1.6 Addendum: Matching with the background field method

In section 1.3, we introduced the idea of integrating out the hard modes from the theory to define the effective theory. Then, in sections 1.4 and 1.5, we implemented our ideas by matching matrix elements of the effective theory to matrix elements of the full theory. The decoupling of hard modes happened in an ad-hoc way, simply by matching up a Lagrangian that did not contain any hard modes to the full theory. In this section, we perform the one-loop matching by actually carrying out the function integral over the hard modes. This can be done by employing the background field method.

1.6.1 Derivation of the formalism

Remember that we want to compute the Wilsonian effective action:

$$\int \mathcal{D}\phi_H \exp\left\{iS(\phi_S,\phi_H)\right\} = \exp\left\{iS_\Lambda(\phi_S)\right\} . \tag{1.1}$$

To this end, we want to find a way of evaluating the functional integral on the left-hand side of explicitely. We can do this even at one-loop order. In the background field method, one separates fields into their classical parts (the background field) and quantum fluctuations around these:

$$\phi = \phi_{\text{cl}} + \eta\,, \tag{1.2}$$

where ϕ_{cl} here denotes the classical field and η denotes the quantum fluctuations. Do not confuse this with the separation of modes we discussed earlier, this step comes in at a later stage. The classical fields are defined by the part of the fields satisfying the classical equations of motion:

$$\left.\frac{\delta\mathcal{L}(\phi)}{\delta\phi}\right|_{\phi=\phi_{\text{cl}}} = 0\,. \tag{1.3}$$

To compute the effective action, we start with the action $iS = i\int d^4x\mathcal{L}(\phi_{\text{cl}}+\eta)$ and expand it around the classical fields:

$$\begin{aligned} iS = i\int d^4x\,\mathcal{L}(\phi) =& i\int d^4x\,\mathcal{L}(\phi_{\text{cl}}) + i\int d^4x \left(\left.\frac{\delta\mathcal{L}(\phi)}{\delta\phi(x)}\right|_{\phi=\phi_{\text{cl}}}\right)\eta(x) \\ &+ \frac{i}{2}\int d^4x\,d^4y\,\eta(x)\left(\frac{\delta^2\mathcal{L}(\phi)}{\delta\phi(x)\,\delta\phi(y)}\right)\eta(y) + \dots \end{aligned} \tag{1.4}$$

The first term in the expansion is the Lagrangian evaluated for classical fields. The linear term in the fluctuations vanishes by definition, since its coefficient is zero by the equations of motion (1.3). The second-order term contains the leading quantum corrections to the classical component. In the Language of Feynman diagrams, this term contains the one-loop corrections.

The next step is to split the fields into hard and soft modes. Next, we want to remove the hard modes from each term in the Lagrangian (1.4), order by order in η. In the leading term, it suffices to substitute hard modes by their classical values. This means, we simply solve the equations of motion for them

$$\frac{\delta L}{\delta \phi_{\mathrm{cl},H}} = \frac{\partial \mathcal{L}}{\partial \phi_{\mathrm{cl},H}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_{\mathrm{cl},H})} \stackrel{!}{=} 0\,. \tag{1.5}$$

and solve for $\phi_{\mathrm{cl},H}$. The evaluation of the quadratic term is slightly more complicated, but still feasible. First, let us write the modes in a vector notation, $\phi_i = (\phi_H, \phi_S)$. Then the quadratic term in (1.4) becomes:

$$\frac{1}{2}\eta \left(\frac{\delta^2 \mathcal{L}(\phi)}{\delta\phi\, \delta\phi}\right)\eta = \frac{1}{2}\eta_i \left(\frac{\delta^2 \mathcal{L}(\phi)}{\delta\phi_i\, \delta\phi_j}\right)\eta_j \equiv \frac{1}{2}\eta_i\, \mathcal{Q}_{ij}\, \eta_j\,. \tag{1.6}$$

Here we have defined the fluctuation operator:

$$\mathcal{Q} = \begin{pmatrix} \Delta_H & X_{SH}^\dagger \\ X_{SH} & \Delta_S \end{pmatrix}. \tag{1.7}$$

The off-diagonal blocks X_{SH} mix soft and hard modes. They should in principle not exist but are a residue of being too simplistic in the way we assigned fields to regions. In our familiar example of a theory of heavy and light scalars, we will have these mixed terms if we identify all light fields with soft modes and all heavy fields with hard modes. We know however that we can have light fields with large virtualities, which we should count as part of the hard field content. To correctly account for this, we simply perform a field redefinition $\eta_i \to \tilde{\eta}_i = V_{ij}\eta_j$ that puts $\mathcal{Q}$ in a block-diagonal form. Then the fields $\tilde{\eta}_H$ contain light fields with hard invariant masses k^2. The transformation matrix V is given by:

$$V = \begin{pmatrix} \mathbf{1} & 0 \\ -\Delta_S^{-1} X_{SH} & \mathbf{1}\,, \end{pmatrix} \tag{1.8}$$

leading to the diagonal operator:

$$\tilde{\mathcal{Q}} = V^\dagger \mathcal{Q} V = \begin{pmatrix} \tilde{\Delta}_H & 0 \\ 0 & \Delta_S \end{pmatrix}, \tag{1.9}$$

with

$$\tilde{\Delta}_H = \Delta_H - X_{SH}^\dagger \Delta_S^{-1} X_{SH}\,. \tag{1.10}$$

Now we can perform the integration over the hard modes:

$$e^{iS_\Lambda} = \int \mathcal{D}\tilde{\eta}_H \,\exp\left\{\frac{i}{2}\int d^4x\, d^4y\, \tilde{\eta}_H \cdot \tilde{\mathcal{Q}}_H \cdot \tilde{\eta}_H\right\} = \left(\det\left[-\frac{i}{2}\tilde{\mathcal{Q}}_H\right]\right)^{-c}, \tag{1.11}$$

where $c = 1/2$ for bosons and $c = -1$ for fermions. In the case of mixed operators, the operator must be further decomposed and diagonalized. Now the action can be written as:

$$S_\Lambda = ic \,\log\det\left(\tilde{\Delta}_H\right). \tag{1.12}$$

Our remaining task is now to evaluate the determinant in the above expression. First note that a determinant can be written as the product of eigenvalues:

$$\det A = \prod_i a_i = \exp\left\{\sum_i \log a_i\right\} = \exp\{\mathrm{Tr}\log A\}\,. \tag{1.13}$$

With this we can expand the trace in momentum eigenstates of the operator:

$$\begin{aligned} S_\Lambda &= ic\,\mathrm{tr}\int \frac{d^dk}{(2\pi)^k}\,\langle k|\log\tilde{\Delta}_H|k\rangle \\ &= ic\,\mathrm{tr}\int d^dx\int\frac{d^dk}{(2\pi)^d}e^{-ikx}\log(\tilde{\Delta}_H)e^{ikx}\,. \end{aligned} \tag{1.14}$$

The symbol $\mathrm{tr}(A)$ indicates that part of the original trace $\mathrm{Tr}(A)$ has been carried out by the sum over momentum eigenstates. The operator $\tilde{\Delta}_H$ is an operator product depending on the coordinate x and partial derivatives ∂:

$$\tilde{\Delta}_H \equiv \tilde{\Delta}_H(x,\partial)\,. \tag{1.15}$$

These derivatives can act on the exponentials and one can show that:

$$\begin{aligned} S_\Lambda &= ic\,\mathrm{tr}\int d^dx\int\frac{d^dk}{(2\pi)^d}e^{-ikx}\log\left(\tilde{\Delta}_H(x,\partial)\right)e^{ikx} \\ &= ic\,\mathrm{tr}\int d^dx\int\frac{d^dk}{(2\pi)^d}\log\left(\tilde{\Delta}_H(x,\partial+ik)\right)\,. \end{aligned} \tag{1.16}$$

1.6.2 A scalar example

It is instructive to evaluate this expression explicitely for a scalar theory. Let us use the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}m^2\varphi^2 + \frac{\lambda_1}{4!}\varphi^4 + \frac{\lambda_2}{4}\varphi^2\Phi^2\,. \tag{1.17}$$

As a first step, let us solve the classical equations of motion to remove Φ at tree-level. We find:

$$\begin{aligned} \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_{\mathrm{cl}})} &= \frac{\partial\mathcal{L}}{\partial\Phi_{\mathrm{cl}}} \\ -\partial^2\Phi_{\mathrm{cl}} &= -M^2\Phi_{\mathrm{cl}} + \frac{\lambda_2}{2}\varphi_{\mathrm{cl}}^2\Phi_{\mathrm{cl}}\,. \end{aligned} \tag{1.18}$$

Since we are only interested in the the low-energy theory, the momentum term can be neglected compared to the mass term. Thus the left-hand side of the above equation vanishes and consequently the classical field Φ_{cl} satisfies

$$\Phi_{\mathrm{cl}} = 0\,. \tag{1.19}$$

There is no tree-level matching in this theory. We can see this diagrammatically when looking at the Lagrangian: There is simply no tree-level diagram with only external φ mediated by a virtual Φ.

Now, let us compute the one-loop effective action. First, we compute the blocks Δ_H, Δ_S and X_{SH}. We obtain:

$$\begin{aligned}\Delta_H(x,\partial) &= -\partial^2 - M^2 + \frac{\lambda_2}{2}\varphi_{\rm cl}^2\,,\\ \Delta_S(x,\partial) &= -\partial^2 - m^2 + \frac{\lambda_1}{2}\varphi_{\rm cl}^2 + \frac{\lambda_2}{2}\Phi_{\rm cl}^2\,,\\ X_{SH} &= 0\,.\end{aligned} \tag{1.20}$$

The last term in the second line vanishes by the equations of motion. Since $X_{SH} = 0$, the matrix V defined in eq. (1.8) is simply identity and $\tilde{\Delta}_H = \Delta_H$. So now we want to compute:

$$\begin{aligned}S_\Lambda &= \frac{i}{2}\int d^dx \int \frac{d^dk}{(2\pi)^d}\log\left(\tilde{\Delta}_H(x,\partial+ik)\right)\\ &= \frac{i}{2}\int d^dx \int \frac{d^dk}{(2\pi)^d}\log\left(k^2 - M^2 - \left(2ik\cdot\partial + \partial^2 - \frac{\lambda_2}{2}\varphi_{\rm cl}^2\right)\right)\\ &= \frac{i}{2}\int d^dx \int \frac{d^dk}{(2\pi)^d}\left\{\log\left(k^2 - M^2\right) + \log\left(1 - \frac{2ik\cdot\partial + \partial^2 - \frac{\lambda_2}{2}\varphi_{\rm cl}^2}{k^2 - M^2}\right)\right\}\end{aligned} \tag{1.21}$$

The first term in the curly brackets does not generate any interactions between the fields $\varphi_{\rm cl}$, so we can neglect it. Focussing on the second term and rewriting the logarithm through its series representation,

$$\log(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k}\,, \tag{1.22}$$

we find

$$S_\Lambda = \frac{i}{2}\int d^dx \int \frac{d^dk}{(2\pi)^d}\log\left(1 - \frac{2ik\cdot\partial + \partial^2 - \frac{\lambda_2}{2}\varphi_{\rm cl}^2}{k^2 - M^2}\right) \tag{1.23}$$

$$= -\frac{i}{2}\int d^dx \sum_{n=1}\frac{1}{n}\int \frac{d^dk}{(2\pi)^d}\left(\frac{2ik\cdot\partial + \partial^2 - \frac{\lambda_2}{2}\varphi_{\rm cl}^2}{k^2 - M^2}\right)^n\,. \tag{1.24}$$

Remember that for the matching, we need to evaluate the integrals in the hard region, meaning $k^2 \sim M^2$. This means that terms of higher order in n are suppressed by the hard scale and that we can truncate the sum at a given order. To obtain the operators up to order $\mathcal{O}(1/\Lambda^2)$, we truncate the sum at $n = 3$. Let us work out the terms, order by order in this expansion. The first order simply gives us:

$$S_\Lambda^{(n=1)} = -\frac{i}{2}\int d^dx \int \frac{d^dk}{(2\pi)^d}\frac{2ik\cdot\partial + \partial^2 - \frac{\lambda_2}{2}\varphi_{\rm cl}^2}{k^2 - M^2}\,. \tag{1.25}$$

The derivatives act on the vacuum, so they vanish. The only contribution is thus

$$S_\Lambda^{(n=1)} = -\frac{i\lambda_2}{4}\int d^dx\,\varphi_{\rm cl}^2 \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2 - M^2}\,. \tag{1.26}$$

To evaluate this and following integrals, let us introduce a handy master forumla: In the hard region, all loop integrals will be of the form:

$$I_{\alpha\beta}(M^2) = \int \frac{d^dk}{(2\pi)^d} \left(\frac{1}{k^2}\right)^{\alpha} \left(\frac{1}{k^2 - M^2}\right)^{\beta}, \tag{1.27}$$

which can be evaluated in closed form to find:

$$I_{\alpha\beta}(M^2) = \frac{i}{16\pi^2} \left(-\frac{1}{M^2}\right)^{2+\alpha+\beta} \left(\frac{4\pi\bar{\mu}^2}{M^2}\right)^{\epsilon} \frac{\Gamma(2-\alpha-\epsilon)\Gamma(\alpha+\beta+\epsilon-2)}{\Gamma(\beta)\Gamma(2-\epsilon)}, \tag{1.28}$$

with $\bar{\mu}$ being the renormalization scale. In the $\overline{\text{MS}}$ scheme, we have

$$\bar{\mu}^2 = \frac{e^{\gamma_E}}{4\pi}\mu^2 . \tag{1.29}$$

Evaluating the expression (1.26) at the renormalization scale $\mu^2 = M^2$ and absorbing the UV divergence into a counterterm, we find the contribution to the Lagrangian

$$\Delta\mathcal{L}_{\text{eff}}^{(n=1)} = \frac{M^2}{2}\frac{\lambda_2}{32\pi^2}\varphi_{\text{cl}}^2 . \tag{1.30}$$

This is a correction to the mass term of the classical field. In the Language of Feynman graphs, this term corresponds to the loop diagram:

$$\rightarrow \qquad \frac{M^2}{2}\frac{\lambda_2}{32\pi^2}\varphi_{\text{cl}}^2 . \tag{1.31}$$

Moving on to the second term in the expansion we have:

$$S_{\Lambda}^{(n=2)} = -\frac{i}{4}\int d^dx \int \frac{d^dk}{(2\pi)^d} \frac{(2ik\cdot\partial + \partial^2 - \frac{\lambda_2^2}{2}\varphi_{\text{cl}}^2)(2ik\cdot\partial + \partial^2 - \frac{\lambda_2^2}{2}\varphi_{\text{cl}}^2)}{(k^2-M^2)^2} . \tag{1.32}$$

In the second bracket, we can drop derivatives, as they act on the vacuum again. Thus:

$$\begin{aligned} S_{\Lambda}^{(n=2)} &= \frac{i\lambda_2}{8}\int d^dx \int \frac{d^dk}{(2\pi)^d} \frac{(2ik\cdot\partial + \partial^2 - \frac{\lambda_2^2}{2}\varphi_{\text{cl}}^2)}{(k^2-M^2)^2}\varphi_{\text{cl}}^2 . \\ &= \int d^dx \int \frac{d^dk}{(2\pi)^d}\left[-\frac{\lambda_2}{4}\frac{(k\cdot\partial)\varphi_{\text{cl}}^2}{(k^2-M^2)^2} + \frac{i\lambda_2}{8}\frac{\partial^2\varphi_{\text{cl}}^2}{(k^2-M^2)^2} - \frac{i\lambda_2^2}{16}\frac{\varphi_{\text{cl}}^4}{(k^2-M^2)^2}\right] . \end{aligned} \tag{1.33}$$

The first and second term in the brackets do not contribute since they represent total derivatives. Only the last term contributes a matching correction to the quartic coupling of the light scalar:

$$\Delta\mathcal{L}_{\text{eff}}^{(n=2)} = -\frac{i\lambda_2^2}{16}\varphi_{\text{cl}}^4\, I_{0\,2}(M^2) . \tag{1.34}$$

Again absorbing the divergence into a counterterm choosing $\mu^2 = M^2$, we find that this integral has no finite term. There is no correction to the quartic coupling at this order in n.

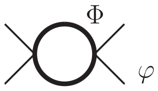
Finally, let us evaluate the third order in our sum. We now have three insertions of the operator $(2ik\cdot\partial+\partial^2-\lambda_2\varphi_{\rm cl}^2/2)$. In the third bracket, we can again drop derivatives as they are acting on nothing. In the first bracket, we can drop derivatives as well, since they only generate terms that are total derivatives. In the middle bracket, the term linear in k can be dropped since the integral over the momentum vanishes by symmetry. Therefore, we have:

$$\begin{aligned} S_\Lambda^{(n=3)} &= -\frac{i}{6}\int d^dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-M^2)^3}\left(-\frac{\lambda_2}{2}\varphi_{\rm cl}^2\right)\left(\partial^2-\frac{\lambda_2}{2}\varphi_{\rm cl}^2\right)\left(-\frac{\lambda_2}{2}\varphi_{\rm cl}^2\right)\\ &= -\frac{i\lambda_2^2}{24}\int d^dx\int\frac{d^dk}{(2\pi)^d}\frac{\varphi_{\rm cl}^2\partial^2\varphi_{\rm cl}^2}{(k^2-M^2)^3}+\frac{i\lambda_2^3}{48}\int d^dx\int\frac{d^dk}{(2\pi)^d}\frac{\varphi_{\rm cl}^6}{(k^2-M^2)^3}\,. \end{aligned} \tag{1.35}$$

This generates a new quartic interaction as well as a six-point interaction:

$$\mathcal{L}_{\rm eff}^{(n=3)} = -\frac{\lambda_2^2}{48(16\pi^2)}\varphi_{\rm cl}^2\partial^2\varphi_{\rm cl}^2+\frac{\lambda_2^6}{96(16\pi^2)}\varphi_{\rm cl}^6\,. \tag{1.36}$$

The quartic operator comes from the one-loop diagram with two heavy scalars in the loop:



$$\rightarrow \qquad -\frac{\lambda_2^2}{48(16\pi^2)}\varphi_{\rm cl}^2\partial^2\varphi_{\rm cl}^2\,. \tag{1.37}$$

One might now wonder how this diagram can give us an expression with three heavy propagators, as it can be seen from the last line in eq. (1.35). The way this comes about is that this is a subleading-power expression in the expansion of external momenta over the heavy mass:

$$\left[\frac{1}{(l+p)^2-m^2}\right]=\frac{1}{l^2-m^2}\left[1-\frac{2l\cdot p+p^2}{l^2-M^2}+\dots\right]. \tag{1.38}$$

The leading piece ends up in the correction to the $\varphi_{\rm cl}^4$ operator, whereas the next-to-leading piece introduces the term we find in our expansion at $n=3$. The p^2 in the numerator of the expansion of the heavy propagator corresponds to the ∂^2 in the effective operator in (1.37).

We have computed the one-loop effective Lagrangian for the theory (1.17). While the expressions can be related to Feynman diagrams, we have done so without computing a single diagram, notably without having to worry about any kinds of symmetry and combinatorial factors. The example we have worked out was lucky enough to have $X_{SH}=0$, meaning that all the loop diagrams contained only heavy scalars. There are no diagrams where we had both light scalars and heavy scalars in the loop. We should conclude this chapter by elaborating on the additional complication that arise in the more general scenario where this is not the case.

For $X_{SH}\neq 0$, the fluctuation operator $\tilde\Delta_H$ contains a term of the form

$$\tilde\Delta_H \supset X_{SH}^\dagger\Delta_S^{-1}X_{SH}\,. \tag{1.39}$$

In this case, we should further separate Δ_S into the soft kinetic term and interactions:

$$\Delta_S = \tilde{\Delta}_S + X_S \,. \tag{1.40}$$

Then we can write the inverse of Δ_S appearing in (1.39) using the Neumann expansion:

$$\Delta_S^{-1} = \sum_{n=0}^{\infty} (-1)^n \left(\tilde{\Delta}_S^{-1} X_S \right)^n \tilde{\Delta}_S^{-1} \,, \tag{1.41}$$

where $\tilde{\Delta}_S^{-1}$ simply gives the propagator of the soft field. One can then work out the full fluctuation operator to a given order in the expansion above. This will reproduce terms in S_H corresponding to diagrams with both heavy and light fields in the loop. These loop integrals can always be evaluated with our master formula (1.28).