coefficients in terms of $f(x)$?

We can use the relations $\cos nx = \dfrac{e^{inx} + e^{-inx}}{2}$ $\quad \sin nx = \dfrac{e^{inx} - e^{-inx}}{2i}$

and exploit the relation

$$\int_{-\pi}^{\pi} e^{imx} e^{-imx}\, dx = 2\pi\, \delta_{mm} \qquad \text{where } \delta_{mm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

In this way we find

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx\, dx = \delta_{nm} \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = \delta_{nm}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \sin mx\, dx = 0$$

$\Rightarrow$ we can multiply the definition of $f(x)$ by $\cos nx$ or $\sin nx$

and integrate over x and we get

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad n = 0, 1, 2 \dots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx \qquad n = 1, 2, 3, \dots$$

The real representation of the trigonometric polynomial

can be turned into

$$f(x) = \sum_{-N}^{N} c_m e^{imx} \qquad \text{complex representation}$$

with

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx$$

The above expression of c_m can be obtained as before by exploiting the orthogonality relation $\int_{-\pi}^{\pi} e^{imx} e^{-inx} = 2\pi \delta_{mn}$

The real and complex representations are fully equivalent: they both contain $2N+1$ independent coefficients:

$a_0, a_1 \dots a_N$

$b_1, b_2 \dots b_N$ $\qquad$ $c_{-N} \dots c_0 \dots c_N$

real rep. $\qquad$ complex rep.

Note that the structure resembles the one we have in linear spaces

$$\underline{f} = \sum_m c_m \underline{e_m} \qquad \underline{e_m} \sim e^{imx} \qquad \text{basis vector}$$

$$c_m = \underline{f} \cdot \underline{e}_m^* = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx$$

projection of f onto the basis vector $\underline{e_m}$

What happens when $N \to \infty$? We need more general linear spaces

(Hilbert spaces, see later)

Note that if $f(x)$ is real $\Rightarrow$ a_m and b_m are real

How about c_m?

If f is real we have $c_{-m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{imx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(x)\left(e^{-imx}\right)^* = c_m^*$

$\Rightarrow \quad c_m^* = +c_m$

- Recall : <u>function series</u>

From analysis 2 we know that a function series is a series where the summands are not just constants but functions.

$$s_m(x) = \sum_{k=1}^{m} f_k(x)$$

The function series is said to converge in $x = x_0$ if the series $\sum_{k=1}^{\infty} f_k(x_0)$ converges, that is $\exists \lim_{m\to\infty} s_m(x_0)$. This is called <u>POINTWISE CONVERGENCE</u>

<u>UNIFORM CONVERGENCE</u> is stronger :

If $\forall \varepsilon > 0 \ \exists N(\varepsilon) \mid x \in A, m > N \Rightarrow |S_m(x) - S(x)| < \varepsilon \Rightarrow$ the series is said to converge uniformly in A to $S(x)$

<u>Crucial point</u> : N depends only on ε and not on $x \in A$

<u>FOURIER SERIES</u>

We can formally consider the function series $\sum_{-\infty}^{+\infty} c_m e^{imx}$ (complex representation)

or $\frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx)$ (real representation)

⊛

In particular if a function $f: \mathbb{R} \to \mathbb{C}$ is integrable over the interval $[-\pi, \pi]$ $\left(f \in L^1_{[-\pi,\pi]}\right)$ and 2π periodic we can define the <u>Fourier coefficients</u> analogously to what done for the trigonometric polynomials

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx \qquad n=0,\pm1,\pm2\ldots$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad n=0,1\ldots \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad n=1,2\ldots$$

At this point the notation

$$f(x) \sim \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

simply means that the Fourier series corresponds to the function $f(x)$ (that is, the coefficients are computed as above) <u>but does not imply any assumption of convergence</u> (this will be discussed later)

For the moment we limit ourselves to note that the integrals defining the Fourier coefficients <u>do not depend on the point</u>. Indeed if $f(x)$ is 2π periodic we have

$$\int_{-\pi+a}^{\pi+a} f(u)\,du = \int_{-\pi}^{\pi} f(u)\,du$$

<u>Proof</u>

$$\int_{-\pi+a}^{\pi+a} f(u)\,du = \int_{-\pi+a}^{-\pi} f(u)\,du + \int_{-\pi}^{\pi} f(u)\,du + \int_{\pi}^{\pi+a} f(u)\,du$$

but

$$\int_{-\pi+a}^{-\pi} f(u)\,du = \int_{-\pi+a}^{-\pi} f(u+2\pi)\,du = \int_{\pi+a}^{\pi} f(t)\,dt = -\int_{\pi}^{\pi+a} f(u)\,du$$

$$\Rightarrow \int_{-\pi+a}^{\pi+a} f(u)\,du = \int_{-\pi}^{\pi} f(u)\,du$$

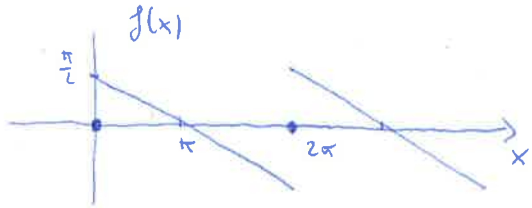
We also note that if the function $f(x)$ has a definite parity, its Fourier series contains either only $\cos(mx)$ or $\sin(mx)$.

If $f(x) = f(-x)$ we have $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{even}} \overbrace{\sin mx}^{\text{odd}}\, dx = 0$ EVEN

If $f(x) = -f(-x)$ we have $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{odd}} \overbrace{\cos mx}^{\text{even}}\, dx = 0$ ODD

EXAMPLE

$$f(x) = \begin{cases} \frac{\pi - x}{2} & 0 < x < 2\pi \\ 0 & x = 0 \end{cases}$$



We have $c_0 = 0$ $\quad c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-imx}\, dx = \frac{1}{2\pi}\int_{0}^{2\pi} f(x)\, e^{-imx} = \frac{1}{2mi}$

$\Rightarrow \quad f(x) \sim \sum_{m=-\infty}^{+\infty} \frac{e^{imx}}{2mi} \quad m \neq 0$ which can be written also as $\sum_{m=1}^{\infty} \frac{\sin mx}{m}$

(indeed $f(x)$ is odd)

+ Under which conditions the Fourier series converges indeed to the function $f(x)$?
Of course this happens trivially when $f(x)$ is a trigonometric polynomial.

We will show that if the function $f(x)$ is continuous there is an average (Fejer average) of the Fourier series which converges uniformly to $f(x)$. This is not sufficient to state the convergence of the Fourier series

example

If a sequence a_n converges, then its averages converge to the same limit

$$\lim_{n\to\infty} a_n = a \quad \Rightarrow \quad \lim_{n\to\infty} \frac{a_1 + a_2 + \dots a_n}{n} = a$$ but the opposite is not true!

$a_n = 1$ n odd, $= 0$ n even $\quad \lim_{n\to\infty} \frac{a_1 + a_2 + \dots a_n}{n} = \frac{1}{2}$ but $\lim_{n\to\infty} a_n$ does not exist!

Fejer average

For a continuous 2π-periodic function $f(x)$ we define the sums

$$S_m(x) = \frac{a_0}{2} + \sum_{k=1}^{m} (a_k \cos kx + b_k \sin kx)$$

a_i, b_i Fourier coefficients of $f(x)$

$$\sigma_m(x) = \frac{S_0(x) + \cdots S_m(x)}{m+1}$$

$\sigma_m(x)$ is called Fejer average

Fejer theorem

The Fejer average of a continuous 2π-periodic function $f(x)$ converges uniformly to $f(x)$

$$\lim_{m \to \infty} \sigma_m(x) = f(x)$$

Before discussing the proof of this theorem we state a corollary:

- A function which is continuous and 2π periodic is uniquely determined by its Fourier series. Indeed, suppose that $f(x)$ and $g(x)$ are continuous and 2π periodic and have the same Fourier series $\Rightarrow$ it follows from the Fejer theorem that they coincide

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The proof of the Fejer theorem can be done by using the Dirichlet and Fejer kernels

Dirichlet kernel

$$\boxed{D_m(x) \equiv \sum_{k=-m}^{m} e^{ikx}} = e^{-imx}(1 + e^{ix} + \cdots e^{2imx}) = e^{-imx} \frac{e^{i(2m+1)x} - 1}{e^{ix} - 1}$$

$$= \frac{e^{i(m+1)x} - e^{-imx}}{e^{ix} - 1} \frac{e^{-i\frac{x}{2}}}{e^{-i\frac{x}{2}}} = \frac{e^{i(m+\frac{1}{2})x} - e^{-i(m+\frac{1}{2})x}}{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}}$$

$$= \frac{\sin(m+\frac{1}{2})x}{\sin\frac{x}{2}}$$

Consider the sums $s_m(x)$ in the complex representation

$$s_m(x) = \sum_{k=-m}^{m} c_k e^{ikx} = \frac{1}{2\pi} \sum_{k=-m}^{m} \int_{-\pi}^{\pi} f(t)\, e^{-ikt} dt \cdot e^{ikx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \underbrace{\sum_{k=-m}^{m} e^{ik(x-t)}}_{D_m(x-t)} dt$$

So the Dirichlet kernel can be used to obtain the sum $s_m(x)$ through an integral over $f(x)$. We can also write

$$s_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u)\, D_m(u)\, du$$

(as seen before the integral over the period does not depend on the chosen interval!)

Fejer kernel

We define
$$\boxed{K_m(x) \equiv \frac{D_0(x) + D_1(x) + \dots D_m(x)}{m+1}}$$

we have

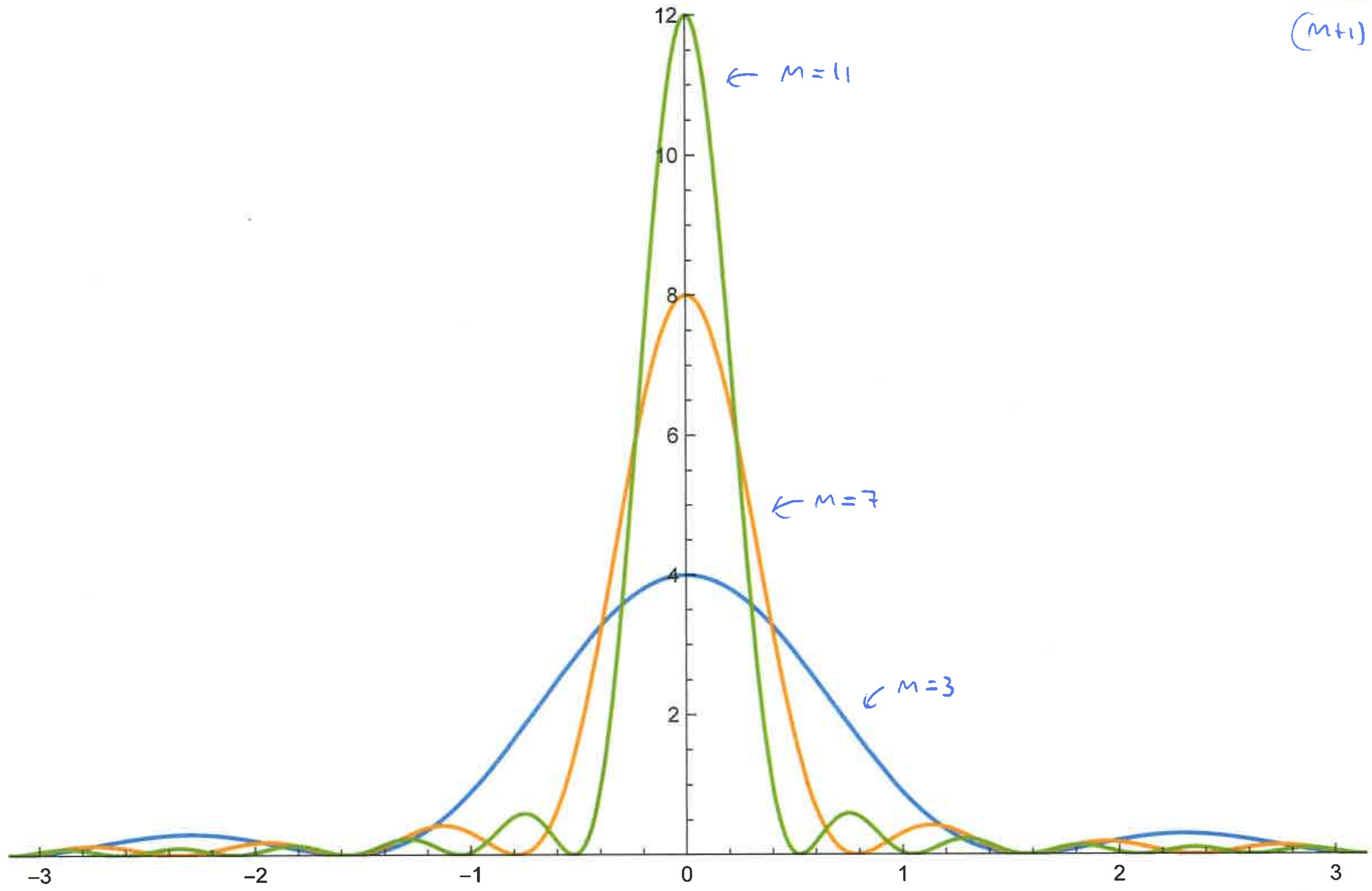
$$K_m(x) = \frac{\sin\frac{x}{2} + \sin\frac{3}{2}x + \dots \sin\frac{2m+1}{2}x}{(m+1)\sin\frac{x}{2}}$$

$$= \operatorname{Im} \frac{e^{i\frac{x}{2}} + e^{i\frac{3}{2}x} + \dots e^{i\frac{2m+1}{2}x}}{(m+1)\sin\frac{x}{2}} = \operatorname{Im} \frac{e^{i\frac{x}{2}}\left(1 + e^{ix} + \dots e^{imx}\right)}{(m+1)\sin\frac{x}{2}}$$

$$= \frac{1}{(m+1)\sin\frac{x}{2}} \operatorname{Im}\left[e^{i\frac{x}{2}} \frac{e^{i(m+1)x} - 1}{e^{ix} - 1}\right] = \frac{1}{(m+1)\sin\frac{x}{2}} \operatorname{Im}\left[\frac{e^{i(m+1)x} - 1}{2i\sin\frac{x}{2}}\right]$$

$$= \frac{1}{2(m+1)\sin^2\frac{x}{2}} \left(1 - \cos(m+1)x\right) = \frac{2\sin^2\frac{m+1}{2}x}{2(m+1)\sin^2\frac{x}{2}}$$

$$K_m(x) = \frac{\sin^2\left(\frac{m+1}{2}x\right)}{(m+1)\sin^2\frac{x}{2}}$$



We now see that $K_M(x)$ allows us to write the Fejer average in a compact way. We have

$$\sigma_M(x) = \frac{s_0(x) + \dots s_M(x)}{M+1}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\,\frac{D_0(u) + \dots D_M(u)}{M+1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\,K_M(u)\,du$$

$K_M(u)$ is more and more peaked around $u=0$ as $M \to \infty$ (see figure)

$\Rightarrow$ $\boxed{K_M(u) \text{ weights } f \text{ around } x \text{ and } \sigma_M \sim f}$

To prove the Fejer theorem we need to show the following properties of $K_M(x)$

1) $K_M(x) \geq 0 \quad \forall x, \forall M$

2) $\displaystyle\int_{-\pi}^{\pi} K_M(x)\,dx = 2\pi \quad \forall M$

3) For each $\varepsilon, \delta > 0$ $\exists\, N(\varepsilon, \delta)$ such that

$$\left.\begin{array}{l} \delta \leq |x| \leq \pi \\ M > N \end{array}\right\} \Rightarrow \quad |K_M(x)| < \varepsilon$$

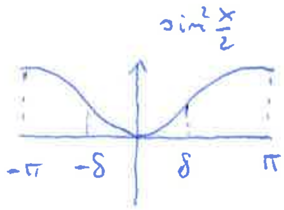
Property 1) is obvious from the expression of $K_M(x)$. As of 2) we have

$$D_M(x) = \sum_{k=-M}^{M} e^{ikx} \Rightarrow \int_{-\pi}^{\pi} D_M(x)\,dx = 2\pi \Rightarrow \int_{-\pi}^{\pi} K_M(x)\,dx = \frac{(M+1)\,2\pi}{M+1} = 2\pi$$

We now prove 3). We fix $\varepsilon, \delta > 0$ and choose $N(\varepsilon, \delta)$ $\Big|$ $(N+1) > \dfrac{1}{\varepsilon \sin^2\frac{\delta}{2}}$

$\Rightarrow$ if $M > N$ we have

$$|K_m(x)| \le \frac{1}{(m+1)\sin^2\frac{x}{2}} < \frac{1}{(N+1)\sin^2\frac{x}{2}} < \frac{\varepsilon \sin^2\frac{\delta}{2}}{\sin^2\frac{x}{2}}$$



but if $\delta \le |x| \le \pi$ $\quad \sin^2\frac{x}{2} > \sin^2\frac{\delta}{2} \Rightarrow |K_m(x)| < \varepsilon$

We are now ready to prove the Fejer theorem.

Proof of Fejer theorem

We start with some preliminary considerations on the function $f(x)$. Since f is continuous and 2π periodic, the Heine-Cantor theorem ensures that, choosing $[-\pi,\pi]$ as a closed interval, f is uniformly continuous

(Heine-Cantor theorem: all functions $f: A\subset\mathbb{R} \to \mathbb{C}$ which are continuous over a compact interval A are uniformly continuous)

Thus, we can say that, given $\varepsilon > 0$ $\exists\, \delta(\varepsilon) \mid |t| < \delta \Rightarrow |f(x) - f(x-t)| < \frac{\varepsilon}{2}$

We now define $M = \max\limits_{x\in\mathbb{R}} |f(x)|$

The property 3) proven above implies that for $\varepsilon, \delta > 0$ $\exists\, N(\varepsilon,\delta)$ such that

$\delta \le |x| \le \pi,\ m > N \Rightarrow |K_m(x)| < \frac{\varepsilon}{4M}$

We now consider the difference $|f(x) - \sigma_m(x)|$ for $m > N$

$$|f(x) - \sigma_m(x)| = \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, K_m(t)\, dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\, K_m(t)\, dt \right|$$

$$= \frac{1}{2\pi}\left| \int_{-\pi}^{\pi} \big(f(x) - f(x-t)\big) K_m(t)\, dt \right|$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - f(x-t)|\, K_m(t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\delta}^{\delta} \underbrace{|f(x)-f(x-t)|}_{<\frac{\varepsilon}{2}} K_m(t)\,dt + \frac{1}{2\pi}\int_{-\pi}^{-\delta} \underbrace{|f(x)-f(x-t)|}_{<2M} K_m(t)\,dt \quad <\frac{\varepsilon}{4M}$$

$$+ \frac{1}{2\pi}\int_{\delta}^{\pi} \underbrace{|f(x)-f(x-t)|}_{<2M} K_m(t)\,dt \quad \hookrightarrow < \frac{\varepsilon}{4M} \qquad < \frac{\varepsilon}{2}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4} = \varepsilon$$

$\Rightarrow \quad |f(x) - \sigma_m(x)| < \varepsilon$ for $m > N$, i.e. $\sigma_m(x)$ converges uniformly to $f(x)$ ■

In general the Fourier series of a continuous periodic function does not converge

One has to make stronger assumptions to prove convergence

Convergence theorem (without proof)

A function $f : (-\pi, \pi) \to \mathbb{C}$ which is PIECEWISE CONTINUOUS and either

(i) its derivative is PIECEWISE CONTINUOUS

or

(ii) f is PIECEWISE MONOTONIC

$\Rightarrow$ The Fourier series of $f(x)$ converges pointwise to the function

$$\tilde{f}(x) = \frac{1}{2}\left(f(x+0) + f(x-0)\right)$$

If $f(x)$ is continuous in $[a,b] \subset [-\pi,\pi]$ $\Rightarrow$ the convergence is uniform

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We now go back to our previous example

$$f(x) = \begin{cases} \frac{\pi - x}{2} & 0 < x < 2\pi \\ 0 & x = 0 \end{cases} \qquad f(x) \sim \sum_{m=1}^{\infty} \frac{\sin mx}{m}$$

the function fulfills the hypotheses of the convergence theorem

Indeed for $x=0$ the series converges to 0 which is the mid point between the left and right limits! For $x=\frac{\pi}{2}$ we get $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots=\frac{\pi}{4}$

PIECEWISE CONTINUITY

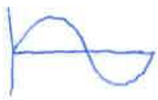
A function $f : [a,b) \to \mathbb{C}$ is PIECEWISE CONTINOUS if it is continuous in every $x \in [a,b)$ except for a finite number of points $x_i \in [a,b)$ $i = 1 \ldots N$ where, however, $\lim_{x \to x_i^+} f(x)$ and $\lim_{x \to x_i^-} f(x)$ exist and they are finite

PIECEWISE MONOTONIC

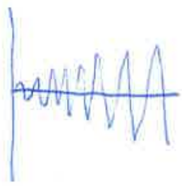
A function $f : [a,b) \to \mathbb{C}$ is PIECEWISE MONOTONIC if we can find a finite set $\{x_i\}$ $i = 1, N$ with $x_1 = a$ $x_i < x_{i+1}$ $x_N = b$ such that f is monotonic in each interval (x_i, x_{i+1}).

Examples:

$f(x) = \sin x$ is PIECEWISE MONOTONIC in $(0, 2\pi)$



$f(x) = x \sin \frac{1}{x}$ is NOT PIECEWISE MONOTONIC in $(0, 2\pi)$



Fourier expansion for functions of arbitrary period

We consider an integrable function of period L, $f(x+L)=f(x)\ \forall x$.

We can define $g(t)=f\left(\frac{Lt}{2\pi}\right)$. The function $g(t)$ has period 2π.

$\Rightarrow$ we can write $$g(t)\approx \frac{a_0}{2}+\sum_{m=1}^{\infty}\left(a_m\cos mt+b_m\sin mt\right)$$

$$a_m=\frac{1}{\pi}\int_0^{2\pi} g(t)\cos mt\,dt \qquad b_m=\frac{1}{\pi}\int_0^{2\pi} g(t)\sin mt\,dt$$

The inverse transformation reads $f(x)=g\left(\frac{2\pi x}{L}\right)$

$$\Rightarrow \boxed{f(x)=\frac{a_0}{2}+\sum_{m=1}^{\infty} a_m\cos\frac{2\pi m x}{L}+b_m\sin\frac{2\pi m x}{L}}$$

$$a_m=\frac{1}{\pi}\int_0^{2\pi} g(t)\cos mt\,dt=\frac{2}{L}\int_0^{L} f(x)\cos\frac{2\pi m x}{L}\,dx$$

$$b_m=\frac{2}{L}\int_0^{L} f(x)\sin\frac{2\pi m x}{L}\,dx$$

EXAMPLE: HEAT CONDUCTION IN A CLOSED ISOLATED CIRCUIT

$u(x,t)$ temperature distribution $\qquad u(x,0)=\phi(x)$

initial condition

physical law: heat equation $\qquad u(x+L,t)=u(x,t)$

$$u_{xx}-\frac{1}{a^2}u_t=0 \qquad \frac{1}{a^2}=c \quad \text{termal capacity}$$

Since the function $u(x,t)$ is periodic with period L we start from the ansatz:

$$u(x,t)=\frac{a_0(t)}{2}+\sum_{m=1}^{\infty}\left(a_m(t)\cos\frac{2\pi m x}{L}+b_m(t)\sin\frac{2\pi m x}{L}\right)$$

We find
$$u_{xx} = -\sum_{m=1}^{\infty}\left[\left(\frac{2\pi m}{L}\right)^2 a_m(t)\cos\frac{2\pi m x}{L} + \left(\frac{2\pi m}{L}\right)^2 b_m(t)\sin\frac{2\pi m}{L}x\right]$$

$$\frac{1}{a^2}u_t = \frac{1}{a^2}\left(\frac{a_0'(t)}{2} + \sum_{m=1}^{\infty}\left(a_m'(t)\cos\frac{2\pi m x}{L} + b_m'(t)\sin\frac{2\pi m x}{L}\right)\right)$$

By comparing term by term we get

$$a_0'(t)=0 \qquad \underbrace{\frac{a_m'(t)}{a_m(t)}}_{(\ln a_m)'} = -\left(\frac{2\pi m}{L}\right)^2 a^2 \qquad \underbrace{\frac{b_m'(t)}{b_m(t)}}_{(\ln b_m)'} = -\left(\frac{2\pi m}{L}\right)^2 a^2$$

These are examples of first order differential equations (see later for a complete treatment) where solutions are

$$a_0(t) = \text{const} \qquad \ln(a_m(t)) = -\frac{4\pi^2 m^2 a^2}{L^2}t + \text{const} \qquad \ln(b_m(t)) = -\frac{4\pi^2 m^2 a^2}{L^2}t + \text{const}$$

$$a_0(t) = A_0 \qquad a_m(t) = A_m e^{-\frac{4\pi^2 m^2 a^2}{L^2}t} \qquad b_m(t) = B_m e^{-\frac{4\pi^2 m^2 a^2}{L^2}t}$$

The coefficients A_0, A_m, B_m must be chosen so as to fulfil the initial condition

$$u(x,0) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m\cos\frac{2\pi m x}{L} + B_m\sin\frac{2\pi m x}{L} \equiv \phi(x)$$

$$\Rightarrow \quad A_m = \frac{2}{L}\int_0^L \phi(x)\cos\frac{2\pi m x}{L}\,dx \qquad m=0,1,\dots$$

$$B_m = \frac{2}{L}\int_0^L \phi(x)\sin\frac{2\pi m x}{L}\,dx \qquad m=1,2\dots$$