

Website: <http://www.physik.uzh.ch/en/teaching/PHY519/>

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**Exercise 1** [Metric of a static star]

We start from the general static, spherical symmetric metric

$$ds^2 = \exp [2\alpha(r)]dt^2 - \exp [2\beta(r)]dr^2 - r^2d\Omega^2. \quad (1)$$

From the normalization of the velocity in the fluid restframe we obtain  $u_t = \sqrt{g_{tt}}$ , which then straightforwardly leads to the energy-momentum tensor

$$T_{\mu\nu} = \text{diag} \{ \exp [2\alpha] \rho, \exp [2\beta] p, r^2 p, r^2 \sin^2 (\theta) p \}. \quad (2)$$

The Ricci scalar for the above metric ansatz read as

$$R = 2 \exp [-2\beta] \left( \partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{2}{r} (\partial_r \alpha - \partial_r \beta) + \frac{1}{r^2} (1 - \exp [2\beta]) \right). \quad (3)$$

For the time component of the Einstein equations we thus have

$$R_{tt} - \frac{1}{2} R g_{tt} = 8\pi G T_{tt} \quad (4)$$

$$- \exp [2(\alpha - \beta)] \left( -\frac{2}{r} \partial_r \beta + \frac{1}{r^2} (1 - \exp [2\beta]) \right) = 8\pi G \rho \exp [2\alpha] \quad (5)$$

$$\frac{1}{r^2} \partial_r m(r) = 4\pi \rho, \quad (6)$$

where  $m(r)$  is defined in Eq. (3) of the exercise sheet. This differential equation can be integrated to obtain the mass within a given radius  $r$

$$m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'. \quad (7)$$

Note that it is different from the integrated energy density, which is obtained integrating the density over the volume element  $\sqrt{\gamma}d^3x$ . Here  $\gamma$  is the determinant of the spatial metric. In contrast to the above, the integrated energy density accounts for the gravitational binding energy as well.

With the replacement of  $m(r)$  we can immediately write down the radial metric element

$$\exp [2\beta] = \left( 1 - \frac{2Gm(r)}{r} \right)^{-1}, \quad (8)$$

which has remarkable similarity with the metric element of the Schwarzschild solution. For the radial component we have

$$R_{rr} - \frac{1}{2} R g_{rr} = 8\pi G T_{rr} \quad (9)$$

$$\frac{2}{r} \partial_r \alpha - \frac{2G}{r^3} \exp [2\beta] m(r) = 8\pi G p \quad (10)$$

thus

$$\frac{d\alpha}{dr} = \frac{4\pi Gr^3 p + Gm(r)}{r[r - 2Gm(r)]} \quad (11)$$

The Bianchi identity or energy-momentum conservation reads as

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma_{\mu\rho}^\mu T^{\rho\nu} + \Gamma_{\mu\rho}^\nu T^{\mu\rho}. \quad (12)$$

While the  $\nu = t$  equation is trivial, the  $\nu = r$  equation leads to

$$\nabla_\mu T^{\mu r} = \partial_r(\exp[-2\beta]p) + T^{rr}(\Gamma_{rr}^r + \Gamma_{\theta r}^\theta + \Gamma_{\phi r}^\phi + \Gamma_{tr}^t) \quad (13)$$

$$+ (\Gamma_{tt}^r T^{tt} + \Gamma_{rr}^r T^{rr} + \Gamma_{\phi\phi}^r T^{\phi\phi} + \Gamma_{\theta\theta}^r T^{\theta\theta}) \quad (14)$$

$$= \exp[-2\beta] \{ (\partial_r p - 2p\partial_r\beta) + p(\partial_r\beta + 2/r + \partial_r\alpha) + (\rho\partial_r\alpha + \partial_r\beta p - 2p/r) \} \quad (15)$$

$$= \exp[-2\beta] \{ (\rho + p)\partial_r\alpha + \partial_r p \} \stackrel{!}{=} 0 \quad (16)$$

From this we obtain

$$(\rho + p) \frac{d\alpha}{dr} = -\frac{dp}{dr}. \quad (17)$$

Plugging in Eq. (11) we finally have

$$\frac{dp}{dr} = -\frac{(\rho + p) [Gm(r) + 4\pi Gr^3 p]}{r[r - 2Gm(r)]}. \quad (18)$$

Now we assume  $\rho = \rho_* = \text{const.}$  out to the surface of the star yielding

$$m(r) = \frac{4\pi}{3} \rho_* r^3 = M \frac{r^3}{R^3}, \quad (19)$$

where  $M$  is the total mass of the star and from now on  $R$  will denote the radius of the star. From Eq. (18) we obtain by separation of variables

$$\frac{dp}{p^2 + \frac{4}{3}\rho_* p + \frac{\rho_*^2}{3}} = \frac{4\pi Gr dr}{\frac{8\pi G}{3} r^2 \rho_* - 1} \quad (20)$$

$$\frac{dp}{(p + \frac{2}{3}\rho_*)^2 - \frac{\rho_*^2}{9}} = \frac{3}{4\rho_*} \frac{\frac{16\pi G}{3} r \rho_* dr}{\frac{8\pi G}{3} r^2 \rho_* - 1} \quad (21)$$

Using that the pressure vanishes at the surface of the star we can integrate to obtain

$$-\frac{1}{2} \frac{3}{\rho_*} \ln \left[ \frac{p' + \frac{2}{3}\rho_* + \frac{1}{3}\rho_*}{p' + \frac{2}{3}\rho_* - \frac{1}{3}\rho_*} \right]_{p'=p}^0 = \frac{3}{4\rho_*} \ln \left[ \frac{8\pi G}{3} r'^2 \rho_* - 1 \right]_{r'=R}^R \quad (22)$$

$$\ln \left[ \frac{p + \rho_*}{3p + \rho_*} \right] = \frac{1}{2} \ln \left[ \frac{2GMR^2 - R^3}{2GMr^2 - R^3} \right]. \quad (23)$$

Solving for  $p(r)$  yields the radial pressure distribution

$$p(r) = \rho_* \frac{\sqrt{R^3 - 2GMR^2} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3\sqrt{R^3 - 2GMR^2}}. \quad (24)$$

For the star of maximum mass the pressure diverges at  $r = 0$  and there is no static solution, i. e. the star collapses. The maximum mass corresponds to the case where the denominator vanishes at  $r = 0$

$$R = 9(R - 2GM), \quad (25)$$

leading to

$$M < \frac{4}{9} \frac{R}{G}. \quad (26)$$

This result remains true for more general equations of state and is known as Buchdahl's theorem. Now it remains to calculate the missing metric coefficients. From Eq. (11) we have

$$\frac{d\alpha}{dr} = -\frac{d \ln(p + \rho_*)}{dr}. \quad (27)$$

The boundary conditions can be set up at the surface of the star, where the interior solution goes over into the exterior Schwarzschild solution, thus  $\rho + p|_{r=R} = \rho_*$  and

$$\exp[\alpha(R)] = \sqrt{1 - \frac{2GM}{R}}. \quad (28)$$

In summary, we have for the metric elements

$$\exp[\alpha] = \frac{3}{2} \sqrt{1 - \frac{2GM}{R}} - \frac{1}{2} \sqrt{1 - \frac{2GM r^2}{R^3}}, \quad (29)$$

$$\exp[-\beta] = \sqrt{1 - \frac{2Gm(r)}{r}} = \sqrt{1 - \frac{2GM r^2}{R^3}}. \quad (30)$$