MMP I
Solution Sheet 2

HS 21
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`https://www.physik.uzh.ch/en/teaching/PHY312`

Issued: 30.09.2021
Due: 07.10.2021

Exercise 1 [Fourier Analysis and Minimization (4 points)]

a)

$$
\begin{aligned}
&f(x) = \pi - |x|, -\pi \leq x \leq \pi, \quad f(-x) = f(x) \Rightarrow b_n = 0 \\
&a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - |x|)dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)dx = \frac{2}{\pi}\left[\pi x - \frac{1}{2}x^2\right]_0^{\pi} = \underline{\underline{\pi}} \\
&a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos(nx)dx = \frac{2}{\pi}\left(\left[(\pi - x)\frac{1}{n}\sin nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin(nx)dx\right) \\
&\quad = -\frac{2}{n^2\pi}\left[\cos(nx)\right]_0^{\pi} = \underline{\underline{\frac{2}{n^2\pi}\left[1 - (-1)^n\right]}}
\end{aligned}
$$

b) We know

$$\int_a^b |f(x) - \sum \alpha_i \phi_i|^2 dx$$

is minimal if we choose $\alpha_i = (f, \phi_i)$, where $\phi_1, \phi_2 ...$ is an orthonormal system, i.e. $(\phi_i, \phi_j) = \delta_{ij}$. Such a system is given by

$$\phi_i = \frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, ...$$

Therefore:

$$
\begin{aligned}
\sum \alpha_i \phi_i &= a\cos(3x) + b\sin(4x) \\
&= \underbrace{a\sqrt{\pi}}_{:=\alpha_1} \cdot \underbrace{\frac{\cos(3x)}{\sqrt{\pi}}}_{:=\phi_1} + \underbrace{b\sqrt{\pi}}_{:=\alpha_2} \cdot \underbrace{\frac{\sin(4x)}{\sqrt{\pi}}}_{:=\phi_2}
\end{aligned}
$$

So:

$$\begin{aligned}
\alpha_1 = (f, \phi_1) &= \int_{-\pi}^{\pi} f(x)\overline{\phi_1(x)}dx \\
&= \int_{-\pi}^{\pi} (\pi - |x|)\frac{\cos(3x)}{\sqrt{\pi}}dx \overset{\text{even}}{=} 2\int_0^{\pi} (\pi - x)\frac{\cos(3x)}{\sqrt{\pi}}dx \\
&= \frac{4}{9\sqrt{\pi}} \overset{!}{=} a\sqrt{\pi} \\
&\Rightarrow a = \frac{\alpha_1}{\sqrt{\pi}} = \underline{\underline{\frac{4}{9\pi}}}
\end{aligned}$$

and

$$\begin{aligned}
\alpha_2 = (f, \phi_2) &= \int_{-\pi}^{\pi} (\pi - |x|)\frac{\sin(4x)}{\sqrt{\pi}}dx = 0 \overset{!}{=} b\sqrt{\pi} \\
&\Rightarrow b = \underline{\underline{0}}
\end{aligned}$$

Exercise 2 [Legendre Polynomials (4 points)]

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n \left[(x^2-1)^n\right]$$

To show: P_n are orthogonal on the interval $[-1, 1] \Leftrightarrow (P_n, P_m) = N_{nm}\delta_{nm}$.
Important relation:

$$\left(\frac{d}{dx}\right)^k (x^2-1)^n\Big|_{x=\pm 1} = ?$$

$$\begin{aligned}
\rightarrow \frac{d}{dx}(x^2-1)^n &= n(x^2-1)^{n-1}2x \\
\left(\frac{d}{dx}\right)^2 (x^2-1)^n &= \underbrace{n(n-1)(x^2-1)^{n-2}2x}_{(x^2-1)\text{term of highest order}} + 2n(x^2-1)^{n-1} \\
&\dots \\
\left(\frac{d}{dx}\right)^k (x^2-1)^n &= n(n-1)\dots(n-k+1)(x^2-1)^{n-k}2x + \dots \\
&\Rightarrow x = \pm 1 \Rightarrow \left(\frac{d}{dx}\right)^k (x^2-1)^n = 0 \ \forall k < n
\end{aligned}$$

Compute the inner product:

$$(P_n, P_m) = \int_{-1}^{1} P_n(x)\overline{P_m}(x)dx = \frac{1}{2^{n+m}n!m!}\int_{-1}^{1} \underbrace{\left(\frac{d}{dx}\right)^n (x^2-1)^n}_{u} \underbrace{\left(\frac{d}{dx}\right)^m (x^2-1)^m}_{v} dx$$

$$\overset{\text{int. by parts}}{=} \frac{1}{2^{n+m}n!m!}\left(\left[\left(\frac{d}{dx}\right)^n (x^2-1)^n \left(\frac{d}{dx}\right)^{m-1} (x^2-1)^m\right]_{-1}^{1} \to 0\right.$$

$$\left. - \int_{-1}^{1} \left(\frac{d}{dx}\right)^{n+1} (x^2-1)^n \left(\frac{d}{dx}\right)^{m-1} (x^2-1)^m dx\right)$$

$$= \underbrace{\ldots}_{i \text{ times}} = \frac{(-1)^i}{2^{n+m}n!m!}\int_{-1}^{1} \left(\frac{d}{dx}\right)^{n+i} (x^2-1)^n \left(\frac{d}{dx}\right)^{m-i} (x^2-1)^m dx$$

$\to$ Boundary terms vanish as long as $m > m - i \Rightarrow$ always for $i > 0$.

Case $m \neq n$: Without loss of generality, choose $m > n$.
$\to$ choose $i = n + 1$ $(m - 1 = m - n - 1 \geq 0)$

$$\Rightarrow \left(\frac{d}{dx}\right)^{2n+1} (x^2-1)^n = \left(\frac{d}{dx}\right)^{2n+1} (x^{2n} + \ldots) = 0$$

$$\Rightarrow (P_n, P_m) = 0 \text{ for } m \neq n$$

Case $m = n$: $\to$ choose $i = n$ $(m - i = m - n = 0)$

$$(P_m, P_n) = \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1} \underbrace{\left(\frac{d}{dx}\right)^{2n} (x^2-1)^n}_{(2n)!} \left(\frac{d}{dx}\right)^0 (x^2-1)^n dx$$

$$= \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n dx = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1} \underbrace{(x+1)^n}_{u} \underbrace{(x-1)^n}_{v'} dx$$

$$\overset{\text{int by parts}}{=} \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\left(\left[(x+1)^n \frac{1}{n+1}(x-1)^{n+1}\right]_{-1}^{1} \to 0 - \frac{n}{n+1}\int_{-1}^{1}(x+1)^{n-1}(x-1)^{n+1}dx\right)$$

$$= \underbrace{\ldots}_{i \text{ times}} = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}(-1)^i \frac{n(n-1)\ldots(n-i+1)}{(n+1)(n+2)\ldots(n+i)}\int_{-1}^{1}(x+1)^{n-i}(x-1)^{n+i}dx$$

$$\overset{\text{choose } i=n}{=} \frac{\cancel{(-1)^{2n}}\cancel{2n!}}{2^{2n}\cancel{(n!)^2}} \frac{\cancel{n!}}{\cancel{(2n)!}\cancel{n!}}\int_{-1}^{1}(x-1)^{2n}dx$$

$$= \frac{1}{2^{2n}}\frac{1}{2n+1}\left[(x-1)^{2n+1}\right]_{-1}^{1} = \frac{1}{2^{2n}}\frac{1}{2n+1}(-1)(-2)^{2n+1} = \underline{\underline{\frac{2}{2n+1}}}$$

$\Rightarrow \underline{\underline{(P_n, P_m) = \frac{2}{2n+1}\delta_{nm}}}$

Exercise 3 [Recursive Relations for Legendre Polynomials (2 points)]

$$\begin{aligned} P_n(x) &= \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n \left[(x^2-1)^n\right] \\ \Rightarrow P_1(x) &= \frac{1}{2}\frac{d}{dx}(x^2-1) = \underline{\underline{x}} \\ P_2(x) &= \frac{1}{8}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}(2(x^2-1)\cdot 2x) = \underline{\underline{\frac{1}{2}\left[3x^2-1\right]}} \end{aligned}$$

$P_n(0)$:

$$\begin{aligned} P_n(x) &= \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n \sum_{k=0}^{n}\binom{n}{k} x^{2k}(-1)^{n-k} \\ &= \frac{(-1)^n}{2^n \cancel{n!}}\sum_{k=0}^{n}\frac{\cancel{n!}}{k!(n-k)!}(-1)^k \frac{d^n}{dx^n}x^{2k} \\ &= \frac{(-1)^n}{2^n}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\cdot \begin{cases} 0 & n > 2k \\ \frac{(2k)!}{(2k-n)!}\underbrace{x^{2k-n}}_{*} & n \le 2k \end{cases} \end{aligned}$$

$\Rightarrow$ as long as x appears in the sum, the term (*) vanishes
$\Rightarrow$ $2k = n$ is the only contributor ($\rightarrow * = x^0$)
$\Rightarrow$ x is always present if n is odd
$\Rightarrow$ $P_n(0) = 0$ if n is odd

$$\begin{aligned} \Rightarrow P_n(x) \underbrace{=}_{n=2k \text{ even}} & \frac{1}{2^n}\frac{(-1)^{n/2} n!}{(\frac{n}{2})!(\frac{n}{2})!} \\ &= (-1)^{\frac{n}{2}}\frac{n!}{2^n\left[(\frac{n}{2})!\right]^2} \\ &= \begin{cases} (-1)^{\frac{n}{2}}\frac{n!}{2^n\left[(\frac{n}{2})!\right]^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \end{aligned}$$

$\rightarrow P_0(0) = 1,\ P_2(0) = -\frac{1}{2}$, etc.

Recursive Relations:

i)

Claim:

$$(xf(x))^{(n)} = xf^{(n)}(x) + nf^{(n-1)}(x)$$

Proof: (by induction)

- $\underline{n = 1:}$

$$(xf)^{(1)} = xf' + 1 \cdot f = xf^{(1)} + 1 \cdot f^{(0)}$$

- $\underline{n \to n+1:}$ assume it holds for n

$$\begin{aligned}(xf)^{(n+1)} &= \frac{d}{dx}(xf)^{(n)} \\ &= \frac{d}{dx}\left[xf^{(n)} + nf^{(n-1)}\right] \\ &= f^{(n)} + xf^{(n+1)} + nf^{(n)} \\ &= xf^{(n+1)} + (n+1)f^{(n)}\end{aligned}$$

ii)

$$\begin{aligned}\frac{d}{dx}P_{n+1}(x) &= \frac{1}{2^{n+1}(n+1)!}\left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right)^{n+1}\left(x^2-1\right)^{n+1} \\ &= \frac{1}{2^{n+1}(n+1)!}\left(\frac{d}{dx}\right)^{n+1} 2x(n+1)(x^2-1)^n \\ &= \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^{n+1} x \cdot \underbrace{(x^2-1)^n}_{=:f(x)} \\ &\underbrace{=}_{i)} \frac{1}{2^n n!}\left[x\left(\frac{d}{dx}\right)^{n+1}(x^2-1)^n + (n+1)\left(\frac{d}{dx}\right)^n (x^2-1)^n\right] \\ &= x\frac{d}{dx}\left[\frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n\right] + (n+1)\left[\frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n\right] \\ &= \underline{\underline{xP_n'(x) + (n+1)P_n(x)}}\end{aligned}$$

$\square$

Exercise 4 [Symmetry Considerations (2 points)]

Additional Theorems:

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$
$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

Let $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$

and consider the symmetry $f(\frac{\pi}{2} - x) = f(\frac{\pi}{2} + x)$, $\forall x \in \mathbb{R}$:

$$\begin{aligned}
&\cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \Big(a_n \cos(\frac{n\pi}{2} - nx) + b_n \sin(\frac{n\pi}{2} - nx) \Big) \\
&= \cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \Big(a_n \cos(\frac{n\pi}{2} + nx) + b_n \sin(\frac{n\pi}{2} + nx) \Big) \\
&\Leftrightarrow \sum_{n=1}^{\infty} \Big\{ a_n \Big[\cos(\frac{n\pi}{2})\cos(nx) + \sin(\frac{n\pi}{2})\sin(nx) \Big] + b_n \Big[\sin(\frac{n\pi}{2})\cos(nx) - \cos(\frac{n\pi}{2})\sin(nx) \Big] \Big\} \\
&= \sum_{n=1}^{\infty} \Big\{ a_n \Big[\cos(\frac{n\pi}{2})\cos(nx) - \sin(\frac{n\pi}{2})\sin(nx) \Big] + b_n \Big[\sin(\frac{n\pi}{2})\cos(nx) + \cos(\frac{n\pi}{2})\sin(nx) \Big] \Big\}
\end{aligned}$$

By this we retain the structure of a Fourier polynomial. We collect the coefficients of $\cos(nx)$ and $\sin(nx)$:

$$\begin{aligned}
&\Leftrightarrow \sum_{n=1}^{\infty} \Big\{ \Big[a_n \cos(\frac{n\pi}{2}) + b_n \sin(\frac{n\pi}{2}) \Big] \cos(nx) + \Big[a_n \sin(\frac{n\pi}{2}) - b_n \cos(\frac{n\pi}{2}) \Big] \sin(nx) \Big\} \\
&= \sum_{n=1}^{\infty} \Big\{ \Big[a_n \cos(\frac{n\pi}{2}) + b_n \sin(\frac{n\pi}{2}) \Big] \cos(nx) + \Big[-a_n \sin(\frac{n\pi}{2}) + b_n \cos(\frac{n\pi}{2}) \Big] \sin(nx) \Big\}
\end{aligned}$$

The functions $\cos(nx)$ and $\sin(nx)$ are independent orthogonal vectors which may be projected on. We obtain

$$a_n \sin(\frac{n\pi}{2}) - b_n \cos(\frac{n\pi}{2}) = 0 \qquad \forall n \in \mathbb{N}$$

We can differentiate between two cases: As $\sin(\frac{n\pi}{2}) = 0$ for n even, and $\cos(\frac{n\pi}{2}) = 0$ for n odd we obtain

$$\begin{cases} a_n \underbrace{\sin(\frac{n\pi}{2})}_{\neq 0} = 0 \quad \text{for } n \text{ odd} \\ b_n \underbrace{\cos(\frac{n\pi}{2})}_{\neq 0} = 0 \quad \text{for } n \text{ even} \end{cases} \Rightarrow \quad \begin{matrix} a_n = 0 \quad (n \text{ odd}) \\ \\ b_n = 0 \quad (n \text{ even}) \end{matrix}$$

For such a symmetry, all the a_n are zero for odd n, while all the b_n are zero for even n.
$\rightarrow$ A simple plot would already be motivation enough: