3.6 Effective chiral Lagrangian at lowest order

In Section 3.4, we studied an example of a renormalizable model, the linear sigma model in (3.16), that can be matched at low energies to a Goldstone Lagrangian (see (3.19)). Contrary to this example, there is no known way of explicitly deriving the low energy QCD effective theory directly from the original QCD Lagrangian. That is, we do not know how to directly match strongly-coupled QCD to a theory of light pseudo-scalar mesons.[1] One must therefore write down the most general possible effective Lagrangian consistent with the chiral symmetries of the original QCD Lagrangian and with C, P and T invariance. The parameters of this effective Lagrangian could in principle be computed from QCD, but in practice, they have to be obtained by comparison with experiment. This is why χPT constitutes an example of a bottom-up EFT.

In the construction of the χPT Lagragian, we consider the QCD Lagrangian in the *chiral limit*, i.e. with massless quarks and no EW interactions, and take three quark flavors (that is $N_f = 3$) that we identify with the lightest SM quarks: u, d and s.[2] As we saw in section 3.2, the QCD Lagrangian in this limit, $\mathcal{L}^0_{\rm QCD}$, respects the following (global) chiral symmetry

$$\mathcal{G}_\chi = SU(3)_L \times SU(3)_R \times U(1)_V \,, \tag{3.39}$$

that at low energies, when QCD becomes strongly coupled, is spontaneously broken by the quark condensate,

$$\langle 0|\bar{u}_L u_R|0\rangle = \langle 0|\bar{d}_L d_R|0\rangle = \langle 0|\bar{s}_L s_R|0\rangle = \Lambda^3 \neq 0 \,, \tag{3.40}$$

down to the diagonal subgroup

$$\mathcal{H}_\chi = SU(3)_V \times U(1)_V \,. \tag{3.41}$$

We can use the CCWZ prescription to build the most general Lagrangian describing the Goldstone bosons arising from the spontaneous symmetry breaking $\mathcal{G}_\chi \to \mathcal{H}_\chi$. As we saw in the previous section, these can be described by

$$\Sigma(\phi) = \xi^2(\phi) = e^{2i\,\Phi(x)/f} \,, \tag{3.42}$$

where f is a normalization constant and[3]

$$\Phi(x) \equiv \phi^a\, T^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix} \,, \tag{3.43}$$

[1] It is interesting to note that there are simpler theories in $1+1$ dimensions where one can exactly relate a strongly coupled theory of fermions (Thirring model) to a weakly coupled theory of scalars (sine-Gordon model) [1].

[2] The reason for this choice will be clear in the next section, where we introduce quark masses explicitly in the chiral Langrangian.

[3] The η_8 field can be identified η meson, up to $\eta - \eta'$ mixing corrections.

with T^a $(a = 1, \ldots, 8)$ being the $SU(3)$ generators with $\langle T^a T^b \rangle = \delta_{ab}/2$, where $\langle . \rangle$ denotes the trace. As we saw, the Σ field transforms under a global $\mathcal{G}_\chi$ transformation as

$$\Sigma \to \Sigma' = g_L^\dagger \, \Sigma \, g_R \, . \tag{3.44}$$

It is easy to construct the most general Lagrangian in terms of the Σ field. The Lagrangian can be organized in terms of increasing powers of momentum or, equivalently, in terms of increasing number of derivatives (the need for an even number of derivatives follows from Lorentz invariance):

$$\mathcal{L}_\chi(\Sigma) = \sum_{n=0}^{\infty} \mathcal{L}_\chi^{(2n)}(\Sigma) \, . \tag{3.45}$$

This Lagrangian contains an infinite number of operators, so we need to apply a power counting to establish a hierarchy between them. Contrary to other types of theories, χPT does not have a small coupling around which to do an expansion. Instead, χPT is an expansion in powers of momenta. Higher-order operators in (3.45) contribute with higher powers of momenta to a given physical amplitude, so at low momenta, their contributions are suppressed. Or in other words, at sufficiently low momenta compared to a certain hadronic scale Λ_χ, an increasing number of derivatives necessarily implies a higher suppression for the associated operators. This let us establish the following power counting

$$\Sigma \sim \mathcal{O}(1) \, , \qquad \qquad \partial_\mu \sim \mathcal{O}\left(\frac{p}{\Lambda_\chi}\right) \, . \tag{3.46}$$

For this counting to make sense, we need to determine the value of the hadronic scale Λ_χ. Since we are ignoring the dynamics of the heavy QCD resonances in our χPT description (see discussion in 3.1), one can argue that the value of Λ_χ should correspond to that of the lightest QCD resonance we are not including: the ρ vector meson, and therefore[4]

$$\Lambda_\chi \sim m_\rho \approx 1 \text{ GeV} \, . \tag{3.47}$$

The most general invariant term with no derivatives must be the product of terms of the form $\langle \Sigma\Sigma^\dagger \ldots \Sigma\Sigma^\dagger \rangle$, where Σ and $\Sigma^\dagger$ alternate. However, $\Sigma\Sigma^\dagger = 1$, so all such terms are constant, and independent of the pion fields. This is just our old result that Goldstone bosons are derivatively coupled. The lowest-order chiral Lagrangian is therefore

$$\mathcal{L}_\chi^{(2)} = \frac{f^2}{4} \, \langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \rangle \, , \tag{3.48}$$

where the prefactor $f^2/4$ is completely fixed by the requirement that the fields in Φ are canonically normalized. Note that this Lagrangian contains as many Σ as $\Sigma^\dagger$

[4] In Section **??**, we will provide another estimate for this scale, based on the consistency of the theory.

fields, showing the parity invariance of QCD. The Lagrangian in (3.50) contains an infinite number of interactions, which are all fixed in terms of a single parameter. Indeed, expanding Σ

$$\Sigma = \mathbb{1} + \frac{2i\Phi}{f} - \frac{2\Phi^2}{f^2} - \frac{4i\Phi^3}{3f^3} + \mathcal{O}\left(\frac{\Phi^4}{f^4}\right), \tag{3.49}$$

in $\mathcal{L}_\chi^{(2)}$, and noting that $\Phi = \Phi^\dagger$, we get

$$\mathcal{L}_\chi^{(2)} = \langle \partial_\mu \Phi \, \partial^\mu \Phi \rangle + \frac{1}{3f^2} \langle [\Phi, \partial_\mu \Phi] \, [\Phi, \partial^\mu \Phi] \rangle + \mathcal{O}\left(\frac{\Phi^6}{f^4}\right), \tag{3.50}$$

where $[\cdot,\cdot]$ is the commutator. As already argued, operators with more than two derivatives are suppressed by the hadronic scale Λ_χ. Using the power counting in (3.46), we can write $\mathcal{L}_\chi$ as

$$\mathcal{L}_\chi = \frac{f^2}{4}\left[\langle \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \rangle + \frac{1}{\Lambda_\chi^2}\mathcal{L}_\chi^{(4)} + \frac{1}{\Lambda_\chi^4}\mathcal{L}_\chi^{(6)} + \dots\right], \tag{3.51}$$

from where the Λ_χ suppression is manifest. The contributions from these additional operators, which are small provided $p \ll \Lambda_\chi$, introduce new unspecified coupling constants that need to be determined experimentally. For the rest of the lecture, we will focus on the lowest-order chiral Lagrangian and postpone the discussion on the higher-order operators to Section **??**.

Computing, for instance, the $\pi\pi$ scattering amplitude is now a trivial perturbative exercise. One then gets the well-known result [2]

$$\mathcal{A}(\pi^+\pi^0 \to \pi^+\pi^0)\big|_{m_q=0} = \frac{(p'_+ - p_+)^2}{f^2}\left[1 + \mathcal{O}\left(\frac{p^2}{\Lambda_\chi^2}\right)\right], \tag{3.52}$$

where p_+ and p'_+ are the momenta of the incoming and outgoing π^+, respectively. The $\mathcal{O}(p^2/\Lambda_\chi^2)$ corrections stem from contributions arising from the next terms in the chiral Lagrangian in (3.51). Similar results to the one above can be obtained for $\pi\pi \to 4\pi, 6\pi, \dots$ from the $\mathcal{O}(\Phi^6/f^4)$ terms in (3.50), without introducing any new parameters. The χPT Lagrangian is therefore extremely predictive.

3.6.1 χPT with non-zero masses and EW interactions

The χPT Lagrangian can be extended to account for non-zero quark masses and electroweak interactions of the complete QCD Lagrangian. These terms break explicitly the global chiral symmetry of $\mathcal{L}^0_{\rm QCD}$ discussed in Section 3.2. While the electromagnetic and weak interactions only introduce a small breaking, and they can be treated as a perturbation, the same is only true for light-quark masses. By noting that χPT is an expansion in inverse powers of the hadronic scale Λ_χ, we can infer the expected size of the mass corrections by naive dimensional analysis

$$\frac{m_{u,d}}{\Lambda_\chi} = \mathcal{O}(10^{-2}), \qquad \frac{m_s}{\Lambda_\chi} = \mathcal{O}(10^{-1}), \qquad \frac{m_c}{\Lambda_\chi} = \mathcal{O}(1), \tag{3.53}$$

where we see that, while these are indeed small perturbations for the u, d and s quarks, this is not the case for the c quark (or any other heavier quark), whose mass completely breaks our power counting.

Before proceeding to systematically include these chiral symmetry breaking effects, it is illustrative to discuss how $m_q \neq 0$ $(q = u, d, s)$ *explicitly* break the chiral symmetry

i) If $m_u = m_d = m_s \Longrightarrow G_\chi \to \mathcal{H}_V = SU(3)_V \times U(1)_V$.

ii) If $m_u = m_d \neq m_s \Longrightarrow G_\chi \to SU(2)_V \times U(1)_V \times U(1)_S$, where $SU(2)_V$ is commonly referred to as *isospin* and $U(1)_S$ (corresponding to strange quark number) as *strangeness*.

iii) If $m_u \neq m_d \neq m_s \Longrightarrow G_\chi \to U(1)_V \times U(1)_Q \times U(1)_S$, where $U(1)_Q$ is the group associated to the electromagnetic interaction.

These explicit breaking patterns provide useful information when defining the symmetries of a realistic chiral Lagrangian. As we can see, while having non-zero masses will partially break some of the (global) symmetries of QCD, there are still certain residual symmetries that remain unbroken, and which should therefore be preserved also in the chiral Lagrangian.

Spurion analysis

The best way to systematically introduce the (small) breakings of the chiral symmetry is by using the so-called spurion analysis. This approach consists in considering the explicit symmetry-breaking terms as *spurions* to which we assign transformation properties such that the symmetry is restored. Since these terms are not necessarily fields, they are not allowed to transform under the symmetry, and the transformations are spurious, hence their name. However this approach serves as a very good organizing principle when translating the breaking of the chiral symmetry in the QCD Lagrangian to our chiral Lagrangian. The most general extension of the QCD Lagrangian in the chiral limit is given by (in terms of the Dirac spinor $q = (u\ d\ s)^\intercal$)

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}^0_{\text{QCD}} + \bar{q}\,\gamma^\mu\,(v_\mu + \gamma_5\,a_\mu)\,q - \bar{q}\,(s - i\gamma_5\,p)\,q\,, \tag{3.54}$$

with $\mathcal{L}^0_{\text{QCD}}$ as in (3.1) and where the v_μ, a_μ, s and p spurions are 3×3 hermitian matrices. We can identify the spurions in terms of the electroweak interactions and quark masses in the full QCD Lagrangian, with $N_f = 3$, by taking:

$$\begin{aligned} r_\mu &\equiv v_\mu + a_\mu = e\,Q\,A_\mu + \dots\,, \\ l_\mu &\equiv v_\mu - a_\mu = e\,Q\,A_\mu + \frac{g}{\sqrt{2}}(W^\dagger_\mu\,T_+ + h.c.) + \dots\,, \\ s &= \mathcal{M} + \dots\,, \\ p &= 0\,. \end{aligned} \tag{3.55}$$

where the dots denote interactions, such as the ones of the Z boson and the Higgs, that are not phenomenological relevant. Here Q and $\mathcal{M}$ denote the quark charge and mass matrices, respectively,

$$Q = \frac{1}{3}\,\mathrm{diag}(2,-1,-1)\,, \quad \mathcal{M} = \mathrm{diag}(m_u, m_d, m_s)\,, \quad T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \qquad (3.56)$$

As we argued before, we assign transformation properties to the spurions to keep the chiral symmetry invariant. Actually, we will go one step further and assign appropriate transformation properties such that the chiral symmetry is promoted to a *local* symmetry of the QCD Lagrangian. This way, we can introduce the SM gauge fields as external spurionic fields. The Lagrangian in (3.54) is invariant under the local $SU(3)_L \times SU(3)_R$ chiral symmetry if

$$\begin{aligned} q_L &\to g_L\, q_L\,, & q_R &\to g_R\, q_R\,, \\ l_\mu &\to g_L\, l_\mu\, g_L^\dagger + i g_L\, \partial_\mu\, g_L^\dagger\,, & r_\mu &\to g_R\, r_\mu\, g_r^\dagger + i g_R\, \partial_\mu\, g_R^\dagger\,, \\ s+ip &\to g_R\,(s+ip)\, g_L^\dagger\,, & s-ip &\to g_L\,(s-ip)\, g_R^\dagger\,. \end{aligned} \qquad (3.57)$$

As anticipated, we can use the transformation properties of these spurions to build a generalized chiral Lagrangian. The first thing to note is that, in order to preserve *local* chiral symmetry in the χPT Lagrangian, we need to promote the derivative to a covariant derivative defined as

$$D_\mu\Sigma = \partial_\mu\Sigma - i r_\mu\Sigma + i\Sigma\, l_\mu\,, \qquad D_\mu\Sigma^\dagger = \partial_\mu\Sigma^\dagger + i\Sigma^\dagger\, r_\mu - i l_\mu\Sigma^\dagger\,. \qquad (3.58)$$

Moreover, local chiral symmetry invariance also allows us to introduce new operators in terms of the field strength tensors

$$F_L^{\mu\nu} = \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu]\,, \qquad F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu]\,. \qquad (3.59)$$

From (3.58), it is clear how we can extend our chiral counting also to the l_μ and r_μ spurions

$$l_\mu, r_\mu \sim \mathcal{O}\left(\frac{p}{\Lambda_\chi}\right)\,, \qquad F_{L,R}^{\mu\nu} \sim \mathcal{O}\left(\frac{p^2}{\Lambda_\chi^2}\right)\,. \qquad (3.60)$$

Concerning the spin-0 spurions, in principle one can formulate χPT as an independent expansion in both derivatives and quark masses. However, it is convenient to combine the two expansions into a single one by making use of the relations between meson and quark masses. As we will see below, the choice of counting rule

$$s, p \sim \mathcal{O}\left(\frac{p^2}{\Lambda_\chi^2}\right)\,, \qquad (3.61)$$

implies the relation $M_\pi^2 \propto m_q$, and leads to the Gell-Mann–Okubo mass formula (3.70), which is well reproduced experimentally.

We can now write a general χPT Lagrangian including the spurion sources. At lowest order in the chiral counting, this Lagrangian reads

$$\mathcal{L}_\chi^{(2)} = \frac{f^2}{4}\,\langle D_\mu\Sigma^\dagger D^\mu\Sigma + \chi\Sigma^\dagger + \Sigma\chi^\dagger\rangle\,, \qquad (3.62)$$

with

$$\chi = 2B\,(s + ip)\,, \tag{3.63}$$

where B is an undetermined constant with dimensions of energy. Thanks to the spurion analysis, it is now trivial to read off the light-meson interactions with the SM gauge fields. For instance, for the electromagnertic interactions, we have (taking the A_μ piece in (3.55) from l_μ and r_μ)

$$\begin{aligned} \mathcal{L}_\chi^{(2)} &\supset -2ie\,A_\mu\,\langle \partial^\mu\,\Phi[Q,\Phi]\rangle + e^2\,A_\mu A^\mu\,\langle [Q,\Phi]\,[Q,\Phi]\rangle \\ &\supset [ie\,A_\mu\,(\pi^+\partial^\mu\pi^- + K^+\partial^\mu K^-) + h.c.] + e^2\,A_\mu A^\mu\,(\pi^+\pi^- + K^+K^-)\,, \end{aligned} \tag{3.64}$$

which defines the QED interactions with the light-mesons.

Connecting the chiral parameters to measurable quantities

The two undetermined quantities of the generalized chiral Lagrangian, f and B, are connected to two fundamental order parameters of the spontaneous chiral symmetry breaking: the pion decay constant f_π, defined by $\langle 0|\bar{u}\,\gamma_\mu\gamma_5\,d|\pi^+\rangle = i\sqrt{2}\,f_\pi\,p_\mu$, and the quark condensate $\langle 0|\bar{q}_L q_R|0\rangle$. Indeed, by differentiating with respect to the external sources we have

$$\begin{aligned} \langle 0|\bar{u}\,\gamma^\mu\gamma_5\,d|\pi^+\rangle &= \langle 0|\frac{\delta\mathcal{L}_{\rm QCD}}{\delta a_\mu^{12}}|\pi^+\rangle = \langle 0|\frac{\delta\mathcal{L}_\chi}{\delta a_\mu^{12}}|\pi^+\rangle \\ &= \frac{if^2}{2}\,\langle 0|\left[\Sigma^\dagger\partial^\mu\Sigma - \Sigma\,\partial^\mu\Sigma^\dagger + \ldots\right]_{12}|\pi^+\rangle \\ &= -\frac{2f}{\sqrt{2}}\,\langle 0|\left[\partial^\mu\phi + \ldots\right]|\pi^+\rangle \\ &\approx \sqrt{2}f\,p_\mu\,, \end{aligned} \tag{3.65}$$

from where we get $f = f_\pi$, up to higher order corrections in the chiral expansion. The value of f_π can be determined experimentally from the W-mediated semileptonic decay of pion $\pi^+ \to \ell\nu$, hence its name, yielding $f = f_\pi = 92.4$ MeV. One could do the same derivation to measure the corresponding Kaon decay constant, defined analogously to f_π and determined experimentally to be $f_K = 114$ MeV. The difference between f_π and f_K is an $\mathcal{O}(p^4)$ effect, which goes beyond the lowest order. However, since higher-order effects are expected to be larger in the case of the Kaon (due to the larger mass of the strange quark), the most natural determination of f at lowest order is provided by f_π.

Proceeding in a similar way with the quark condensate, we find that

$$\begin{aligned} \langle 0|\bar{q}_L^i q_R^j|0\rangle &= -\,\langle 0|\frac{\delta\mathcal{L}_{\rm QCD}}{\delta(s-ip)^{ij}}|0\rangle = -\,\langle 0|\frac{\delta\mathcal{L}_\chi}{\delta(s-ip)^{ij}}|0\rangle \\ &= -\frac{f^2}{2}B\,\langle 0|\Sigma_{ij} + \ldots|0\rangle \\ &= -\frac{f^2}{2}B\,\langle 0|\delta_{ij} + \ldots|0\rangle \\ &\approx -\frac{f^2}{2}B\,\delta_{ij}\,. \end{aligned} \tag{3.66}$$

Note that contrary to f, B is not an observable quantity, so we cannot determine its value experimentally. However, we can relate the product $B \times m_q$ to the meson masses. Expanding the $\langle \chi\Sigma^\dagger + \Sigma\chi^\dagger \rangle$ piece in $\mathcal{L}_\chi^{(2)}$, and taking $s = \mathcal{M}$ and $p = 0$, we have

$$\begin{aligned}\mathcal{L}_\chi^{(2)} \supset \frac{f^2}{4} \langle \chi\Sigma^\dagger + \Sigma\chi^\dagger \rangle &= \frac{f^2}{4} 2B \langle \mathcal{M}(\Sigma^\dagger + \Sigma) \rangle \\ &= 2B \left[\frac{f^2}{2} \langle \mathcal{M} \rangle - \langle \mathcal{M}\Phi^2 \rangle + \frac{1}{3f^2} \langle \mathcal{M}\Phi^4 \rangle + \mathcal{O}\left(\frac{\Phi^6}{f^4} \right) \right] .\end{aligned} \tag{3.67}$$

The constant term $Bf^2 \langle \mathcal{M} \rangle$ is unphysical and can be dropped. From the term quadratic in the light-meson fields, we find[5]

$$\begin{aligned} M_{\pi^0}^2 &= M_{\pi^+}^2 = 2B\,\hat{m} + \mathcal{O}(\epsilon)\,, & M_{K^+}^2 &= B\,(m_u + m_s)\,, \\ M_{K^0}^2 &= B\,(m_d + m_s)\,, & M_{\eta_8}^2 &= \frac{2}{3} B\,(\hat{m} + 2m_s) + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{3.69}$$

with $\hat{m} = (m_u + m_d)/2$. Since we have four meson masses written in terms of three quark masses, we can obtain an absolute prediction for one of the meson masses in terms of the others. In the limit $m_u = m_d$, this is the famous Gell-Mann–Okubo mass formula [3, 4]

$$4M_{K^0}^2 = 3M_{\eta_8}^2 + M_{\pi^0}^2\,, \tag{3.70}$$

which gives 0.99 GeV^2 = 0.92 GeV^2 if we take $M_{\eta_8} = M_\eta$. The validity of the Gell-Mann–Okubo relation provides an important consistency check for our power counting assignment in (3.61). Moreover, it shows that $\mathcal{O}(m_q^2)$ corrections to the meson masses (which would arise from higher-order operators in $\mathcal{L}_\chi$) are small. In addition to $\mathcal{O}(m_q^2)$ corrections, the mass relations in (3.69) are affected by QED effects. At leading order in the chiral expansion, they only depend on meson charges and we can write[6]

$$\begin{aligned} M_{\pi^0}^2 &= 2B\,\hat{m} + \mathcal{O}(\epsilon, m_q^2) & M_{\pi^+}^2 &= 2B\,\hat{m} + \alpha\Delta_{\text{em}} + \mathcal{O}(m_q^2, \alpha m_q)\,, \\ M_{K^0}^2 &= B\,(m_d + m_s) + \mathcal{O}(m_q^2)\,, & M_{K^+}^2 &= B\,(m_u + m_s) + \alpha\Delta_{\text{em}} + \mathcal{O}(m_q^2, \alpha m_q)\,, \\ M_{\eta_8}^2 &= \frac{2}{3} B\,(\hat{m} + 2m_s) + \mathcal{O}(\epsilon, m_q^2)\,, \end{aligned} \tag{3.71}$$

Although the absolute value of the quark masses cannot be determined within χPT, this theory does provide information about the quark-mass ratios. Neglecting

[5] The $\mathcal{O}(\epsilon)$ isospin-breaking corrections, with

$$\epsilon = \frac{B}{4} \frac{(m_u - m_d)^2}{m_s - \hat{m}}\,, \tag{3.68}$$

originate from a small mixing term between the π^0 and the η_8 fields.

[6] QED corrections to the meson masses are computed in Section ??.

the small $\mathcal{O}(\epsilon, m_q^2, \alpha m_q)$ corrections, we find

$$\begin{aligned} \frac{m_d - m_u}{m_u + m_d} &= \frac{(M_{K^0}^2 - M_{K^+}^2) - (M_{\pi^0}^2 - M_{\pi^+}^2)}{M_{\pi^0}^2} = 0.29\,, \\ \frac{m_s - m_u}{m_u + m_d} &= \frac{M_{K^0}^2 - M_{\pi^0}^2}{M_{\pi^0}^2} = 12.6\,. \end{aligned} \tag{3.72}$$

Interestingly, the three light-quark masses turn out to be very different. From the expressions above, we have

$$\frac{m_u}{m_d} = 0.55\,, \qquad \frac{m_s}{m_d} = 20.3\,. \tag{3.73}$$

As we anticipated at the beginning of this section (see (3.53)), quark-mass corrections are dominated by m_s, which is large compared with m_u and m_d. Also, note that even though the difference $m_d - m_u$ is not small compared with the individual up- and down-quark masses, isospin breaking effects are controlled by the small ratio $(m_u - m_d)/\Lambda_\chi$, and therefore the isospin symmetry $SU(2)_V$ turns out to be a very good (approximate) symmetry of the strong interactions.

The new interactions in the Lagrangian of (3.62) introduce mass corrections to the $\pi\pi$ scattering amplitude derived in (3.52). Including these mass corrections (see (3.67)), we find now the full result [2]

$$\mathcal{A}(\pi^+\pi^0 \to \pi^+\pi^0) = \frac{(p'_+ - p_+)^2 - M_\pi^2}{f^2}\left[1 + \mathcal{O}\left(\frac{p^2}{\Lambda_\chi^2}\right)\right]\,. \tag{3.74}$$

Since $f = f_\pi$ is fixed from the pion decay, this result is an absolute prediction of χPT.

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