Exercise 1. ***Green's function***

Compute the retarded and advanced Green functions defined as

$$G_k^{(\pm)}(\vec{r}-\vec{r}') = \langle\vec{r}|G_0^{(\pm)}(E_k)|\vec{r}'\rangle = \langle\vec{r}|(E_k - H_0 \pm i0^+)^{-1}|\vec{r}'\rangle. \tag{1}$$

and prove that the adopted prescription $E \to E \pm i0^+$ leads to the expected asymptotic behaviour of the Green functions, namely

$$G_k^{(+)}(\vec{r}) \approx \frac{e^{+ikr}}{r}, \tag{2}$$

$$G_k^{(-)}(\vec{r}) \approx \frac{e^{-ikr}}{r}. \tag{3}$$

Exercise 2. ***Retarded Green operator***

Consider the state

$$|\psi^+, t\rangle \equiv \lim_{t'\to-\infty} i\hbar G^+(t-t')|\psi_0, t'\rangle$$

where G^+ is the retarded Green operator to the full Hamiltonian $H = H_0 + V$ and $|\psi_0, t'\rangle$ is a free state.

(a) Show that $|\psi^+, t\rangle$ satisfies the Schrödinger equation $i\hbar\partial_t|\psi^+, t\rangle = H|\psi^+, t\rangle$ with the full Hamiltonian and approaches the free state $|\psi_0, t\rangle$ for $t \to -\infty$

(b) Show that $|\psi^+, t\rangle$ can be written as

$$|\psi^+, t\rangle = |\psi_0, t\rangle + \int dt' G^+(t-t')V|\psi_0, t'\rangle$$

Hint: prove first

$$\partial_{t'} i\hbar G^+(t-t')|\psi_0, t'\rangle = -\delta(t-t')|\psi_0, t'\rangle - G^+(t-t')V|\psi_0, t'\rangle$$

and then integrate with respect to t'.

(c) Show that the relation in (b) is equivalent to

$$|\psi_\alpha^+\rangle = (1 + G^+(E)V)|\psi_\alpha^0\rangle$$

where $|\psi_\alpha^+\rangle$ and $|\psi_\alpha^0\rangle$ satisfy $H|\psi_\alpha^+\rangle = E_\alpha|\psi_\alpha^+\rangle$ and $H_0|\psi_\alpha^0\rangle = E_\alpha|\psi_\alpha^0\rangle$ respectively.

Exercise 3. ***S matrix***

In the lecture, the first two terms of the S-matrix $S_{\beta\alpha} \equiv \langle\psi_\beta^0|S|\psi_\alpha^0\rangle$ have been computed as

$$S_{\beta\alpha} = \delta(\beta-\alpha) - 2i\pi\delta(E_\alpha - E_\beta)V_{\beta\alpha} + \dots$$

Compute the third and fourth term in this expansion.