MMP I
Solution Sheet 4

HS 21
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Exercise 1 [Vector space (4 points)]

$$
\begin{aligned}
V &= \mathbb{C} && \text{(vector space)}\\
\langle\cdot|\cdot\rangle &= V \times V \to \mathbb{C} && \text{(scalar product)}\\
\langle x|y\rangle &= \overline{\langle y|x\rangle}\\
\langle x|\lambda y\rangle &= \overline{\lambda}\langle x|y\rangle\\
\langle \lambda x|y\rangle &= \lambda\langle x|y\rangle
\end{aligned}
$$

Norm $||x|| = \sqrt{\langle x|x\rangle}$ $\qquad |\langle x|y\rangle|^2 = \langle x|y\rangle\overline{\langle x|y\rangle}$

For any system $\{y_k \in V|\langle y_k|y_l\rangle = \delta_{kl}\}$ (orthonormal) show the following relations:

a) Bessel Identity (2 points)

$$
\begin{aligned}
&\sum_{k=1}^{n}|\langle x|y_k\rangle|^2 + ||x - \sum_{k=1}^{n}\langle x|y_k\rangle y_k||^2\\
&= \sum_k |\langle x|y_k\rangle|^2 + \left\langle x - \sum_k \langle x|y_k\rangle y_k \,\middle|\, x - \sum_l \langle x|y_l\rangle y_l \right\rangle\\
&= \sum_k |\langle x|y_k\rangle|^2 + \langle x|x\rangle - \left\langle x \middle| \sum_l \langle x|y_l\rangle y_l \right\rangle\\
&\quad - \left\langle \sum_k \langle x|y_k\rangle y_k \middle| x \right\rangle + \left\langle \sum_k \langle x|y_k\rangle y_k \middle| \sum_l \langle x|y_l\rangle y_l \right\rangle\\
&= \sum_k |\langle x|y_k\rangle|^2 + ||x||^2 - \sum_l \left\langle x \middle| \langle x|y_l\rangle y_l \right\rangle\\
&\quad - \sum_k \left\langle \langle x|y_k\rangle y_k \middle| x \right\rangle + \sum_k \sum_l \left\langle \langle x|y_k\rangle y_k \middle| \langle x|y_l\rangle y_l \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_k |\langle x|y_k\rangle|^2 + ||x||^2 - \sum_l \overline{\langle x|y_l\rangle}\langle x|y_l\rangle \\
&\quad - \sum_k \langle x|y_k\rangle\langle y_k|x\rangle + \sum_k \sum_l \langle x|y_k\rangle\langle y_k|y_l\rangle\overline{\langle x|y_l\rangle} \\
&= \sum_k |\langle x|y_k\rangle|^2 + ||x||^2 - \sum_l |\langle x|y_l\rangle|^2 - \sum_k |\langle x|y_k\rangle|^2 + \sum_k |\langle x|y_k\rangle|^2 \\
&= ||x||^2
\end{aligned}
$$

b) Bessel Inequality (0.5 points)

$$||x||^2 \geq \sum_{k=1}^{n} |\langle x|y_k\rangle|^2$$

We can use a)
$||x||^2 = \sum_{k=1}^{n} |\langle x|y_k\rangle|^2 + ||x - \sum_{k=1}^{n}\langle x|y_k\rangle y_k||^2$
$||x - \sum_{k=1}^{n}\langle x|y_k\rangle y_k||^2 \geq 0$ by definition
$\Rightarrow ||x||^2 \geq \sum_{k=1}^{n} |\langle x|y_k\rangle|^2$.

c) Schwarz inequality (1.5 points)

$$||x|| \cdot ||y|| \geq |\langle x|y\rangle| \quad \forall x, y \in V.$$

If $y = 0$, the inequality is trivially satisfied. If $y \neq 0$, by definition,

$$||x - \lambda y||^2 \geq 0,$$

and therefore,

$$
\begin{aligned}
0 \leq ||x - \lambda y||^2 &= \langle x - \lambda y | x - \lambda y\rangle \\
&= \langle x|x\rangle - \lambda\langle y|x\rangle - \overline{\lambda}\langle x|y\rangle + |\lambda|^2\langle y|y\rangle.
\end{aligned}
$$

Let's choose

$$\lambda = \frac{\langle x|y\rangle}{\langle y|y\rangle}.$$

Then

$$
\begin{aligned}
&0 \leq ||x||^2 - \frac{\langle x|y\rangle}{\langle y|y\rangle}\langle y|x\rangle - \frac{\overline{\langle x|y\rangle}}{\langle y|y\rangle}\langle x|y\rangle + \frac{|\langle x|y\rangle|^2}{|\langle y|y\rangle|^2}\langle y|y\rangle \\
&= ||x||^2 - \frac{|\langle x|y\rangle|^2}{||y||^2}.
\end{aligned}
$$

Therefore

$$||x||^2||y||^2 \geq |\langle x|y\rangle|^2 \quad \Rightarrow \quad ||x|| \cdot ||y|| \geq |\langle x|y\rangle.$$

Exercise 2 [Fourier Transform and Fourier Integral (2 points)]

$$f(x) = \begin{cases} 0, & |x| > a \\ 2a - 2|x|, & |x| \leq a \end{cases}$$

Cosine FT:

$$\begin{aligned}
\hat{f}_{\cos}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x)\cos(kx)dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (2a - 2|x|)\cos(kx)dx \\
&= \frac{4}{\sqrt{2\pi}} \int_{0}^{a} (a - x)\cos(kx)dx \\
&= \frac{4}{\sqrt{2\pi}} \left(\left[(a-x)\frac{1}{k}\sin(kx) \right]_0^a - \int_0^a (-1)\frac{1}{k}\sin(kx)dx \right) \\
&= \frac{4}{\sqrt{2\pi}k} \int_0^a \sin(kx)dx = -\frac{4}{\sqrt{2\pi}k^2}\left[\cos(ka) - 1\right]
\end{aligned}$$

Exponential FT:

$$\begin{aligned}
\hat{f}_{\exp}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x)e^{-ikx}dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (2a - 2|x|)e^{-ikx}dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{0} (2a - 2|x|)e^{-ikx}dx + \int_0^a (2a - 2|x|)e^{-ikx}dx \right] \\
&= \frac{2}{\sqrt{2\pi}} \left[\int_0^a (a-x)e^{ikx}dx + \int_0^a (a-x)e^{-ikx}dx \right] \\
&= \frac{2}{\sqrt{2\pi}} \int_0^a (a-x)\left[e^{ikx} + e^{-ikx} \right] dx \\
&= \frac{4}{\sqrt{2\pi}} \int_0^a (a-x)\cos(kx)dx = -\frac{4}{\sqrt{2\pi}k^2}\left[\cos(ka) - 1\right]
\end{aligned}$$

Exercise 3 [Finite Wave (4 points)]

a)

$$f(t) = \begin{cases} \sin(\omega_0 t), & -t_0 < t < t_0 \\ 0, & \text{elsewhere} \end{cases}, \quad t_0 = \frac{N\pi}{\omega_0}$$

$$
\begin{aligned}
g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-t_0}^{t_0} \sin(\omega_0 t) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-t_0}^{t_0} \left(\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \right) e^{-i\omega t} dt \\
&= \frac{1}{2i\sqrt{2\pi}} \int_{-t_0}^{t_0} \left[e^{i(\omega_0 - \omega)t} - e^{-i(\omega_0 + \omega)t} \right] dt \\
&= \frac{1}{2i\sqrt{2\pi}} \left[\frac{1}{i(\omega_0 - \omega)} \; e^{i(\omega_0 - \omega)t} \Big|_{-t_0}^{t_0} - \frac{1}{-i(\omega_0 + \omega)} \; e^{-i(\omega_0 + \omega)t} \Big|_{-t_0}^{t_0} \right] \\
&= \frac{1}{i\sqrt{2\pi}} \left[\frac{\sin\left((\omega_0 - \omega) \frac{\pi N}{\omega_0} \right)}{(\omega_0 - \omega)} - \frac{\sin\left((\omega_0 + \omega) \frac{\pi N}{\omega_0} \right)}{(\omega_0 + \omega)} \right]
\end{aligned}
\tag{1, 2}
$$

We could go on:

$$
\begin{aligned}
g(\omega) &= \frac{1}{i\sqrt{2\pi}} \left[\frac{\sin\left((\omega_0 - \omega) \frac{\pi N}{\omega_0} \right)}{(\omega_0 - \omega)} - \frac{\sin\left((\omega_0 + \omega) \frac{\pi N}{\omega_0} \right)}{(\omega_0 + \omega)} \right] \\
&= \frac{-1}{i\sqrt{2\pi}} (-1)^N \sin\left(\frac{\omega}{\omega_0} \pi N \right) \left[\frac{1}{\omega_0 - \omega} + \frac{1}{\omega_0 + \omega} \right] \\
&= i\sqrt{\frac{2}{\pi}} (-1)^N \sin\left(\frac{\omega}{\omega_0} \pi N \right) \frac{\omega_0}{\omega_0^2 - \omega^2}
\end{aligned}
\tag{3}
$$

As a physical example for such a function, consider the equation of a driven harmonic oscillator: $m\ddot{x} + c\dot{x} + kx = F(t)$, with a sinusoidal external force $F(t) = F_0 e^{i\omega t}$.

The solutions relax towards $x(t) = Ae^{i\omega t}$, where, in the case of no damping, $(c = 0)$, $A = \frac{F_0/m}{|\omega_0^2 - \omega^2|}$.

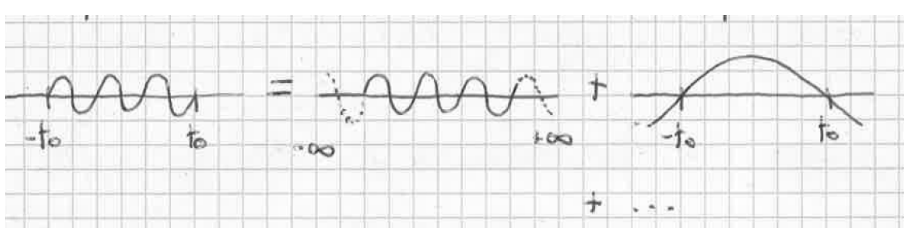
Here, $\omega_0 = \sqrt{k/m}$ is the angular frequency of the undamped, and non-driven, harmonic oscillator.

For $\omega \to \omega_0$ the system gets into **resonance**. This presents some similarities with our curve $g(\omega)$ for $\omega \to \omega_0$.

More generally, resonances are described by the so-called "Lorentzian-function".

b) $g(\omega)$: distribution of frequencies ω of all the oscillations used to build up $f(t)$.

For $f(t) = \sin(\omega_0 t)$ only one mode $\omega = \omega_0$ is needed, but since in our problem the oscillation is restricted to $(-t_0, t_0)$, a whole variety of modes is needed to build up $f(t)$.



c) **Case 1** $|\omega| \ll |\omega_0|$

$g(\omega) \approx \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(\pi N)}{\omega_0} - \frac{\sin(\pi N)}{\omega_0}\right] = 0$

Case 2 $|\omega - \omega_0| \ll |\omega_0|$ (or, better: $N|\omega - \omega_0| \ll |\omega_0|$)

consider the first term: $\frac{\sin\left((\omega-\omega_0)\frac{\pi N}{\omega_0}\right)}{\omega-\omega_0} = \frac{\pi N}{\omega_0} \frac{\sin\left((\omega-\omega_0)\frac{\pi N}{\omega_0}\right)}{(\omega-\omega_0)\frac{\pi N}{\omega_0}} \approx \frac{\pi N}{\omega_0}$

on the other side, the second term: $\frac{-\sin\left((\omega+\omega_0)\frac{\pi N}{\omega_0}\right)}{\omega+\omega_0} = -\frac{1}{\omega+\omega_0} \sin\left((2 + \frac{\omega-\omega_0}{\omega_0})\pi N\right) \approx -\frac{1}{2\omega_0}\sin(2\pi N) = 0$

Therefore, $g(\omega) \approx \frac{1}{i\sqrt{2\pi}} \frac{\pi N)}{\omega_0}$

Similarly, for $N|\omega + \omega_0| \ll |\omega_0|$ (i.e, $\omega \approx -\omega_0$), $g(\omega) \approx \frac{-1}{i\sqrt{2\pi}} \frac{\pi N)}{\omega_0}$

Case 3: $|\omega| \gg |\omega_0|$ (or better: $|\omega| \to \infty$)

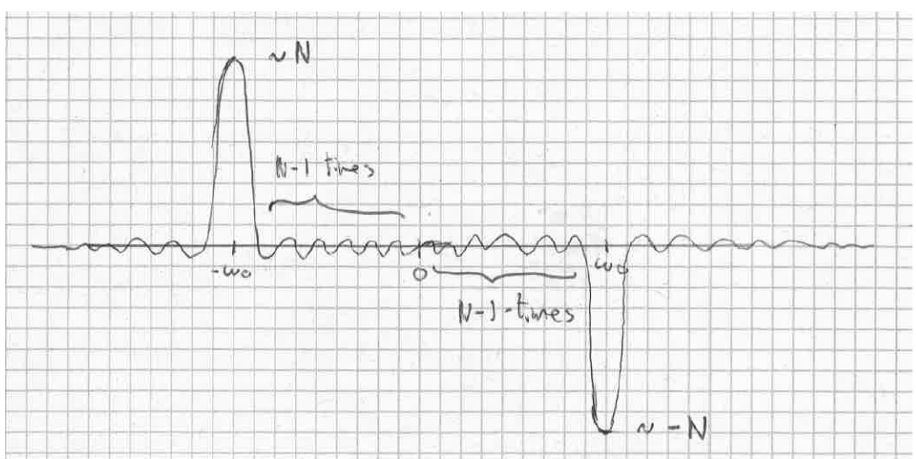
$g(\omega) \approx \frac{1}{i\sqrt{2\pi}\omega_0} \left[\frac{\sin\left((\omega_0-\omega)\frac{\pi N}{\omega_0}\right)}{-\omega/\omega_0} - \frac{\sin\left((\omega_0+\omega)\frac{\pi N}{\omega_0}\right)}{\omega/\omega_0}\right] \to 0$ since both denominators $\to \infty$.

For the other zeroes of $g(\omega)$, we can use the formulation $g(\omega) = i\sqrt{\frac{2}{\pi}}(-1)^N \sin\left(\frac{\omega}{\omega_0}\pi N\right) \frac{\omega_0}{\omega_0^2-\omega^2}$.

We see that $g(\omega) = 0$ for $\frac{\omega}{\omega_0}N = z$, with $z \in \mathbb{Z}$, $z \neq -N, N$

In particular, there will be $N-1$ zeroes in the interval $(-\omega_0, 0)$ and in the interval $(0, \omega_0)$, and an infinite number of zeroes in $(-\infty, -\omega_0)$ and $(\omega_0, +\infty)$.

Taking into account all the collected information, we can sketch $\mathrm{Im}\,(g(\omega))$ as follows:



d) For $N \to \infty$, $g(\omega)$ becomes more and more peaked at $\omega = \pm\omega_0$. The number of zeroes in $(-\omega_0, \omega_0)$ approaches infinity $\to$ kind of delta distribution.