Second Order Relativistic Perturbation: Gravitational Waves From Nonlinearity

Magnoli Christopher

Supervisor: Professor Dr. Yoo Jaiyul

Thesis submitted to

University of Zurich

in partial fulfilment for the award of the degree of

MASTER OF SCIENCE

PHYSICS

Faculty of Science
Physik-Institut, Universität Zürich
Winterthurerstrasse 190, 8057 Zürich, Schweiz

June 1, 2023

Abstract

Gravitational waves induced by scalar curvature fluctuations are a possible source of the cosmological background of gravitational waves. After providing an overview of the theory behind this study, we offer a detailed explanation covering topics starting from the metric perturbation and concluding with the relative energy density of scalar-induced gravitational waves. We consider a spatially flat Friedmann-Lemaître-Robertson-Walker space and use the conformal Newtonian gauge. In our study, we analytically examined the scalar-induced gravitational wave spectrum in the periods of radiation and matter domination. In comparison, numerical analysis was developed for the matter domination period. For the latter, we considered two approaches, which led to different results.

Acknowledgements

I would like to take this opportunity to thank my family and close friends for the great support I received during my years of study, leading up to the presentation of this thesis. This support has sustained me through both, the good and the difficult times, motivating me to give my best and pursue my dreams. Without them, I would not have been able to get to this point and experience the thousand unforgettable moments of these past years. This is why I want to dedicate this thesis to you, thank you very much for everything.

A special thanks go to all my fellow students, with whom I was able to engage in discussions, exchange ideas and dream about physics topics. Moments of fun and learning have led me to love this wonderful subject more and more.

I also want to thank the Physics Department and the Institute of Computational Science of the University of Zurich, for the lectures and teaching I received. Especially Prof. Prasenjit Saha, who introduced me to Astrophysics and Cosmology and Prof. Jaiyul Yoo, who transmitted his passion for theoretical Cosmology to me. Thank you both so much for passing on your energy and passion, I hope to continue to collaborate with like-minded people in the future.

Finally, I would also like to thank, again, Prof. Jaiyul Yoo for the opportunity to write this thesis with him, and for the help he has given me over the last few months. As well as his group (Sandra, Matteo, and Peter), who helped me and with whom I have always enjoyed talking and sharing moments.

Contents

Part I
Introduction

The aim of the following thesis in Theoretical Cosmology is to describe cosmological gravitational waves induced by scalar perturbations. In order to understand this, we must first ask ourselves: What is cosmology? What do we mean by cosmological gravitational waves? Why should they interest us? In the following lines, we aim to answer these questions by introducing the main concepts.

Cosmology is a branch of physics and seeks to study the origin, the evolution of the Universe [1], and its large-scale structure. Where with the term 'Universe', we mean all that physically exists [2]. Cosmology as scientific study began with the discovery of Einstein's static Universe (1917); the observational discovering of the linear relation between redshift and distance, indicating the expansion of the Universe, *Hubble* [3]; and the development of the Friedmann-Lemaître models with the Robertson-Walker metric for a non-static description of the Universe [2].

Einstein's general relativity theory of 1915 provided the basis for the theoretical description of modern cosmology [4], as the Einstein's field equations govern the evolution of the Universe. The theory of general relativity could be used to construct fully self-consistent models for the Universe as a whole. One of the objectives of Einstein was to incorporate into the structure of general relativity what he called Mach's Principle. This means that, the local inertial frame of reference should be determined by the large-scale distribution of matter in the Universe. There was, however, a further problem, first noted by Newton, that for a static model the Universes are unstable under gravity [5]. Soon after the Einstein's theory, Friedmann worked on models of the expanding Universe and Lemaître developed the concepts of redshift and Big Bang [4]. These theoretical models solve the problem of instability, and form the basis framework of what is referred to as the standard, ΛCDM model. Supporting Hubble's observations.

In 1916 and 1918, Einstein found that at the linear order, the weak-field equations have wave solutions [6, 7]. Gravitational waves are ripples in spacetime, their formation can be caused by astrophysical phenomenon related to accelerating masses [8] or by cosmological phenomenon related to the early Universe, such as quantum fluctuations of the scalar field or phase transition, which result in perturbations of the spacetime metric [9, 10].

It took almost a century from the prediction of Einstein for the direct detection of gravitational waves by the Laser Interferometer Gravitational-Wave Observatory (LIGO) in 2016 [11]. This observation of gravitational waves emitted from the binary black hole merger, GW150914, opened a new era in astrophysics and cosmology. Gravitational waves offer a unique way to observe and study the Universe that differs from traditional methods such as radio and optical telescopes. The observation of gravitational waves will provide information about the early Universe's physical processes and its evolution as violent events occurred in the Universe. Currently, with ground-based observatories we are able to observe gravitational waves emitted by astrophysical sources (near to us and with high energy), but they do not have sufficient resolution to observe cosmological gravitational waves. Space-based observatories (such as LISA [12]) can aspire to observe such waves.

Cosmological gravitational waves are assumed to permeate the Universe in the form of a stochastic background of gravitational waves; i.e. a superposition of gravitational waves produced by a multitude of sources [13, 14]. Many efforts have been undertaken in order to characterise the stochastic background [13, 14, 15, 16, 17], only to cite some works. The possible origin of a stochastic background can be traced back to different cosmological event, such as inflation, phase transitions, cosmic strings, and binary system (like primordial black holes). Therefore, the observation of this background could provide a smoking gun evidence of the inflationary period and other theory about the primordial Universe. Gravitational waves arising from the inflationary period are assumed to be of two kinds: the so-called primordial gravitational waves, produced directly by the tensor perturbation arising from the quantum fluctuations, and the induced gravitational waves, produced in later times, once scalar and tensor perturbations, produced by the quantum fluctuations, enter the horizon.

As we are going to explain in cosmology, scalar, and tensor perturbations are more interesting than the vector ones, as the latter decay exponentially in an expanding Universe. To theoretically calculate induced gravitational waves, a non-linear approach to the theory of relativistic perturbations is required. The formulation of Einstein's second-order (or higher) field equations yields a nonlinear term, which can be regarded as the source of the wave. Considerable work on the second-order perturbation has been proposed by *Noh, and Hwang* [18] and by *Gong et al.* [19], of primary importance for the developing of the calculations in this thesis. Many works on induced gravitational waves have been developed over the past two decades, e.g. *Baumann et al.* [20], *Hwang et al.* [21], *Gong* [22], *Sipp, and Schäfer* [23], to name only a few of them.

In order to deal with scalar-induced gravitational waves, we begin by introducing some theoretical concepts necessary to contextualise both cosmology in general and more specifically cosmological gravitational waves. After that, we present our calculations of scalar-induced gravitational waves.

The first theoretical concept introduced are the Einstein's field equations, resulting from general relativity, Sec. 2. We, first, describe the geometric part of these, and then focus on the matter part. Einstein's equations are necessary to describe the dynamic evolution of gravitational waves. Next, we explain the standard models of cosmology, Sec. 3. Introducing, Robertson-Walker geometry and Friedmann-Lemaître-Robertson-Walker Cosmology. This is the geometry that will be used as the background solution, where we assume a spatially flat case.

These two topics are the foundations of modern cosmology, from which we can start building different theories. Once they have been described, we can deepen the discussion, introducing more advanced topics. Like relativistic perturbation theory, described in detail in Sec. 4. As we have already introduced, the Universe can be described to a good approximation by the background solution (FLRW), which is based on the cosmological principle: the Universe is homogeneous and isotropic. Nevertheless, ignoring local characteristics, the Universe is not completely homogeneous and isotropic; for a more precise description, inhomogeneities, and anisotropies must be introduced. This can be done by perturbing the background metrics and quantities of the cosmological fluid. In this section, we therefore define scalar, vector, and tensor perturbations; and the important concept of the gauge transformation. In order to get the contribution of scalar-induced gravitational waves on the relative energy density of cosmological gravitational waves, the concept of power spectrum is needed, Sec. 5. This important tool allows us to obtain the relation at different points of a given quantity.

After this, it is necessary to characterise the period of cosmological inflation, Sec. 6, from which, due to the quantum oscillations, scalar perturbations (which will be the source of the induced gravitational waves) and tensor perturbations (corresponding to the primordial gravitational waves) arise. Finally, we treat gravitational waves more precisely, Sec. 7. We first formally define gravitational waves in a Minkowski space, so that the fundamental concepts of gravitational waves can be introduced. The decision to show these concepts with the example of a Minkowski space lies in the simplicity of treating gravitational waves in this space. In such a way, it is easier to understand the reasons for choosing transverse-teaceless gauge conditions, the transition from real to the associated Fourier space, and polarisation. After that, it is useful to understand, which are the various sources of cosmological gravitational waves, so as to have a better understanding of what is expected to be observed with space-based telescopes, such as LISA and Einstein Telescope (ET).

At this point, we have everything we need to deal with scalar-induced gravitational waves. In the calculation part, adopting a geometric approach, we derive the evolution equations, Sec. 8, without fixing any gauge condition. That is, starting with the metric and the cosmological fluid, we derive the Einstein's field equations. Once the evolution equations are obtained, we calculate the general solutions of the gravitational wave equation, in radiation and matter domination epochs 9. By performing a gauge transformation of the variables, we find the most general form of the perturbation variables, which include the dependence of the gauge choice. To solve this problem, we must fix a gauge, i.e. choose a correspondence between the real Universe and the background Universe. Our choice is the conformal Newtonian gauge (or also known as the zero-shear gauge), Sec. 10. Having obtained the solutions of the wave equation, we can describe the power spectrum, Sec. 11. At this point we have everything we need to find the relative energy density of gravitational waves Sec. 12. In this last section, we present the results both analytically and numerically.

Part II
Theory

2 General Relativity

General relativity is a theory developed by Albert Einstein in 1911. He derived a formula for the light deflection in a gravitational field [24], but the complete mathematical formalism was presented five years later in [25]. In this theory, we consider gravity as a manifestation of spacetime's curved geometry rather than a force. *Wheeler* in [26] described this central idea of general relativity as:

'Spacetime tells matter how to move; matter tells spacetime how to curve.'

The principle of equivalence of inertia and gravitation tells us how an arbitrary physical system responds to an external gravitational field; this is the physical basis of general relativity, i.e., 1) inertial and gravitational masses are equal; 2) gravitational forces are equivalent to inertial forces; 3) in a local inertial frame, we experience the laws of special relativity without gravitation. Therefore, the Einstein Equivalence Principle states: In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments [27]. Mathematically, the curvature can be described with differential manifolds.

2.1 Set up: Manifold and Curvature

By manifold, we mean a space consisting of patches that locally look like $\mathbb{R}^n$ and are smoothly sewn together. In order to define geometrical features on a manifold, we need to introduce the metric. Recalling the quote of Wheeler, the curvature is nothing but the manifestation of gravity. Therefore, in general relativity, it is crucial to study curved manifolds. To do that, we need the metric. Formally, the metric $g_{\mu\nu}$ is a mathematical object that assigns a value to each point in spacetime. It is a symmetric $(0, 2)$-tensor which is nondegenerate. The curvature can be found from it and is described by the Riemann tensor $R^{\mu}_{\nu\rho\sigma}$. This tensor characterises the intrinsic curvature of spacetime at each point. Hereafter, we want to understand better how to get the Riemann tensor and derive other proper quantities related to its curvature.

Let us consider the metric associated with the spacetime manifold. The canonical form of the metric is $g_{\mu\nu} = (-1, +1, +1, +1)$, i.e. we consider a Lorentzian metric. Therefore, an infinitesimal distance between two points is defined as

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}. \tag{1}$$

An object that can be derived from the metric is the Christoffel symbol. This symbol describes the variation in basis vectors from one point to another in a coordinate system. It measures the rate of change of the covariant basis with respect to the coordinate variable. The Christoffel symbol is described as

$$\Gamma^{\mu}_{\nu\rho} := \frac{1}{2} g^{\mu\sigma} \left(\partial_{\rho} g_{\nu\sigma} + \partial_{\nu} g_{\rho\sigma} - \partial_{\sigma} g_{\nu\rho} \right). \tag{2}$$

This symbol is also known as connection, because it describes how the tangent spaces at different points in spacetime are connected. From the connection, we can find the Riemann curvature tensor, or simply Riemann tensor, $R^{\mu}_{\nu\rho\sigma}$. This tensor provides an intrinsic way of describing the curvature of a surface

$$R^{\mu}_{\nu\rho\sigma} := \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\varepsilon}_{\nu\sigma}\Gamma^{\mu}_{\rho\varepsilon} - \Gamma^{\varepsilon}_{\nu\rho}\Gamma^{\mu}_{\sigma\varepsilon}. \tag{3}$$

Contracting it, we get the Ricci tensor and the Ricci scalar

$$R_{\mu\nu} := R^{\rho}_{\mu\rho\nu}, \qquad\qquad R := R^{\mu}_{\mu}. \tag{4}$$

The Riemann tensor, Ricci tensor, and Ricci scalar are intrinsic curvature measures [27].

2.2 Einstein's Field Equations

This section aims to derive the Einstein's equations. Here, we will derive these using the Einstein-Hilbert action approach. In order to do that, we need to define an appropriate Lagrangian for general relativity.

In general relativity, the dynamical variable is the metric tensor $g_{\mu\nu}$, since it is the most fundamental field describing spacetime geometry. *Hilbert* [28] proposed a Lagrangian resulting from the Ricci scalar, i.e. the curvature. The Lagrangian has to be invariant, so its value has to be independent of the choice of the coordinate system. In contrast to classical mechanics, where the action contains only the first derivatives of the dynamical variables, the Lagrangian associated with the Einstein-Hilbert action must involve at least second derivatives of the metric. Because there is no non-trivial scalar invariant constructed out of first derivatives of the metric tensor, as it is always possible to find a coordinate system in the neighbourhood of any point, where all first derivatives of the metric tensor are vanishing [27, 29]. Working with second derivatives means considering all 20 components of the Riemann tensor. Considering a local inertial frame, we obtain 14 curvature invariants, which are also coordinate-independent. Ricci's scalar is the only one of these 14 that is linear in the second derivative of the metric [29].

We now define the total action for general relativity as

$$S \equiv S_{EH} + S_M = \int d^n x \;\; \mathcal{L}_{EH} + \int d^n x \;\; \mathcal{L}_M, \tag{5}$$

where S_{EH} and $\mathcal{L}_{EH}$ are respectively the Einstein-Hilbert's action and the Lagrangian density, the vacuum solution (only geometry). While the two quantities with subscript M correspond to the action and Lagrange density for matter fields.

For both cases, we define the Lagrange density as

$$\mathcal{L}_i = \sqrt{-g} L_i, \tag{6}$$

where $i = EH, M$; $g := \det(g_{\mu\nu})$ is the determinant of the metric tensor and L_i are the Lagrange functions. We start considering the geometric Lagrange function described by the Einstein-Hilbert Lagrange function

$$L_{EH} := \frac{c^3}{16\pi G}\left(R - 2\Lambda\right), \tag{7}$$

where R is the Ricci scalar and Λ is the cosmological constant. The Ricci scalar can be written in terms of the Ricci tensor and the metric tensor, such that the Lagrange density yields:

$$\mathcal{L}_{EH} = \frac{c^3}{16\pi G}\sqrt{-g}\left(R - 2\Lambda\right) = \frac{c^3}{16\pi G}\sqrt{-g}\left(g^{\mu\nu}R_{\mu\nu} - 2\Lambda\right). \tag{8}$$

Since the Einstein-Hilbert action cannot be expressed in terms of covariant derivatives of the metric tensor, it turns out more convenient to describe the action under small variation of the metric instead of working with the Euler-Lagrange equations [27]. Therefore, we get

$$\begin{aligned}\delta S_{EH} &= \frac{c^3}{16\pi G}\delta\int_V d^n x \;\; \sqrt{-g}\left(g^{\mu\nu}R_{\mu\nu} - 2\Lambda\right) \\ &= \frac{c^3}{16\pi G}\int_V d^n x \;\; \left[\left(\delta\sqrt{-g}\right)(R-2\Lambda) + \sqrt{-g}\left(\delta g^{\mu\nu}\right)R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\left(\delta R_{\mu\nu}\right)\Lambda\right] \\ &= \frac{c^3}{16\pi G}\left[\int_V d^n x \;\; \left(\delta\sqrt{-g}\right)(R-2\Lambda) + \int_V d^n x \;\; \sqrt{-g}\left(\delta g^{\mu\nu}\right)R_{\mu\nu} + \int_V d^n x \;\; \sqrt{-g}g^{\mu\nu}\left(\delta R_{\mu\nu}\right)\right].\end{aligned} \tag{9}$$

First, we consider the variation of the square root of the determinant $\delta\sqrt{-g}$. In order to solve it, we use the following to relations:

$$\begin{aligned}&\frac{d}{d\lambda}(\det(A(\lambda)) = \det(A(\lambda))\cdot Tr\left(A^{-1}\frac{d}{d\lambda}A(\lambda)\right) \qquad \text{(Jacobi's formula)}, \\ &\frac{d}{d\lambda}\mathbb{I} = \left(\frac{d}{d\lambda}A(\lambda)\right)A^{-1}(\lambda) + \left(\frac{d}{d\lambda}A^{-1}(\lambda)\right)A(\lambda) \overset{!}{=} 0,\end{aligned} \tag{10}$$

and so we get

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{11}$$

Then, we need to work on the third term of Eq. 9. The variation of the Ricci tensor is described by

$$\delta R_{\mu\nu} = \delta R^{\lambda}_{\mu\lambda\nu} = \delta\left[\partial_\rho\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\rho} + \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\nu\rho}\Gamma^\rho_{\sigma\mu}\right]. \tag{12}$$

Now, as we want to compute the Ricci tensor at any point p, we can choose a coordinate system centred in p, such that $x(p) = 0$ and $\Gamma^\rho_{\mu\nu}(0) = 0$ [30]. This choice is not a problem since when dealing with tensors, a coordinate system is as good as another [29]. According to this choice, the variation of the contracted curvature tensor becomes

$$\delta R_{\mu\nu} = \partial_\rho\left(\delta\Gamma^\rho_{\mu\nu}\right) - \partial_\nu\left(\delta\Gamma^\rho_{\mu\rho}\right). \tag{13}$$

This is a tensor relation, since according to *Jetzer* [30], the difference of two connections $\delta\Gamma^\rho_{\mu\nu}$ is a tensor, and it holds in every coordinate system. Thus, it can be formulated in terms of covariant derivatives, and so the Palatini identity [31] is achieved

$$\delta R_{\mu\nu} = \nabla_\rho\left(\delta\Gamma^\rho_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^\rho_{\mu\rho}\right). \tag{14}$$

Then, since the covariant derivative of the metric tensor is equal to zero, we get

$$\begin{aligned}g^{\mu\nu}\delta R_{\mu\nu} &= \nabla_\rho\left(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}\right) - \nabla_\nu\left(g^{\mu\nu}\delta\Gamma^\rho_{\mu\rho}\right) \\ &\equiv \nabla_\rho\omega^\rho \\ &= \partial_\rho\omega^\rho + \Gamma^\rho_{\rho\sigma}\omega^\sigma,\end{aligned} \tag{15}$$

where the Christoffel symbol $\Gamma^{\rho}_{\rho\sigma}$ can be expressed as

$$\begin{aligned}\Gamma^{\rho}_{\rho\sigma} &= \frac{1}{2} g^{\mu\nu}\left(\partial_\sigma g_{\rho\lambda} + \partial_\rho g_{\sigma\lambda} - \partial_\lambda g_{\rho\sigma}\right) \\ &= \frac{1}{\sqrt{-g}} \partial_\sigma \sqrt{-g}.\end{aligned} \tag{16}$$

Therefore, Eq. 15 becomes

$$g^{\mu\nu} \delta R_{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\rho \left(\sqrt{-g} w^\rho\right). \tag{17}$$

Plugging this result into Eq. 9 and using Gauss's theorem, we obtain that if $\delta g^{\mu\nu}$ vanishes outside the volume of integration, then it is vanishing on the boundary as well [30]. In contrast, if we do not consider a finite region (hence an infinite volume of integration), we can set the contribution to zero by making the variation vanish at infinity [27]. Therefore, the third integral of Eq. 9 does not contribute

$$\int_V d^n x \ \sqrt{-g} g^{\mu\nu} \left(\delta R_{\mu\nu}\right) = \int_V d^n x \ \partial_\rho \left(\sqrt{-g} w^\rho\right) = \int_{\partial V} d0_\rho \sqrt{-g} w^\rho = 0, \tag{18}$$

where $d0_\rho$ is the coordinate normal to the boundary ∂V. Putting the results of Eq 11 and Eq. 18 together, we can rewrite Eq. 9 as

$$\begin{aligned}\delta S_{EH} &= \frac{c^3}{16\pi G} \int_V d^n x \ \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}\right) \delta g^{\mu\nu} \\ &= \frac{c^3}{16\pi G} \int_V d^n x \ \sqrt{-g} \left(G_{\mu\nu} + \Lambda g_{\mu\nu}\right) \delta g^{\mu\nu},\end{aligned} \tag{19}$$

where $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor. According to Carroll [27], the functional derivative of the action satisfies

$$\delta S = \int d^n x \ \sum_i \left(\frac{\delta S}{\delta \Phi^i} \delta \Phi^i\right), \tag{20}$$

where $\{\Phi^i\}$ is a complete set of fields being varied. Then, the Einstein's field equation in vacuum (i.e. if $\mathcal{L}_M = 0$) can be derived from Eq. 19, and is

$$\frac{16\pi G}{c^3} \frac{1}{\sqrt{-g}} \frac{\delta S_{EH}}{\delta g^{\mu\nu}} = G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{21}$$

Now considering a small variation of the metric on the action of matter, we get

$$\begin{aligned}\delta S_M &= \delta \int_V d^4 x \ \mathcal{L}_M \\ &= \int_V d^4 x \ \left(\delta \sqrt{-g}\right) L_M + \sqrt{-g} \left(\delta L_M\right).\end{aligned} \tag{22}$$

Following Carroll's book [27], the energy-momentum tensor can be directly defined as

$$T_{\mu\nu} := -2 \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{23}$$

To conclude, based on the principle of least action, applied to the general action defined in Eq. 5, we obtain

$$\delta S = \delta S_{EH} + \delta S_M = 0, \tag{24}$$

combining Eq. 19 and Eq. 22 with Eq. 24, we get

$$\delta S = \int_V d^n x \ \sqrt{-g} \left[\frac{c^4}{16\pi G} \left(G_{\mu\nu} + \Lambda g_{\mu\nu} \right) + \delta \mathcal{L}_M \right] \delta g^{\mu\nu} \tag{25}$$

Therefore, recalling the functional derivative of the action Eq. 20, the Einstein's field equations yield

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{26}$$

In some cases, expressing Einstein's field equations in other terms is useful. By multiplying Eq. 26 with the metric $g^{\mu\nu}$, we obtain the relation between the Ricci scalar and the Energy-Momentum tensor trace

$$R = -8\pi G T + 4\Lambda. \tag{27}$$

So, we can rewrite Eq. 26 as

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) + \Lambda g_{\mu\nu}. \tag{28}$$

The usefulness of writing Einstein's field equations is that the RHS can be written directly as the source of the curvature, e.g. compare Sec. 3.2.

2.3 Energy-Momentum Tensor

In the previous subsection, we defined the equations that relate spacetime geometry to the matter content, i.e. Einstein's field equations, which are described by geometric tensors, analysed in detail above, and by the energy-momentum tensor. The latter will be treated differently here. One of the first to deal with this issue was Eckart in his paper [32], and following his reasoning, we will define this tensor.

The physical meaning of the tensor components can be obtained by introducing an observer, which allows us to describe these components in terms of projections.

These can be along timelike or spacelike directions associated with the observer. Those of the former type can be obtained by contracting the indices with the four-velocity of the observer u_μ. While those along spacelike directions are defined by the $\mathcal{H}_{\mu\nu}$ operator [33]. Where the following relations hold

$$\mathcal{H}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \qquad u^\mu u_\mu = -1, \qquad \mathcal{H}_{\mu\nu} u^\nu. \tag{29}$$

According to Eckart [32], we can get the following three components:

$$\rho = T_{\mu\nu} u^\mu u^\nu, \tag{30}$$

$$q_\mu = -T_{\rho\sigma} u^\rho \mathcal{H}^\sigma_\mu, \tag{31}$$

$$s_{\mu\nu} = T_{\rho\sigma} \mathcal{H}^\rho_\mu \mathcal{H}^\sigma_\nu, \tag{32}$$

where ρ is the invariant energy density, q_μ is the heat flow or energy flux and $s_{\mu\nu}$ is the stress tensor. Now we can separate the stress tensor into the hydrostatic (isotropic) pressure and the anisotropic stress tensor as

$$s_{\mu\nu} = \pi_{\mu\nu} + p\mathcal{H}_{\mu\nu}. \tag{33}$$

Therefore, the Energy-Momentum tensor can be written as

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\mathcal{H}_{\mu\nu} + 2q_{(\mu}u_{\nu)} + \pi_{\mu\nu}, \tag{34}$$

which is the most general form for a single fluid. To consider different fluids, one must add the individual energy-momentum tensor to derive the total energy-momentum tensor [34]. In Eq. 26 and Eq. 28, we have explicitly kept the term relating to the cosmological constant. It is common to put it on the right-hand side as a part of the energy-momentum tensor

$$\rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G}. \tag{35}$$

Therefore, the energy density ρ_Λ and the isotropic pressure p_Λ of the cosmological constant are part of the total energy density ρ and the total isotropic pressure p, respectively.

For this reason we can rewrite Eq. 26 (and analogously, Eq. 28) as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{36}$$

3 Standard Models of Cosmology

The Universe is composed of countless celestial bodies, such as galaxies, stars, and planets. Despite this inhomogeneity and apparent complexity, cosmologists have assumed that the large-scale Universe is simple. This idea is described by the cosmological principle that the Universe can be considered spatially homogeneous and isotropic on a large scale. This principle is not a physical law and is not exact, but rather an approximation, a hypothesis, which we believe holds true on a large scale. As explained by *Riddle* [35], this is a property of the Universe as a whole that breaks down on a local scale. The concepts of homogeneity and isotropy can be explained as that there is neither a position nor a preferential direction in the Universe, respectively. The concept of spatial isotropy of the Universe, mathematically, can be understood as a 3-dimensional spherical symmetric hypersurface [36].

As *Milne* [37] states, this assumption follows from the belief that 'not only the laws of nature but also the events that occur in nature, the world itself, must appear the same to all observers.' From current observations, this hypothesis appears to be correct [38], a prime example being the cosmic microwave background (CMB). The CMB allows us to understand that the temperature of the Universe (excluding small-scale variations) is almost the same at any point in the observable Universe. Before discussing a cosmological model based on this principle, let us put the cosmological principle in other terms, as proposed by *Mo et al.* [38]. They propose to explain the cosmological principle as the existence of a fundamental observer at each point, for which the Universe appears isotropic. Introducing this fundamental observer is possible to solve the problem that two observers in relative motion with each other cannot observe the Universe as isotropic.

FLRW Universes are models based on the cosmological principle, exhibiting perfect homogeneity and isotropy. These, of course, cannot be exact models of the Universe, as they cannot show the inhomogeneities associated with astronomical structures. More realistic models are derived from perturbations of the FLRW Universes. To quote *Ellis et al.* [39], "The 'almost FLRW' models are the standard models of cosmology at the present time".

Hereafter, we consider the Robertson-Walker geometries 3.1, employed in the Friedmann-Lemaître world models (FLRW models) 3.2. As we are going to learn in the following section, a Universe is called FLRW if and only if the following conditions are valid everywhere [39]:

1. shear, vorticity and acceleration are zero everywhere;
2. the total matter contained in the Universe must have the form of a perfect fluid, i.e. no anisotropic pressure and no energy flux;
3. all kinematics and non-zero stress-energy scalars must be functions of time only.

3.1 The Robertson-Walker Geometry

According to *Weinberg* [36][1], the metric describing a homogeneous and spherically symmetrical space can be described by the following relation

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu . \tag{37}$$

[1]NB: In this book, Weinberg uses a convention of the $(+ - --)$ signature. While we use $(- + ++)$, so our equations are modified in order to follow our signature choice.

In his book, he specifies that, as previously written, symmetry is not relative to spacetime but to a subspace of it. In our case, this subspace is described by hypersurfaces with the time coordinate fixed. In the following, we propose a general description based on Chapter 13 of *Weinberg* [36], without going into the demonstration presented in the book. We define a space with dimension N and its maximally symmetric subspaces of dimension M. We can distinguish these subspaces with $N - M$ coordinates, v^i and localise the points in each subspace with M coordinates u^α.

We can say that, subspaces with constant v^i are maximally symmetric if the metric of the space is invariant under the influence of a group of infinitesimal transformations

$$\begin{aligned} u^\alpha &\to u'^\alpha = u^\alpha + \epsilon\xi^\alpha(u,v), \\ v^i &\to v'^i = v^i, \end{aligned} \tag{38}$$

with $M(M+1)/2$ independent Killing vectors ξ^i.[2] Thus, it is always possible to choose the u-coordinates so that the metric Eq. 37 can be separated as follows:

$$ds^2 = g_{ij}(v)dv^i dv^j + f(v)\tilde{g}_{\alpha\beta}(u)du^\alpha du^\beta, \tag{39}$$

where $g_{ij}(v), f(v)$ and $\tilde{g}_{\alpha\beta}(u)$ are functions of the v- or u-coordinate, alone. In order to find the RW metric, we are interested in the cases where the space dimension is $N = 4$, and the dimension of the subspaces is $M = 3$. We start considering the second term in the RHS of Eq. 39,

$$d^2\sigma = \tilde{g}_{\alpha\beta}(u)du^\alpha du^\beta. \tag{40}$$

According to *Weinberg* [36] the Riemann curvature tensor of the maximally symmetric metric $\tilde{g}_{\alpha\beta}$ is

$$\begin{aligned} {}^{(3)}R_{\alpha\beta\gamma\delta} &= \frac{{}^{(3)}R}{n(n-1)}\left(\tilde{g}_{\delta\beta}\tilde{g}_{\alpha\gamma} - \tilde{g}_{\beta\gamma}\tilde{g}_{\alpha\delta}\right) \\ &\equiv K\left(\tilde{g}_{\delta\beta}\tilde{g}_{\alpha\gamma} - \tilde{g}_{\beta\gamma}\tilde{g}_{\alpha\delta}\right), \end{aligned} \tag{41}$$

where ${}^{(3)}R$ is the Ricci scalar, n is the dimension of the space and $K \equiv \frac{{}^{(3)}R}{n(n-1)}$ is the Gaussian curvature on the 3-dimensional space. This scalar curvature is constant for every space with dimension $n > 2$. Contracting two indices on the Riemann curvature tensor to get the Ricci tensor, we obtain the following relation:

$$R_{\alpha\beta} = 2K\tilde{g}_{\alpha\beta}. \tag{42}$$

We define the following change of coordinate system

$$\int \sqrt{g(v)}dv \equiv -t, \tag{43}$$

$$u^1 \equiv r\sin\theta\cos\varphi, \tag{44}$$

$$u^2 \equiv r\sin\theta\sin\varphi, \tag{45}$$

$$u^3 \equiv r\cos\theta, \tag{46}$$

[2]The Killing vectors describe the symmetries of a system

where we opted to describe the spatial subspaces by spherical coordinates; because of the cosmological principle. Then, according to this coordinate transformation Eq. 43, Eq. 39 can be written as

$$\begin{aligned} ds^2 &= -dt^2 + f(t)\left[e^{2b(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \\ &\equiv -dt^2 + f(t)\left[e^{2b(r)}dr^2 + r^2 d\Omega^2\right] \\ &= -dt^2 + f(t)d\sigma^2, \end{aligned} \tag{47}$$

where the exponential factor is to generalise the metric as much as possible in order to preserve spherical symmetry [27]. [3] Therefore, the metric of the 3-space is

$$\tilde{g}_{rr} = e^{2b(r)}, \quad \tilde{g}_{\theta\theta} = r^2, \quad \tilde{g}_{\varphi\varphi} = r^2\sin^2\theta, \tag{48}$$

from which the Ricci tensors can be calculated

$$\tilde{R}_{rr} = \frac{2}{r}\frac{db(r)}{dr}, \quad \tilde{R}_{\theta\theta} = e^{-2b(r)}\left(r\frac{db(r)}{dr} - 1\right) + 1, \quad \tilde{g}_{\varphi\varphi} = e^{-2b(r)}\left[\left(r\frac{db(r)}{dr} - 1\right) + 1\right]\sin^2\theta. \tag{49}$$

By including the radial component of the metric Eq. 48 and the Ricci tensor 49 in the relation given in Eq. 42, we can characterise the exponential function

$$e^{2b(r)} = \frac{1}{1 - Kr^2}, \tag{50}$$

and hence, we rewrite Eq. 47 as

$$ds^2 = -dt^2 + a(t)\left[\frac{1}{1 - Kr^2}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{51}$$

where $a(t) = \sqrt{f(v)}$ is the scale factor.

We, then, define the scalar curvature $K = \delta \cdot k$, where $k \in (-1, 0, +1)$ is the normalized curvature. In order to express the metric in terms of the normalised curvature, we parametrise the radial coordinate so that $r \to \sqrt{\delta}r$ and the metric of 3-dimensional space becomes

$$d\sigma^2 = \frac{1}{\delta}\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right). \tag{52}$$

To conclude, let us consider the Universes described by the three values of k. It is useful to define a new radial coordinate

$$d\chi = \frac{dr^2}{\sqrt{1 - kr^2}}, \tag{53}$$

from which we get

$$r = \begin{cases} \sin\chi & \text{, if } k = +1; \\ \chi & \text{, if } k = 0; \\ \sinh\chi & \text{, if } k = -1. \end{cases} \tag{54}$$

By including the factor δ in the scale factor $a^2(t)$, we can write the metric as

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + S_k(\chi)d\Omega^2\right]. \tag{55}$$

[3]Compare chapter 5 and 8 of *Carroll* [27] for additional information about the exponential function.

where

$$S_k(\chi) = \begin{cases} \sin^2 \chi & \text{, if } k = +1; \\ \chi^2 & \text{, if } k = 0; \\ \sinh^2 \chi & \text{, if } k = -1. \end{cases} \tag{56}$$

The metric with $k = +1$ describes a closed Universe (metric of the three-sphere), the one with $k = 0$ a flat Universe (metric of a flat Euclidean space) and the last one $k = -1$ an open Universe (metric of the hyperboloid), a proof can be found in *Kolb, and Turner* [40]. It turns out useful to work with the conformal time, defined as $\eta = a \cdot t$, such that Eq. 55 becomes

$$ds^2 = a^2(\eta) \left[-d\eta^2 + d\chi^2 + S_k(\chi) d\Omega^2 \right]. \tag{57}$$

Usually, the spacetime is considered to be spatially flat, so from now on, when we refer to the (FL)RW-metric, we consider the case where $k = 0$:

$$ds^2 = a^2(\eta) \left(-d\eta^2 + \bar{g}_{\alpha\beta} dx^\alpha dx^\beta \right), \tag{58}$$

where $\bar{g}_{\alpha\beta}$ is the 3-dimensional Euclidean metric, with signature $(+++)$.

3.2 The Friedmann-Lemaître-Robertson-Walker Cosmology

The dynamics of the expanding Universe appeared implicitly in the metric described in the previous subsection. To make this time dependence explicit, we must solve the Einstein's field equations [40]. The first consideration is that due to the symmetries arising from the Robertson-Walker metric, the energy-momentum tensor takes the form of a perfect fluid, the components dependent only on time coordinates [41]. Hence, we know that the energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = \rho u_\mu u_\nu + p \mathcal{H}_{\mu\nu} \tag{59}$$

where $\mathcal{H}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projection tensor; and it fulfils the energy and momentum conservation $T_{\mu\nu;\nu} = 0$ [42]. According to Eq. 51, we find that the metric components are

$$g_{00} = -1, \quad g_{\alpha\beta} = a^2(t) \widetilde{g}_{\alpha\beta}, \tag{60}$$

where $\tilde{g}_{\alpha\beta}$ is given in Eq. 48. The only non-vanishing Christoffel symbols associated with this metric are

$$\Gamma^0_{\alpha\beta} = a(t)\dot{a}(t)\widetilde{g}_{\alpha\beta}, \quad \Gamma^\alpha_{0\beta} = \frac{\dot{a(t)}}{a(t)} \delta^\alpha_\beta, \quad \Gamma^\alpha_{\beta\gamma} = \widetilde{\Gamma}^\alpha_{\beta\gamma}, \tag{61}$$

from which we get the Ricci tensor

$$R_{00} = -3\frac{\ddot{a}(t)}{a(t)}, \quad R_{0\alpha} = 0, \quad R_{\alpha\beta} = \widetilde{R}_{\alpha\beta} + \left[a(t)\ddot{a}(t) + 2\dot{a}^2(t) \right] \widetilde{g}_{\alpha\beta}, \tag{62}$$

where from Eq. 42 we have $\widetilde{R}_{\alpha\beta} = 2k\widetilde{g}_{\alpha\beta}$.

Here, as anticipated, it is worth considering the second formulation of Einstein's field equations 28. The source of Ricci's curvature tensor is

$$\begin{aligned} S_{\mu\nu} &:= T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \\ &= (\rho + p)\, u_\mu u_\nu + \frac{1}{2} (\rho - p)\, g_{\mu\nu}. \end{aligned} \tag{63}$$

Hence, According to Eq. 28, which can be stated as

$$R_{\mu\nu} = 8\pi G S_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{64}$$

the Friedmann-Lemaître equations yield

$$\text{00-component:} \qquad \dot{H}(t) + H^2(t) = \frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{65}$$

$$\alpha\beta\text{-component:} \qquad H^2(t) = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{66}$$

3.3 The ΛCDM Model

The concordance model of cosmology is a Euclidean Universe that today is dominated by a cosmological constant Λ and non-baryonic cold dark matter (CDM), with initial perturbations generated by inflation in the very early Universe. This model is known as flat ΛCDM model [43]. This model assumes the FLRW cosmology as the background framework. As we discuss in the following sections, The real Universe is not homogeneous and isotropic; instead, there are small deviations from the cosmological principle described well by perturbations. These perturbations arise from the quantum fluctuation during inflation and become the seeds for the large-scale structure.

4 Relativistic Perturbation Theory

As we have treated previously in Sec3.1, the FLRW metric Eq. 58 is generally used to describe the spatially homogeneous and isotropic Universe. For instance, neglecting local features, the cosmic microwave background is homogeneous and isotropic up to an accuracy of $\delta T/\bar{T} = 10^{-5}$, if one needs more accuracy, it is necessary to take into account inhomogeneity and anisotropy. As for the moment, it has not been possible to have exact solutions to Einstein's field equations due to their highly nonlinear nature; perturbation methods have been developed [44]. In general, one considers the FLRW metric as the background Universe and parameterises the perturbations to the homogeneous and isotropic Universe to describe the real (inhomogeneous and anisotropic) Universe.

General covariance states that there is no preferred coordinate system in nature. This introduces the concept of diffeomorphism, or, as we will explain, of gauge. In theories with general covariance (like general relativity), the gauges give rise to non-physical degrees of freedom. For this reason, one must fix the gauges or extract invariant quantities to obtain physical results [45]. Additionally, splitting the variables into a background's part and a perturbation's part is not covariant. Because of gauge dependence, it causes the presence of coordinate artefacts and gauge modes in the calculations [44]. In perturbation theory, one always treats two spacetime manifolds, the physical one $\mathcal{M}$ and the background one $\mathcal{M}_0$. All perturbations are defined by comparing the quantities in the physical spacetime with those in the fictitious background. Therefore the perturbation's quantities depend directly on the background's choice. The background is called fictitious because it is a reference to carry out perturbative analyses and has nothing to do with the nature [38, 45, 46]. Each physical variable Q is written in the form:

$$Q(\tilde{p}) = Q_0(p) + \sum_{i=1}^{\infty} \delta Q(p)^{(i)} = Q_0(p) + \Delta Q(p), \tag{67}$$

where $Q(\tilde{p})$ in the LHS of 67 is a field over $\mathcal{M}$ with $\tilde{p} \in \mathcal{M}$. In contrast, $Q_0(p)$, the background quantity, and $\delta Q(p)^{(i)}$, the i-th perturbation of the quantity in the RHS of 67 are fields over $\mathcal{M}_0$ with $p \in \mathcal{M}_0$. The two points are implicitly considered to be the same but defined on two different manifolds.

In general relativity, points of $\mathcal{M}$ and $\mathcal{M}_0$ are unrelated. However, from Eq. 67, one identifies the assumption of the existence of a map, which permits to compare a physical quantity in the two spacetimes.

Following [45], [46], [47] and partially [44] an infinitesimal parameter λ is introduced for the perturbation, as well as a $(4+1)$-dimensional manifold $\mathcal{N} = \mathcal{M} \times \mathbb{R}$, where $dim(\mathcal{M}) = 4$ and $\lambda \in \mathbb{R}$. Then the background spacetime is described by $\mathcal{M}_0 = \mathcal{N}|_{\lambda=0}$ and on the other hand the physical spacetime $\mathcal{M} \equiv \mathcal{M}_\lambda = \mathcal{N}|_\lambda$. Thus, the manifold $\mathcal{N}$ is foliated in by four-dimensional submanifolds $\mathcal{M}_\lambda$, and these are diffeomorphic to $\mathcal{M}$ and $\mathcal{M}_0$. Then one defines a chart in which x^A, with $A = 0, ..., 4$, are coordinates on each $\mathcal{M}_\lambda$ and $x^4 \equiv \lambda$. Now we consider a set of field equations

$$\varepsilon[Q_\lambda] = 0 \tag{68}$$

on the physical spacetime for the physical variables Q_λ on $\mathcal{M}_\lambda$. The field equation represents the Einstein equation. If the tensor field Q_λ is given on each $\mathcal{M}_\lambda$, then it is extended to a tensor field on $\mathcal{N}$. Such tensor fields are tangent to each $\mathcal{M}_\lambda$. We define a normal form and its dual, $(d\lambda)_a$ and $(\partial/\partial\lambda)^a$, respectively, such that they satisfy $(d\lambda)_a (\partial/\partial\lambda)^a = 1$. They are normal to any tensor field extended from the tangent space on each $\mathcal{M}_\lambda$. The set consisting in the normal form, its dual and the basis of the tangent space on each $\mathcal{M}_\lambda$ is the basis of the tangent space of $\mathcal{N}$.

In order to compare an arbitrary tensor field Q on $\mathcal{M}_\lambda$ with Q_0 on $\mathcal{M}_0$, one needs to identify the points of the physical spacetime with those of the background. Mathematically, the point identification map corresponds to a diffeomorphism, $\varphi_\lambda : \mathcal{N} \to \mathcal{N}$ so that $\varphi_\lambda|_{\mathcal{M}_0} : \mathcal{M}_0 \to \mathcal{M}_\lambda$. This correspondence specified a vector field X, which generates a flow φ on $\mathcal{N}$ in which φ_λ belongs. We define a pull-back φ_λ^* of the diffeomorphism φ_λ. Pulling-back a quantity means evaluating it on $\mathcal{M}_0$ in order to describe the quantity Q in the same coordinate for each λ. Let $p \in \mathcal{M}_0$

$$\begin{aligned} Q(\varphi_\lambda(p)) &= (\varphi_\lambda^* Q)(p) \\ &= Q(p) + \Delta Q_\lambda. \end{aligned} \tag{69}$$

ΔQ_λ in the left-hand side can be Taylor expanded to get

$$\Delta Q_\lambda = \sum_{k=1}^{+\infty} \frac{\lambda^k}{k!} \delta^k Q, \tag{70}$$

where

$$\delta^k T := \left[\frac{d^k (\varphi_\lambda^* Q)}{d\lambda^k} \right]_{\lambda=0,\mathcal{M}_0}. \tag{71}$$

Then the perturbation is nothing but:

$$\Delta Q(p) = Q(p) - \bar{Q}(p), \tag{72}$$

where $Q(p)$ represents the pulled-back quantity and $\bar{Q}(p)$ the background quantity. Important to note that we could not have done the subtraction if we had not pulled Q back to $\mathcal{M}_0$.

4.1 Decomposition of quantities

Hereafter, we define the decomposition of the geometric quantities and the cosmological fluid accordingly to Eq. 72. As explained above, this equation allows us to compare a quantity in the physical spacetime with the same quantity in the background spacetime, which results in a perturbation. So by rearranging the above equation, we can decompose the real quantity into the quantity in the background and its perturbations.

Then we can define the metric as the combination of the background metric, i.e. the FLRW metric, described by Eq. 58 and the perturbation terms as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \tag{73}$$

where $\delta g_{00} = A$, $\delta g_{0\alpha} = B_\alpha$ and $\delta g_{\alpha\beta} = C_{\alpha\beta}$, therefore, it can be expressed in the following way:

$$\begin{aligned} g_{00} &= -a^2 (1 + 2A), \\ g_{0\alpha} &= -a^2 B_\alpha, \\ g_{\alpha\beta} &= a^2 (\bar{g}_{\alpha\beta} + 2C_{\alpha\beta}), \end{aligned} \tag{74}$$

where A,B_α and $C_{\alpha\beta}$ are perturbation variables based on the 3-metric $\bar{g}_{\alpha\beta}$. Then, the time-like four velocity (the motion of an observer)

$$u^0 = \frac{1}{a}\left(1 - A\right), \qquad u^\alpha = \frac{1}{a}U^\alpha, \tag{75}$$

where U^α is based on the 3-metric $\bar{g}_{\alpha\beta}$. The Energy-Momentum tensor quantities can be decomposed into the background, and the perturbations as

$$\rho = \bar{\rho} + \delta\rho, \qquad p = \bar{p} + \delta p, \qquad q_\alpha = aQ_\alpha, \qquad \pi_{\alpha\beta} = a^2\Pi_{\alpha\beta}, \tag{76}$$

where a bar over the density energy and the isotropic pressure stand for the background fluid quantity. As in the case of the metric perturbations, the four-velocity perturbation, the quantities $\delta\rho$, δp, Q_α and $\Pi_{\alpha\beta}$ are based on the 3-metric $\bar{g}_{\alpha\beta}$.

All the perturbation quantities expressed in Eq. 74, Eq. 75 and 76 can be Taylor expanded up to any order, according to Eq. 70.

4.2 Scalar Vector Tensor decomposition

Another decomposition of the perturbation variables is that into scalars, vectors and tensors, valid at all perturbation orders. This tool is very useful in perturbation theory, especially at the linear order. In this case, the scalar, vector, and tensor quantities do not mix and evolve independently.

$$\begin{aligned}
&A = \alpha, \qquad B_\alpha = \beta_{,\alpha} + B^{(v)}_\alpha, \qquad C_{\alpha\beta} = \varphi\bar{g}_{\alpha\beta} + \gamma_{,\alpha|\beta} + C^{(v)}_{(\alpha|\beta)} + C^{(t)}_{\alpha\beta} \\
&U^\alpha = -U^{,\alpha} + U^{(v)\alpha}, \qquad \delta\rho, \qquad \delta p, \qquad Q_\alpha = Q_{,\alpha} + Q^{(v)}_\alpha \\
&\Pi_{\alpha\beta} = \frac{1}{a^2}\left(\Pi_{,\alpha|\beta} - \frac{1}{3}\bar{g}_{\alpha\beta}\Delta\Pi\right) + \frac{1}{a}\Pi^{(v)}_{(\alpha|\beta)} + \Pi^{(t)}_{\alpha\beta}.
\end{aligned} \tag{77}$$

The decomposition in scalar, vector, and tensor is interesting because it lets us consider the three kinds of perturbation separately. For example, the scalar perturbations of the metric, which coupled with the density matter's fluctuations ρ cause the large-scale structure we observe today; the tensor perturbations describe the gravitational waves; while the vector perturbations decay exponentially in an expanding Universe, the reason why they are less interesting to study.

They mix at nonlinear order, and the quantities' evolution becomes more complicated. In any case, as we are going to consider scalar-induced gravitational waves, this decomposition is necessary in order to study only the scalar contribution to the source of gravitational waves.

4.3 Gauge transformation

As introduced previously, the main idea of relativistic perturbation theory is to find an approximated solution of the EFEs, considering small deviation/perturbation from a known background [46]. A physical quantity is defined on a different manifold with respect to the same quantity on the background. We need to evaluate the two quantities at the same point to perform calculations. Aiming to do that, we introduced a diffeomorphism and a pull-back of it. In perturbation theory, this map is called gauge choice -or sometimes point identification map- and this one-to-one correspondence between points in the physical and background spacetimes has to be chosen to define perturbation quantities [45, 46, 47]. The gauge choice is not unique, and one can find different correspondences, even for the same background spacetime, because of general coordinate transformations on the perturbed spacetime [38].

A gauge transformation is a change of the gauge choice, or in other words, a change of the correspondence between the real Universe and the background Universe [48]. This transformation preserves the structure of a theory and aims to remove redundant degrees of freedom [49]. Following *Malik, and Matra* [44], we formalise this transformation by considering two different diffeomorphisms $\varphi_\lambda : \mathcal{M}_0 \to \mathcal{M}_\lambda$ and $\psi_\lambda : \mathcal{M}_0 \to \mathcal{M}_\lambda$, which are the two gauge choices and connect the background with the λ-leaf of the foliation. They are, respectively, the integral curves of the vector fields X and Y defined on $\mathcal{N}$. Thus, the vector fields are everywhere transverse to $\mathcal{M}_\lambda$, and points on the same curve are considered to be the same point with respect to the gauge choice. One can define a diffeomorphism $\Phi_\lambda : \mathcal{M}_0 \to \mathcal{M}_0$

$$\Phi_\lambda := \varphi_{-\lambda} \circ \psi_\lambda \tag{78}$$

for each value of λ. Under this gauge transformation, the resulting transformation of a generic tensor field T (where the indices are omitted for seek of clarity) is given by

$$\Phi^*_\lambda T = \left(\varphi_{-\lambda} \circ \psi_\lambda\right)^* T = \psi^*_\lambda \varphi^*_{-\lambda} T = \exp\{\lambda \pounds_Y\} \exp\{-\lambda \pounds_X\} T. \tag{79}$$

For the Lie Algebra of a Lie group, we can use the Baker-Campbell-Hausdorff (BCH) formula

$$\log\left(\exp\{A\}\exp\{B\}\right) = A + B + \frac{1}{2}\left[A, B\right] + ... \tag{80}$$

in order to simplify Eq. 79, and get

$$\begin{aligned}
\Phi^*_\lambda T &= \exp\left\{\lambda\left(\pounds_Y - \pounds_X\right) + \frac{1}{2}\lambda^2\left[\pounds_X, \pounds_Y\right] + ...\right\} T \\
&= \exp\left\{-\sum_{k=1}^{\infty} \frac{1}{k!}\lambda^k \pounds_{\zeta_{(k)}}\right\} T \\
&= T - \left(\lambda \pounds_{\zeta_{(1)}} + \frac{1}{2}\lambda^2 \pounds_{\zeta_{(2)}}\right) T + \frac{1}{2}\lambda^2 \pounds^2_{\zeta_{(1)}} T + \mathcal{O}(3)
\end{aligned} \tag{81}$$

where T is evaluated on $\mathcal{M}_0$, $\zeta_{(1)} := X - Y$ and $\zeta_{(2)} := [X, Y]$ and in the last equality we Taylor expanded the exponential. The definition of the ζ-field is different to the one proposed by *Malik* [44]. We choose to rewrite the change of coordinates evaluated at the same point, with only positive signs, as proposed in *Magi, and Yoo* [48]. We can simplify further the expression including λ in the variable and defining the vector field $\zeta^\mu := \zeta^\mu_{(1)} + \frac{1}{2}\zeta^\mu_{(2)}$

$$\Phi^*_\lambda T = T - \pounds_\zeta T + \frac{1}{2}\pounds^2_\zeta T, \tag{82}$$

where all the tensor fields T are evaluated on the same point. The map generated by Φ enables us to relate the two coordinate systems. This transformation takes the point p with coordinates $x^\mu(p)$ to the point $q = \Phi_\lambda(p)$ with coordinates $x^\mu(q)$. Thus, according to Eq. 81 we get

$$\begin{aligned}
x^\mu(q) &= \exp\{-\pounds_\zeta|_p\} x^\mu(p) \\
&= x^\mu(p) - \zeta^\mu(p) + \frac{1}{2}\zeta^\mu_{,\nu}(p)\zeta^\nu(p).
\end{aligned} \tag{83}$$

We want to express Eq. 83 at the same point. In order to do that, we introduce a new coordinate system $\tilde{x}^\mu$ such that the coordinates of q in the new coordinates are the same as p in the old: $\tilde{x}^\mu(q) = x^\mu(p)$.

Rearranging Eq. 83 we obtain

$$x^\mu(p) = x^\mu(q) + \zeta^\mu(p) - \frac{1}{2}\zeta^\mu_{,\nu}(p)\zeta^\nu(p), \tag{84}$$

where $\zeta^\mu(p)$ can be Taylor expanded around q

$$\begin{aligned}\zeta^\mu(p) &= \zeta^\mu(q) + \frac{\partial\zeta^\mu(q)}{\partial x^\nu(q)}\frac{\partial x^\nu(p)}{\partial x^\rho(q)}\left(\zeta^\rho(p) - \frac{1}{2}\zeta^\rho_{,\sigma}(p)\zeta^\sigma(p)\right)\\ &= \zeta^\mu(q) + \zeta^\mu_{,\nu}(q)\zeta^\nu(q).\end{aligned} \tag{85}$$

The substitution of Eq. 85 into Eq. 84 results in the change of coordinates arising from the gauge transformation, i.e.

$$\tilde{x}^\mu(q) = x^\mu(q) + \zeta^\mu(q) + \frac{1}{2}\zeta^\mu_{,\nu}(q)\zeta^\nu(q). \tag{86}$$

4.3.1 Second Order Gauge Transformation

As we have seen in Eq. 82 any tensor of order (n, m) $T^{\mu_1\cdots\mu_n}_{\nu_1\ldots\nu_m}$ the gauge transformation is described as follows:

$$\tilde{T}(x^p) = T(x^p) - \pounds_\zeta T(x^p) + \frac{1}{2}\pounds^2_\zeta T(x^p) \tag{87}$$

where $\tilde{T}(x^p) := \Phi^*_\lambda T(x(p))$ and we omitted the indices for seek of clarity. The Lie derivative for such a generic tensor T is defined as follows:

$$\begin{aligned}\pounds_\zeta T^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m} = \zeta^\rho\partial_\rho T^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m} &- \left[(\partial_\rho\zeta^{\mu_1})\, T^{\rho\mu_2\ldots\mu_n}_{\nu_1\ldots\nu_m} + \ldots + (\partial_\rho\zeta^{\mu_n})\, T^{\mu_1\ldots\mu_{n-1}\rho}_{\nu_1\ldots\nu_m}\right]\\ &+ \left[(\partial_{\nu_1}\zeta^\rho)\, T^{\mu_1\ldots\mu_n}_{\rho\nu_2\ldots\nu_m} + \ldots + (\partial_{\nu_1}\zeta^\rho)\, T^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_{m-1}\rho}\right]\end{aligned} \tag{88}$$

Then we define a new gauge field ξ such that up to the second order one can write ζ as

$$\zeta^\mu = \xi^\mu - \frac{1}{2}\xi^\nu\partial_\nu\xi^\mu \equiv \xi^\mu - \frac{1}{2}\xi^\nu\xi^\mu_{,\nu} \tag{89}$$

Thus the gauge transformations for scalar, vectorial and tensorial fields are:

$$\begin{aligned}\tilde{T} &= T - \zeta^\mu T_{,\mu} + \zeta^\mu_{,\nu}\zeta^\nu T_{,\mu}\\ &= T - \xi^\mu T_{,\mu} + \xi^\mu_{,\nu}\xi^\nu T_{,\mu} + \frac{1}{2}\xi^\mu\xi^\nu T_{,\mu\nu}\end{aligned} \tag{90}$$

$$\begin{aligned}\tilde{T}_\mu &= T_\mu - \left(\zeta^\nu T_{\mu,\nu} + \zeta^\nu_{,\mu}T_\nu\right) + \frac{1}{2}\left(T_{\mu,\nu\rho}\zeta^\nu\zeta^\rho + \zeta^\rho_{,\mu}\zeta^\nu T_{\rho,\nu} + \zeta^\rho_{,\nu}\zeta^\nu T_{\mu,\rho} + \zeta^\rho\zeta^\nu_{,\mu}T_{\nu,\rho} + \zeta^\rho_{,\nu}\zeta^\nu_{,\mu}T_\rho + \zeta^\rho\zeta^\nu_{,\mu\rho}T_\nu\right)\\ &= T_\mu - \left(\xi^\nu T_{\mu,\nu} + \xi^\nu_\mu T_{,\nu}\right) + \frac{1}{2}T_{\mu,\nu\rho}\xi^\nu\xi^\rho + \xi^\rho_{,\nu}\xi^\nu_{,\mu}T_\rho + \xi^\rho\left(\xi^\nu_{,\rho}T_{\mu,\nu} + \xi^\nu_{,\mu}T_{\nu,\rho} + \xi^\nu_{,\mu\rho}T_\nu\right)\end{aligned} \tag{91}$$

$$\begin{aligned}\tilde{T^\mu} &= T^\mu - \left(\zeta^\nu T_{\mu,\nu} - \zeta^\mu_{,\nu}T^\nu\right) + \frac{1}{2}\left[\zeta^\nu\left(T^\mu_{,\nu\rho}\zeta^\rho + \zeta^\rho_{,\nu}T^\nu_{,\rho} - \zeta^\mu_{,\rho}T^\rho_{,\nu} - \zeta^\mu_{,\nu\rho}T^\nu\right) + \zeta^\mu_\nu\zeta^\nu_{,\rho}T^\rho\right] \\ &= T^\mu - \left(\xi^\nu T^\mu_{,\nu} - +\xi^\mu_{,\nu}T^\nu\right) + \xi^\rho\left(\frac{1}{2}T^\mu_{,\nu\rho}\xi^\nu + \xi^\nu_{,\rho}T^\mu_{,\nu} - \xi^\mu_{,\nu}T^\nu_{,\rho} - \xi^\mu_{,\nu\rho}T^\nu\right)\end{aligned} \tag{92}$$

$$\begin{aligned}\widetilde{T}_{\mu\nu} =& T_{\mu\nu} - \left(T_{\mu\nu,\rho}\zeta^\rho + 2\zeta^\rho_{,(\mu}T_{\nu)\rho}\right) + \frac{1}{2}\left(\bar{T}_{\mu\nu,\rho\sigma}\zeta^\rho\zeta^\sigma + \bar{T}_{\mu\nu,\rho}\bar{\zeta}^\rho_{,\sigma}\zeta^\sigma + 2\zeta^\sigma\zeta^\rho_{,(\mu}\bar{T}_{\nu)\rho,\sigma}\right. \\ &\left. + 2\zeta^\sigma\bar{T}_{\rho(\mu}\zeta^\rho_{,\nu)\sigma} + 2\zeta^\sigma_{,\rho}\zeta^\rho_{,(\mu}\bar{T}_{\nu)\sigma} + 2\zeta^\rho\zeta^\sigma_{,(\mu}\bar{T}_{\nu)\sigma,\rho} + 2\zeta^\rho_{,\mu}\zeta^\sigma_{,\nu}\bar{T}_{\rho\sigma}\right. \\ =& T_{\mu\nu} - \left(T_{\mu\nu,\rho}\xi^\rho + 2T_{\rho(\mu}\xi^\rho_{,\nu)}\right) + \xi^\sigma\left(2\xi^\rho_{,(\nu}\bar{T}_{\mu)\rho,\sigma} + 2\bar{T}_{\rho(\mu}\xi^\rho_{,\nu)\sigma} + \frac{1}{2}\bar{T}_{\mu\nu,\rho\sigma}\xi^\rho + \bar{T}_{\mu\nu,\rho}\xi^\rho_{,\sigma}\right) \\ &+ 2\bar{T}_{\rho(\mu}\xi^\sigma_{,\nu}\xi^\rho_{,\sigma} + \bar{T}_{\rho\sigma}\xi^\rho_{,\mu}\xi^\sigma_{,\nu}\end{aligned} \tag{93}$$

$$\begin{aligned}\widetilde{T}^\mu_\nu =& T^\mu_\nu - \left(\zeta^\rho T^\mu_{\nu,\rho} - \zeta^\mu_{,\rho}T^\rho_\nu + \zeta^\rho_{,\nu}T^\mu_\rho\right) + \frac{1}{2}\left[\zeta^\sigma\zeta^\rho\bar{T}^\mu_{\nu,\rho\sigma} + \zeta^\sigma\zeta^\rho_\sigma\bar{T}^\mu_{\nu,\rho} + \zeta^\sigma\left(\zeta^\rho_{,\nu\sigma}\bar{T}^\mu_\rho - \zeta^\mu_{,\rho\sigma}\bar{T}^\rho_\nu\right)\right. \\ &\left. +2\zeta^\sigma\left(\zeta^\rho_{,\nu}\bar{T}^\mu_{\rho,\sigma} - \zeta^\mu_{,\rho}\bar{T}^\rho_{\nu,\sigma}\right) + \zeta^\rho_{,\sigma}\left(\zeta^\mu_{,\rho}\bar{T}^\sigma_\nu + \zeta^\sigma_{,\nu}\bar{T}_s\rho^\mu\right) - 2\zeta^\mu_{,\rho}\zeta^\sigma_{,\nu}\bar{T}^\rho_\sigma\right] \\ =& T^\mu_\nu\left(T^\mu_{\nu,\rho}\xi^\rho - \xi^\mu_{,\rho}T^\rho_\nu + \xi^\rho_{,\nu}T^\mu_\rho\right) + \frac{1}{2}\xi^\sigma\xi^\rho\bar{T}^\mu_{\nu,\rho\sigma} + \xi^\sigma\xi^\rho_{,\sigma}\bar{T}^\mu_{\nu,\rho} + \xi^\sigma\left(\xi^\rho_{,\nu\sigma}T^\mu_{,\rho} - \xi^\mu_{,\rho\sigma}\bar{T}^\rho_\nu\right) \\ &+ \xi^\sigma\left(\xi^\rho_{,\nu}\bar{T}^\mu_{\rho,\sigma} - \xi^\mu_{,\rho}T^\rho_{\nu,\sigma}\right) + \xi^\rho_{,\sigma}\xi^\sigma_{,\nu}\bar{T}^\mu_\rho - \xi^\mu_{,\rho}\xi^\sigma_{,\nu}\bar{T}^\rho_\sigma\end{aligned} \tag{94}$$

$$\begin{aligned}\tilde{T}^{\mu\nu} =& T^{\mu\nu} - \left(T^{\mu\nu}{}_{,\rho}\zeta^\rho - 2T^{\rho(\mu}\zeta^{\nu)}{}_{,\rho}\right) + \frac{1}{2}\left(\bar{T}^{\mu\nu}{}_{,\rho\sigma}\zeta^\rho\zeta^\sigma + \bar{T}^{\mu\nu}{}_{,\rho}\zeta^\rho_{,\sigma}\zeta^\sigma - 2\zeta^\rho\bar{T}^{\sigma(\mu}_{,\rho}\zeta^{\nu)}{}_{,\sigma}\right. \\ &\left. -2\bar{T}^{\rho(\mu}\zeta^{\nu)}_{,\rho\sigma}\zeta^\sigma - 2\bar{T}^{\rho(\mu}_{,\sigma}\zeta^{\nu)}_{,\rho}\zeta^\sigma + 2\bar{T}^{\rho(\mu}\zeta^{\nu)}_{,\sigma}\zeta^\sigma_{,\rho} + 2\bar{T}^{\rho\sigma}\zeta^\mu_{,\rho}\zeta^\nu_{,\sigma}\right) \\ =& T^{\mu\nu} - \left(T^{\mu\nu}{}_{,\rho}\xi^\rho - 2T^{\rho(\mu}\xi^{\nu)}_{,\rho}\right) + \xi^\sigma\left(\frac{1}{2}T^{\mu\nu}{}_{,\rho\sigma}\xi^\rho + \bar{T}^{\mu\nu}{}_{,\rho}\xi^\rho_{,\sigma} - 2\bar{T}^{\rho(\mu}\xi^{\nu)}{}_{,\rho\sigma} - 2\bar{T}^{\rho(\mu}_{\sigma}\xi^{\nu)}{}_{,\rho}\right) \\ &+ \bar{T}^{\rho(\mu}\xi^{\nu)}_{,\sigma}\xi^\sigma_{,\rho} + \bar{T}^{\rho\sigma}\xi^\mu_{,\rho}\xi^\nu_{,\sigma}\end{aligned} \tag{95}$$

4.3.2 Conformal Newtonian gauge

Let us consider the gauge transformation of the scalar components of the metric, as defined in Eq. 77, i.e: $\alpha, \beta, \varphi,$ and γ. The gauge transformation at the linear order of the metric tensor is given by Eq. 93 up to the first order

$$\widetilde{\delta g}_{\mu\nu} = \delta g_{\mu\nu} - \left(\bar{g}_{\mu\nu,\rho}\xi^\rho + 2\bar{g}_{\rho(\mu}\xi^\rho_{,\nu)}\right). \tag{96}$$

Therefore the gauge transformation of the scalar components of the linear order perturbation of the metric are

$$\widetilde{\alpha} = \alpha - \xi^{0\prime} - \mathcal{H}\xi^0, \tag{97}$$

$$\widetilde{\beta} = \beta + \xi' + \xi^0, \tag{98}$$

$$\widetilde{\varphi} = \varphi - \frac{1}{3}\nabla^2\xi + \mathcal{H}\xi^0, \tag{99}$$

$$\widetilde{\gamma} = \gamma + \xi. \tag{100}$$

There are several kinds of gauge choices in Cosmology, one of these is called Conformal Newtonian gauge. The choice, in the conformal Newtonian gauge, is to fix the scalar perturbations β and γ to zero. This means that, we choose the spatial conditions to be $\beta = 0 = \gamma$ and the temporal condition $\chi = a(\gamma' + \beta) = 0$ [18]. This gauge is also known as zero-shear gauge, as the scalar shear is given by $\chi = 0$. From the previous relations, this means that we fix

$$\begin{aligned} \xi &= -\gamma = 0, \\ \xi^0 &= -\beta + \gamma' + \xi' = 0. \end{aligned} \tag{101}$$

They return a diagonal metric tensor for the scalar perturbations, which simplifies calculations [49]. We define $\psi := \alpha_\chi$ and $\phi := \varphi_\chi$, then the metric can be written as

$$ds^2 = -a^2(1 + 2\psi)d\eta^2 + a^2(1 + 2\phi)\bar{g}_{\alpha\beta}dx^\alpha dx^\beta. \tag{102}$$

According to *Clifton et al.* [49], this is the only gauge choice, that can be fully specified in both cosmological perturbation theory and post-Newtonian theory and appears to be valuable in the linear and nonlinear regimes of cosmology.

5 Power Spectrum

This section will introduce an essential tool in cosmology: the power spectrum. This importance lies in the fact that, it is a powerful and basic statistical measure that describes the distribution of mass and light in the Universe [50]; moreover, the quantities that we can calculate for a given model and then compare with observations are precisely the power spectra [51].

As we have discussed, the Universe is well described by the standard models of cosmology (i.e. the FLRW models). These predict an early phase when the distribution of matter fluctuations is very close to a Gaussian and later times when these fluctuations become non-Gaussian on small scales due to non-linear gravitational evolution while remaining Gaussian on very large scales. This is crucial for describing the power spectrum since a Gaussian field is fully described by its two-point correlation function [52], and the power spectrum is the harmonic transformation of the two-point correlation function, which corresponds to the Fourier transformation, in the case of a spatially flat Universe [51].

We now consider FLRW background perturbations, which are treated as random variables (stochastic variables), and the observations determine the statistical properties of these distributions, according to *Ellis et al.* [39]. Let, for instance, $A(\mathbf{x})$ be such a stochastic perturbative variable. Then the two-point correlation function is defined as

$$\xi(x) = \langle\, A(\mathbf{x})A(\mathbf{x}')\,\rangle, \tag{103}$$

this corresponds to the statistical average [39, 51]. In the following, we consider a spatially flat spacetime so that the harmonic transformation reduces to a Fourier transformation since we consider the curvature to be zero in this thesis.

Then the two-point correlation function in real space, described in Eq. 103, in Fourier space defines the power spectrum

$$\langle\, A(\mathbf{k})A(\mathbf{k}')\,\rangle = (2\pi)^3\,\delta(\mathbf{k}+\mathbf{k}')P_A(k). \tag{104}$$

There is a more intuitive quantity related to it, the so-called dimensionless power spectrum, defined as

$$\Delta_A^2(k) = \frac{k^3}{2\pi^2}P_A(k), \tag{105}$$

this quantity has the same dimension as the variable [51]. Regarding the specific case of this thesis, we are more interested in the power spectrum of primordial fluctuations. As we discuss in the inflation section Sec. 6, in the simplest models, these fluctuations are almost purely adiabatic and close to a scale-invariant power spectrum [53]. In the following, we consider the primordial scalar power spectrum, i.e. the power spectrum of scalar perturbations. We now consider the variable to be the Bardeen potential Φ. Commonly, the parameterization of the power spectrum is assumed to be a power-law of the form

$$\Delta_\Phi^2(k) = A_\Phi(k_0)\left(\frac{k}{k_0}\right)^{n_s(k_0)-1}, \tag{106}$$

where k_0 is the pivot scale, $n_s(k_0)$ is the scalar spectral index and $A_\Phi(k_0)$ is the power spectrum amplitude, which determines the variance of the fluctuations [54]. Nowadays, it is more common to work with the primordial curvature ζ power spectrum, which is closely related to the scalar perturbation primordial power spectrum by

$$\Delta_\Phi^2 = \frac{4}{9}\Delta_\zeta^2. \tag{107}$$

Hence, we can write the power spectrum in terms of the curvature as

$$\Delta_\zeta^2(k) = A_\zeta \left(\frac{k}{k_0}\right)^{n_s-1}, \tag{108}$$

In the previous two equations, we have omitted the pivot dependence of the power spectrum amplitude and scalar spectral index for clarity. These two variables are the parameters of the cosmological model and are fitted to the data [55]. According to the Planck mission, from the 2018 data, the best-fits are

$$A_\zeta(k_0 = 0.05 \text{ Mpc}^{-1}) = \left(2.101^{+0.031}_{-0.034}\right) \times 10^{-9}, \tag{109}$$

$$n_s(k_0 = 0.05 \text{ Mpc}^{-1}) = 0.9649 \pm 0.0042, \tag{110}$$

at 68% c.l. [56].

6 Inflation

The purpose of this section is to add a new piece to the puzzle describing the Universe. Initial conditions are required in the standard hot Big Bang model of Cosmology, they are problematic in several ways. In 1980, Alan Guth proposed a possible solution to two of these problems: the horizon and flatness problems. This solution is called cosmological inflation [57]. In this theory, the Universe undergoes a period of rapid expansion driven by a large cosmological constant in the false vacuum [58].

There are two versions of inflation: The original idea was that the Universe was trapped in a state of false vacuum and could escape through a tunnel to the real vacuum. This process would have created bubbles of real vacuum that could have filtered and reheated the Universe. Unfortunately, e.g. *Guth* himself, and *Weinberg* showed in [59] that this could not happen for the needed parameters of the potential. The problem was solved (by *Linde* [60] and independently by *Albrecht, and Steinhardt* [61]) by introducing a scalar field: the inflaton, and relying on fine-tuning the inflaton potential. This resulted in a slow rolling inflation with $N_e > 60$ e-folding [62]. There are other models, but we will focus on the models with a single scalar field.

In slow-roll inflation, the inflationary period begins with the inflation in a region where the actual potential is very flat, called the false vacuum. A (metastable) false vacuum is naturally created in a theory containing scalar fields. This vacuum represents a temporary minimum energy density condition, as *Weinberg* explained in [63]. This corresponds to a local minimum higher than the global minimum (true minimum), see Fig. 1. The properties of this false vacuum are that its pressure is large and negative. In general relativity, pressures, like energy densities, create gravitational fields. A positive pressure creates an attractive gravitational field, while a negative pressure creates a repulsive field, which is the force responsible for this period of strong expansion [64].

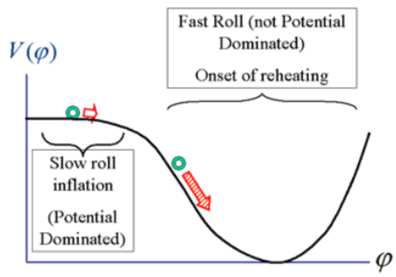

Figure 1: *The figure shows a scalar field slowly rolling down the flat region of a potential $V(\phi)$. In this case, the kinetic energy of the scalar field is very low. If the potential energy is greater than the kinetic one, then the pressure is negative, cf. Eq 116, and thus we are exactly in the inflationary epoch. When the field reaches the steepest region, it falls rapidly towards the global minimum of the potential and begins to oscillate about it [58]. (Figure credits: Albrecht [65]).*

At the end of the inflationary period, as shown in Fig. 1, if the scalar field reaches the absolute minimum, and starts to oscillate. The decay of the scalar field (inflaton) to the other, lighter fields to which it couples will damp these oscillations [40]. The thermalisation of the created particles heats up the Universe, and a Universe dominated by radiation is settled [40, 58]. This period may be considered to be constituted by two phases: the preheating and the reheating phases. In the first phase, particle creation results from parametric resonance.

In contrast, in the subsequent phase, particle creation results from the decreasing amplitude of the inflaton oscillation around the potential minimum, as the energy of the inflaton is transferred to other fields [66]. It is important to remark that there is not, at the moment, a clear understanding of the period connecting the inflation and the Big Bang Nucleosynthesis [67]. However, we are expecting that some process of reheating has to occur in order to connect the slow-roll inflation theory to the standard Hot Big-Bang theory [68]. Such a process is necessary because inflation would leave the Universe empty of matter and with a much lower temperature than expected for the radiation-dominated era in the Hot Big Bang Theory [69].

Before dealing mathematically with the slow-roll inflation model, we want to point out that it could be interesting to understand how the introduction of this rapid expansion into the early period of the Universe manages to solve some important problems of the standard model of the hot Big Bang. This part will not be dealt here, as it is not highly relevant to the development of the thesis [4]

6.1 Cosmological inflation: slow rolling single scalar field

For inflation, we need matter with the property of negative pressure, i.e. a scalar field describing a scalar particle of spin-0. Such particles possess a potential energy that can be very slowly reduced as the Universe expands [72]. The scalar field in this period is called inflaton, and we denote it as ϕ.

$$\phi = \phi_{cl} + \delta\phi_{qm}, \qquad \text{with} \qquad \delta\phi_{qm} \ll \phi_{cl}, \tag{111}$$

where we have adopted the *Kolb, and Turner* convention [40], in which the quantum scalar field is a correction of the classical scalar field, i.e. a perturbation.

We will begin by considering only classical dynamics and then analyse the evolution of ϕ_{cl}, where we will omit the subscript for the sake of clarity. After that, we will treat quantum fluctuations around the classical background evolution.

The dynamics of a scalar field minimal coupled to gravity is governed by the action

$$S = S_{EH} + S_\phi = \int d^4x\sqrt{-g}\left[\frac{c^4}{16\pi G}R - \left(\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + V(\phi)\right)\right]. \tag{112}$$

Similar to the procedure done in Section 2.2, we can define the energy-momentum tensor as in Eq. 23 and obtain for a scalar field

$$T^{(\phi)}_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}\phi_{,\rho}\phi^{,\rho} + V(\phi)\right) \tag{113}$$

We assume spatial homogeneity of the classical background field for the inflation, as proposed by *Kolb, and Turner* [40], Eq. 113 can be written as

$$T^{(\phi)\mu}{}_{\nu} = -\delta^\mu_0\delta^0_\nu\phi^{,0}\phi_{,0} + \delta^\mu_\nu\left(\frac{1}{2}\phi_{,0}\phi^{,0} - V(\phi)\right). \tag{114}$$

[4]For this reason, we invite interested readers to consult books on cosmology, such as *Moh et al.* [38]; *Mukhanov* [70]; *Ryden* [71]; *Kolb. and Turner* [40]; *Dodelson* [43], ...

From this, we can obtain the energy density $\rho \equiv -T^0_0$ and the isotropic pressure $p = \frac{1}{3}T^\alpha_\alpha$.

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{115}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{116}$$

For the purpose of showing how a scalar field can lead to negative pressure, we can write the resulting equation of state from the previous two equations

$$p_\phi = w_\phi \rho_\phi, \quad \text{with} \quad w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{117}$$

This equation shows that if the potential energy $V(\phi)$ dominates over the kinetic energy $\frac{1}{2}\dot{\phi}^2$, then $w_\phi < 0$.

The equation of motion for the classical scalar field can be obtained by the conservation of energy-momentum

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi}(\phi) = 0. \tag{118}$$

The second term of the equation results from the expansion of the Universe. According to *Kolb, and Turner* [40], there is an additional term in the equation of motion, which does not follow from the Lagrangian density as we have considered; this is the term resulting from the decay of the inflaton in the other fields to which it is weakly coupled. Hence, the previous equation can be described as follows

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi}(\phi) + \Gamma_\phi \dot{\phi} = 0, \tag{119}$$

where Γ_ϕ is the decay width and plays a role only during the scalar field's oscillation. For a formal treatment of the decay width and the Lagrangian density needed, one can read *Dolgov, and Linde* [73]; *Dolgov, and Hansen* [74]; *Abbott et al.* [75] or summarise in *Lazarides* [76].

During the slow-roll phase the $\ddot{\phi}$ term is negligible compared to the friction term [77], then Eq. 118 becomes

$$\dot{\phi} = -\frac{V_{,\phi}(\phi)}{3H}, \tag{120}$$

then

$$\ddot{\phi} = -\frac{V_{,\phi\phi}(\phi)\dot{\phi}}{3H(\phi)} + \frac{V_{,\phi}(\phi)}{3H^2(\phi)}H'(\phi)\dot{\phi}. \tag{121}$$

At this point, taking into account the fact that the pressure must be negative and therefore from Eq. 116, we know that the kinetic energy must be much smaller than the potential energy $\dot{\phi} \ll V(\phi)$, we can obtain the slow-roll conditions

$$\frac{(V_{,\phi}(\phi))^2}{V(\phi)} \ll H^2, \tag{122}$$

and

$$|V_{,\phi\phi}(\phi)| \ll H^2. \tag{123}$$

We now define the slow-roll parameters

$$\varepsilon := \frac{d}{dt}\frac{1}{H} \quad \text{and} \quad \eta := \left(\frac{\ddot{\phi}}{H\dot{\phi}}\right), \tag{124}$$

and in terms of the derivative of the potential ε_V and η_V as

$$\varepsilon_V := \frac{M_{pl}^2}{2}\left(\frac{V_{,\phi}(\phi)}{V(\phi)}\right)^2 \quad \text{and} \quad \eta_V := M_{pl}^2\left(\frac{V_{,\phi\phi}(\phi)}{V(\phi)}\right), \tag{125}$$

where $M_{pl}^{-2} := 8\pi G$. Then the following relations hold

$$\varepsilon = \varepsilon_V\left(1 - \frac{4}{3}\varepsilon_V + \frac{2}{3}\eta_V\right) \simeq \varepsilon_V \quad \text{and} \quad \eta_V \simeq \varepsilon + \eta. \tag{126}$$

The slow-roll parameter ε quantifies how much the Hubble rate H changes with time during inflation [78]. In general, inflationary models differ slightly from a de Sitter phase (which has $\varepsilon = 0$, and this discrepancy is captured by this parameter [79]. The slow-roll approximation, i.e.

$$\varepsilon \ll 1 \quad \text{and} \quad \eta \ll 1, \tag{127}$$

implies

$$\ddot{a} > 0, \quad \dot{\phi} \ll V(\phi), \quad |\ddot{\phi}| \ll |H\dot{\phi}|. \tag{128}$$

These lead to the slow-roll dynamics [79]

$$H^2 \simeq \frac{1}{3M_{pl}^2}V(\phi), \quad 3H\dot{\phi} \simeq -V_{,\phi}(\phi), \quad \rho_\phi + p_\phi \simeq \left(\frac{V_{,\phi}(\phi)}{3H}\right)^2. \tag{129}$$

As soon as the slow-roll approximation 127 fails, the inflation ends [78].

6.2 Quantum fluctuations as source of primordial power spectrum

In the following, we will study the effect of quantum fluctuations around the classical inflaton dynamics described so far. These fluctuations during inflation are the sources of the primordial power spectrum, i.e. the seeds of all structures in the Universe [80]. In our discussion of fluctuations, we will stop at the second-order action perturbation, which corresponds to the action of a free-field, since at the linearised level, there are no mode interactions (as can be clearly seen from Eq. 132) [81]. But, it is essential to note that, in single-field inflationary models, the theory is of interactive fields [34]. Thus

$$S = S_{(0)} + S_{(2)} + \mathcal{O}(S_{(3)}), \tag{130}$$

where background action defines classical dynamics, while quadratic action specifies free-field action and everything included in $\mathcal{O}(S_{(3)})$ establish interactions.

Now, in order to simplify the calculations, we fix the comoving gauge [34]

$$0 = v_\phi = \frac{\delta\phi}{\phi'}, \quad \phi(x) = \bar{\phi}(t), \quad \zeta := \varphi_v = \varphi_{\delta\phi}. \tag{131}$$

According to, for example *Baumann* [80]; *Maldacena* [82]; *Yoo* [79], the quadratic action has the following form

$$S_{(2)} = \frac{1}{2}\int dt d^3\mathrm{x}\, a^3 \frac{\dot{\phi}^2}{H^2}\left[\dot{\zeta}^2 - \frac{1}{a^2}(\nabla\zeta)^2\right] = \frac{1}{2}\int d\tau d^3\mathrm{x}\left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right] \tag{132}$$

where ζ is the comoving curvature perturbation, τ the conformal time, and v is the canonically-normalised Mukhanov-Sasaki variable

$$v := z\zeta, \quad \text{and} \quad z^2 := a^2\frac{\dot{\phi}^2}{H^2} = 2a^2\varepsilon. \tag{133}$$

The Euler-Lagrangian equation for the field v of Eq. 132 is

$$\begin{aligned} \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu v)} - \frac{\partial \mathcal{L}}{\partial_\mu v} = 0 \\ \implies \left(\partial_\tau^2 - \nabla^2 + m^2\right) v = 0, \end{aligned} \tag{134}$$

where $m^2 = -z''/z$. This is known as the Klein-Gordon equation.

According to *Mukhanov* [83], the quantisation of the dynamical system described by Eq. 132 is analogous to the quantisation of a scalar field with time-dependent mass m. Then we define the momentum conjugate with v

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial v'} = v', \tag{135}$$

such that the corresponding Hamiltonian is

$$\begin{aligned} H = \int d^3x\, \mathcal{H} &= \int d^3x\ (\pi v' - \mathcal{L}) \\ &= \frac{1}{2}\int d^3x\ \left[\pi^2 + (\nabla v)^2 + m^2 v^2\right]. \end{aligned} \tag{136}$$

Replacing v and π by the corresponding operators $\hat{v}$ and $\hat{\pi}$ and by imposing the canonical commutation relation, we get the following expression for $\hat{v}$ [84]

$$\hat{v}(\tau,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\hat{v}_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} = \int \frac{d^3k}{(2\pi)^3}\left[\hat{a}_{\mathbf{k}} v_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{k}} v^*_k(\tau)e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{137}$$

where $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ are the annihilation and creation operators which satisfy:

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = \left[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\right] = 0 \quad \text{and} \quad \left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\right] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}'). \tag{138}$$

Substituting Eq. 137 into the operator analogue of the Klein-Gordon equation 134, we find the so-called Mukhanov-Sasaki equation

$$v''_k + w_k^2 v_k = 0, \tag{139}$$

where $w_k^2 = k^2 + m^2$ effective frequency, and v_k must satisfy the normalization condition

$$v'_k(\tau)v^*_k(\tau) - v_k(\tau)v^{*\prime}_k(\tau) = 2i. \tag{140}$$

This condition comes from the fact that, since $v_k(\tau)$ and $v_k^*(\tau)$ are linear independent ad form a basis of the solutions of Eq. 139, the Wronskian $W(v_k, v_k^*) \neq 0$. Then, since the Wronskian is time-independent, the mode function $u_k(\tau)$ can always be normalised by the previous condition [85].

Quantum states are constructed by defining the vacuum state $|0\rangle$ through the annihilation operator

$$\hat{a}_k|0\rangle = 0, \tag{141}$$

and the creation operator to produce excited states

$$|n_k\rangle = \sqrt{\frac{2E_k}{n!}}\hat{a}_k^\dagger|0\rangle, \tag{142}$$

where $\sqrt{2E_k}$ aims to make it Lorentz invariant and $E_k \equiv |v_k'|^2 + k^2|v_k|^2$ [79].

We now aim to find a vacuum state for the fluctuation. For this purpose, we must fix the mode functions $v_k(\tau)$ with a second condition so that the expected value of the Hamiltonian is minimised in the vacuum state. Since the normalisation given by Eq. 140 is insufficient to fix the solutions, we must find initial conditions for v_k. As explained by *Baumann* [80], at sufficiently early times, all modes of cosmological interest were deep inside the horizon and thus are unaffected by gravity, and their frequencies w_k are independent of time. In this limit, the Mukhanov-Sasaki equation becomes that of a simple harmonic oscillator. The only solution that interests us is the one describing the minimum energy state. Accordingly, we set the initial conditions

$$\lim_{\tau\to-\infty} v_k = \frac{1}{\sqrt{2k}}e^{-ik\tau}. \tag{143}$$

The two conditions completely define a set of mode functions, and the last one defines a unique vacuum, the Bunch-Davies vacuum.

The only thing left to do is to solve the Mukhanov-Sasaki equation with the initial conditions just found, and then compute the power spectrum, which is nothing but the quantum zero-point fluctuations. According to the definition of z given in Eq. 133, we can compute the mass term of the effective frequency. We recall that the effective frequency is defined as

$$w_k^2 = k^2 + m^2, \quad \text{with} \quad m = -\frac{z''}{z}.$$

Then the exact solution of the mass term is

$$\frac{z''}{z} = (aH)^2\left[2 - \varepsilon + \frac{3}{2}\varepsilon_2 - \frac{1}{2}\varepsilon\varepsilon_2 + \frac{1}{4}\varepsilon_2^2 + \varepsilon_2\varepsilon_3\right], \tag{144}$$

where

$$\varepsilon \equiv -\frac{\dot{H}}{H^2}, \quad \varepsilon_2 \equiv \frac{\dot{\varepsilon}}{H\varepsilon}, \quad \varepsilon_3 \equiv \frac{\dot{\varepsilon_2}}{H\varepsilon_2}. \tag{145}$$

We begin by considering a de Sitter space, $a = -(H\tau)^{-1}$ and $\varepsilon = \varepsilon_2 = \varepsilon_3 = 0$. Then the Mukhanov-Sasaki equation Eq. 139 becomes

$$v_k'' + \left(k^2 - \frac{2}{\tau^2}\right)v_k = 0, \tag{146}$$

and its solution is

$$v_k(\tau) = \alpha\frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + \beta\frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right). \tag{147}$$

Considering the initial conditions, Eq. 143, we can fix the coefficients and find the unique mode function

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right). \tag{148}$$

Then, the zero-point fluctuation is given by

$$\begin{aligned}
\langle\, \hat{v}_{\mathbf{k}}\hat{v}_{\mathbf{k}'}\,\rangle &= \langle\, 0|\hat{v}_{\mathbf{k}}\hat{v}_{\mathbf{k}'}|0\,\rangle \\
&= \langle\, 0|\left(\hat{a}_{\mathbf{k}}v_k(\tau) + \hat{a}^\dagger_{\mathbf{k}}v^*_k(\tau)\right)\left(\hat{a}_{\mathbf{k}'}v'_k(\tau) + \hat{a}^\dagger_{\mathbf{k}'}v^*_{k'}(\tau)\right)|0\,\rangle \\
&= v_k(\tau)v^*_{k'}(\tau)\langle\, 0|\left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\right]|0\,\rangle \\
&\equiv (2\pi)^3 P_v(k)\delta(\mathbf{k}+\mathbf{k}').
\end{aligned} \tag{149}$$

On superhorizon scales, the power spectrum of the field v approaches

$$P_v(k) = \frac{(aH)^2}{2k^3}. \tag{150}$$

The power spectrum of the comoving curvature perturbation is calculated at the horizon crossing of the superhorizon limit. Thus we get

$$\Delta^2_\zeta = \frac{k^3}{2\pi^2}P_\zeta(k) = \frac{k^3}{2\pi^2}\frac{P_v(k)}{z^2} = \frac{H^2}{8\pi^2\varepsilon}, \tag{151}$$

where Δ^2_ζ is the dimensionless power spectrum of the comoving curvature perturbation. It is important to observe that in a perfect de Sitter spacetime, a problem arises in defining the curvature fluctuations: as de Sitter means that the slow-roll parameter $\varepsilon = 0$ and, therefore, z vanishes as well. This is due to the fact that a perfectly de Sitter inflation never ends, so the curvature perturbation is meaningless.

Turning our attention to the slow-roll inflation case, we solve the Mukhanov-Sasaki equation to find the power spectrum (as before).
From the definition of the slow-roll parameter ε in cosmic time, we get the following exact relation in conformal time

$$\begin{aligned}
&\varepsilon = \frac{d}{dt}\left(\frac{1}{H}\right) = \frac{1}{a}\frac{d}{d\tau}\left(\frac{a}{\mathcal{H}}\right) = 1 - \frac{d}{d\tau}\left(\frac{1}{aH}\right), \quad \text{assuming } \varepsilon = \text{const} \\
&\implies -\frac{1}{aH} = (1-\varepsilon)\tau.
\end{aligned} \tag{152}$$

Therefore, to the first term in ε we find the relation

$$aH = -\frac{1}{\tau}\left[1 + \varepsilon + \mathcal{O}(2)\right]. \tag{153}$$

According to *Stewart, and Lyth* [86], the standard result, considering constant slow-roll parameters

$$\frac{z''}{z} = \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right), \quad \text{with} \quad \nu := \frac{3-\varepsilon}{2(1-\varepsilon)}. \tag{154}$$

The result obtained allows us to solve analytically the Mukhanov-Sasaki equation 139 and derive the result in terms of Hankel functions [86]

$$v_k(\tau) = \sqrt{-\tau}\left[\alpha H_\nu^{(1)}(-k\tau) + \beta H_\nu^{(2)}(-k\tau)\right]. \tag{155}$$

Again, through the initial conditions, Eq. 143, we fix the coefficient and find the solution

$$\zeta_k(\tau) = \frac{1}{z(\tau)}v_k(\tau) = \frac{1}{z(\tau)}\sqrt{-\tau\frac{\pi}{2}}H_\nu^{(1)}(-k\tau). \tag{156}$$

Then up to the first order $z \simeq \tau^{1/2-\nu}$ and following the procedure described in Eq. 149, we are able to find the dimensionless power spectrum of the comoving curvature perturbation in the superhorizon limit ($k \ll aH$)

$$\Delta_\zeta^2 = \frac{k^3}{2\pi^2}P_\zeta = 2^{2\nu-3}\frac{\Gamma^2(\nu)}{\Gamma^2(3/2)}(1-\varepsilon)^{2\nu-1}\left(\frac{H^2}{8\pi^2\varepsilon}\right). \tag{157}$$

In this section, we described the scalar fluctuations during inflation and the associated power spectrum, assuming that the slow-roll parameters are constant. *Stewart, and Lyth* [86] developed a more general result, in which these parameters are assumed to be small but not constant. In the next section, when we discuss gravitational wave sources, Sec. 7.2, we discuss tensor fluctuations resulting from inflation.

7 Gravitational Waves

Shortly after the final formulation of Einstein's field equations, Einstein predicted the existence of gravitational radiation. In 1916 and 1918, he found that at the linear order, the weak-field equations had wave solutions [6, 7]. According to *Hughes* [87], gravitational waves are not only a consequence of Einstein's GR theory, but they are necessary in any relativistic theory of gravity. Gravitational waves are ripples in spacetime. Their formation can be caused by astrophysical phenomena related to accelerating masses [8] or by cosmological phenomenon related to the early Universe, such as quantum fluctuations of the scalar field or phase transition, which result in perturbations of the spacetime metric [9, 10].

At the beginning of 2016, one hundred years after the prediction, LIGO-Virgo Collaboration announced that their detectors simultaneously observed a transient gravitational wave signal emitted from the binary black hole merger GW150914 [11, 88], with their ground-based gravitational wave detectors. This observation also proved the unexpected existence of binary systems of black holes. Other binary systems have been discovered in recent years (see [11, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100]). The implications do not only stop at the discovery of new astrophysical objects but also tests in fundamental physics and cosmology [101], especially with space-based gravitational wave detectors, such as LISA [12], DECIGO [102]. In addition to these individual gravitational wave sources, the Universe is permeated by stochastic gravitational wave background [14]. Observations of gravitational waves from the stochastic cosmological background could have an important impact on the cosmology of the early Universe and, consequently, on the corresponding energy ranges.

The general idea is that, while particles are coupled to the plasma, information about the Universe is not measurable due to subsequent interactions. Once they decouple from the plasma, they preserve information of that instant (e.g. the cosmic microwave background). Therefore, the earlier a particle decouples, i.e. the weaker the interaction of a particle, the higher the energy scale in which it leaves the thermal equilibrium. Considering gravitational waves, we can do the following reasoning: the interaction rate is given by $\Gamma = n\sigma|v|$, where the density number $n \sim T^3$, for mass-less particles, assuming that the typical velocity is $|v| \sim c = 1$ and the cross-section of the process is given by $\sigma \sim G^2T^2 \sim (T/M_{pl}^2)^2$. On the other hand, we have the Hubble parameter, which is $H \sim T^2/M_{pl}$ during the radiation-dominated era. Thus, we obtain

$$\left(\frac{\Gamma}{H}\right)_{graviton} \sim \left(\frac{T}{M_{pl}}\right)^3, \tag{158}$$

where the Planck mass is $M_{pl} \sim 10^{19}$GeV. Therefore, the gravitons are decoupled below the Planck scale [103]. Our knowledge of the Universe allows us to conclude that, since the temperature of the Universe after inflation is always lower than this value, the gravitational waves propagate freely immediately after being generated [101].

As we have seen, there are different sources of gravitational waves. They can be of relativistic astrophysical origin (e.g. merging of astrophysical objects) or cosmological origin, produced in the early times and resulting in a stochastic gravitational wave background, produced by numerous weak and unresolved sources [10, 13, 104, 14]. In this thesis, we will focus on the cosmological gravitational waves, more precisely on the scalar induce gravitational waves. However, some of the considerations we will make are precisely the same for astrophysical gravitational waves. This is because the differences between the two types of gravitational waves are few and related to the 'distance' at which they occur and the wave's frequency. Astrophysical gravitational waves correspond to events relatively close to us, while cosmological ones correspond to events typically occurring in the primordial Universe. In addition, the formers have higher frequencies than the latter [105].

Typical sources of cosmological gravitational waves in the early Universe are quantum fluctuations during inflation (typically referred to as primordial gravitational waves), resonance during the reheating period, phase transitions, and primordial fluctuations. Our focus is on gravitational waves that result from these fluctuations. It's important to note that these waves can lead to the formation of primordial black holes [106].

In the following, we will discuss the meaning of stochastic background and explain some possible sources of cosmological gravitational waves, focusing on the one arising from slow-roll inflation and the one induced by primordial fluctuation. After that, we will formally define the gravitational wave, deriving the equation of motion (i.e. the gravitational wave equation) at the linear order in a Minkowski spacetime, explaining the gauge choices that one can take, the polarisations and showing how gravitational waves can affect the spacetime. In conclusion, we consider the gravitational waves induced by scalar perturbations in more detail.

7.1 Formalism: Gravitational waves in a Minkowski background

The following is mostly based on [101, 107, 108, 109]. Let us initially consider a Minkowski spacetime, with metric $\eta_{\mu\nu}$ and signature $(-+++)$. Then for a weak field, we can assume a small perturbation from the Minkowski metric (cf. Eq. 72)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{159}$$

where $|h_{\mu\nu}| \ll 1$. Then, we consider the Einstein's field equations formulation, where the cosmological constant is put in the RHS of it as part of the energy-momentum tensor. Consider the Einstein's field equation as expressed in Eq. 36

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

Now, in order to find the Ricci tensor and scalar, we have to compute the affine connection (Christoffel symbols) to the linear-order

$$\begin{aligned} \Gamma^{\mu}_{\nu\rho} :&= \frac{1}{2} g^{\mu\sigma} \left(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma} \right) \\ &= \frac{1}{2} \eta^{\mu\sigma} \left(h_{\nu\sigma,\rho} + h_{\rho\sigma,\nu} - h_{\nu\rho,\sigma} \right) + \mathcal{O}(h^2), \end{aligned} \tag{160}$$

then the Riemann tensor is

$$\begin{aligned} R^{\mu}_{\nu\rho\sigma} :&= \Gamma^{\mu}_{\nu\sigma,\rho} - \Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\varepsilon}_{\nu\sigma}\Gamma^{\mu}_{\rho\varepsilon} + \Gamma^{\varepsilon}_{\nu\rho}\Gamma^{\mu}_{\sigma\varepsilon} \\ &= \Gamma^{\mu}_{\nu\sigma,\rho} - \Gamma^{\mu}_{\nu\rho,\sigma} + \mathcal{O}(h^2) \\ &= \frac{1}{2} \eta^{\mu\tau} \left(h_{\sigma\tau,\nu\rho} + h_{\nu\rho,\sigma\tau} - h_{\rho\tau,\nu\sigma} - h_{\nu\sigma,\rho\tau} \right) + \mathcal{O}(h^2). \end{aligned} \tag{161}$$

As final step, we compute the Ricci tensor

$$\begin{aligned} R_{\mu\nu} &:= R^{\rho}_{\mu\rho\nu} \\ &= \frac{1}{2}\left[2h^{\rho}_{(\mu,\nu)\rho} - \Box h_{\mu\nu} - h_{,\mu\nu}\right], \end{aligned} \tag{162}$$

and the Ricci scalar

$$R \equiv R^{\mu}_{\mu} = h^{,\mu\nu}_{\mu\nu} - \Box h, \tag{163}$$

where $h = h^{\mu}_{\nu}$ is the trace of the metric perturbation, the partial derivative is denoted with a comma $\partial_{\mu} x \equiv x_{,\mu}$ (respectively $\partial^{\mu} x \equiv \eta^{\mu\nu}\partial_{\nu} x \equiv x^{,\nu}$), and the d'Alembert operator $\Box \equiv \partial^{\mu}\partial_{\mu}$.

Therefore, the Einstein's field equation in Eq. 36 becomes

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\bar{h}_{\rho\sigma}{}^{,\rho\sigma} - 2\bar{h}^{\rho}_{(\mu,\nu)\rho} = -16\pi G T_{\mu\nu}. \tag{164}$$

In the previous equation, we have defined

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \tag{165}$$

which is known as the trace-reversed metric perturbation because $\bar{h} = -h$. Since the fields are covariant, we are allowed to perform coordinate transformation, which deviates only slightly from Minkowski coordinate [108],

$$x^{\mu} \to \widetilde{x}^{\mu} = x^{\mu} + \zeta^{\mu} = x^{\mu} + \xi^{\mu}, \tag{166}$$

where in the second equality, we performed the change of gauge field according to Eq. 89. The idea is to perform a gauge transformation (as we have seen in section 4). Therefore, we can extrapolate the linear order terms from the relation we already derived for the second order. We find that for the metric perturbation on a Minkowski background, the gauge transformation yields

$$\widetilde{h}_{\mu\nu} = h_{\mu\nu} - 2\xi_{(\mu,\nu)}, \tag{167}$$

and the trace-reversed gauge transformation yields

$$\widetilde{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} + \eta_{\mu\nu}\xi^{\rho}_{,\rho} - 2\xi_{(\mu,\nu)}. \tag{168}$$

If $\xi_{\mu,\nu} \lesssim |h_{\mu\nu}|$ the condition $|h_{\mu\nu}| \ll 1$ is preserved [109], this means that only slowly varying coordinate transformations are symmetries of the linearised theory[5]. The gauge freedom 167 allows us to fix the gauge, and through a coordinate transformation, we can choose the Lorentz gauge

$$\bar{h}_{\mu\nu}{}^{,\nu} = 0 \iff 2h_{\mu\nu}{}^{,\nu} = h_{,\mu}. \tag{169}$$

One can easily demonstrate that this gauge choice is always possible (cf. [101, 109]). Choosing only the coordinate system, which satisfies the Lorentz gauge condition 169, the Einstein's field equations 164 can be written as

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{170}$$

[5]It is not necessary that they are infinitesimal as proposed by Caprini and Figueroa [101], it is enough that the variation is much smaller than 1 [109].

Due to the symmetry $h_{\mu\nu} = h_{\nu\mu}$, we are left with ten degrees of freedom, with the gauge choice 169 we can impose four additional conditions, such that we have six degrees of freedom. Let us solve the partial differential equation 170: the homogeneous equation is important, as it corresponds to the vacuum equation of motion. We will focus for the moment on this equation, and later, we will consider the inhomogeneous part. Therefore, the vacuum equation of motion is

$$\Box \bar{h}_{\mu\nu} = 0, \tag{171}$$

in the vacuum case, the Lorentz gauge condition 169 does not fix completely the gauge. Therefore, the Lorentz condition is not altered by a further coordinate transformation, which satisfies

$$\Box \xi_{\mu} = 0. \tag{172}$$

This implies that we can choose the functions ξ_μ to impose four additional conditions on the metric perturbation $h_{\mu\nu}$ [109]. Thus, we choose ξ_μ, so that the trace and the spatial-temporal components of the metric perturbation vanish, $\bar{h} = 0 = \bar{h}_{0\alpha}$. Thus, the vacuum case described in Eq. 171 is equivalent to

$$\Box h_{\mu\nu} = 0. \tag{173}$$

Combining the Lorentz condition with the new conditions $\bar{h} = 0 = \bar{h}_{0\alpha}$, the only non-vanishing components are the spatial ones $h_{\alpha\beta}$. Therefore, we can define a new gauge choice, which is called transverse-traceless gauge (TT gauge) as

$$h_{0\mu} = 0, \qquad h^{\alpha}_{\alpha} = 0, \qquad h^{,\beta}_{\alpha\beta} = 0. \tag{174}$$

With the choice of the TT-gauge we are left with only two degrees of freedom, fixing completely the gauge freedom. In this case, the metric perturbation contains only the physical radiative degrees of freedom [101].

The solutions of Eq. 173 are plane waves

$$h_{\mu\nu}(x) = h_{\mu\nu}(t, \mathbf{x}) = \sum_P h^P_{\mathbf{x}}(t) e^P_{\mu\nu}(\mathbf{x}), \tag{175}$$

which can be expanded in the Fourier space

$$\begin{aligned} h_{\mu\nu}(x) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{ikx} h_{\mu\nu}(k) \\ &= \sum_P \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{ikx} e^P_{\mu\nu}(\mathbf{k}) h^P_{\mathbf{k}}(t), \end{aligned} \tag{176}$$

where $e^P_{\mu\nu}$ are the polarization tensors with $P = \{+, \times\}$, $k^\mu = (\omega/c, \mathbf{k})$ and $k^\mu k_\mu = 0$. We define the direction of propagation $\hat{\mathbf{k}} = \mathbf{k}/k$.

Since the temporal and temporal-spatial components of the metric perturbation are vanishing, according to the TT gauge choice, we can consider only the pure spatial component of $h_{\mu\nu}$. The Lorentz condition for plane waves becomes $|\hat{\mathbf{k}}|^\alpha h_{\alpha\beta} = 0$. Without any loss of generality, we impose that the direction of propagation $\hat{\mathbf{k}}$ lays along the z-axis, and under the symmetry and traceless conditions of the metric perturbation, we have

$$h_{\alpha\beta}(t, x) = \sum_{P=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{ikx} e^P_{\mu\nu}(\hat{\mathbf{k}}) h^P_{\mathbf{k}}(t), \tag{177}$$

where the component of the polarization tensor $e^{+}_{\alpha\beta}$ and $e^{\times}_{\alpha\beta}$ are defined

$$e^{+}_{\alpha\beta}(\hat{\mathbf{k}}) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\alpha\beta} , \quad e^{\times}_{\alpha\beta}(\hat{\mathbf{k}}) \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\alpha\beta} . \tag{178}$$

Let us now consider the effect of gravitational waves. Since the two possible polarisations have the same behaviour and differ only in the fact that one is rotated by 45° with respect to the other, we can consider only one of them. So, without loss of generality, let us consider the polarisation $P = +$. The solution in real space (Eq. 175) for a wave propagating along the z-axis corresponds to

$$\begin{aligned} h_{\alpha\beta}(t,z) &= h^{+}(t)e^{+}_{\alpha\beta}(\mathbf{x}) \\ &= h^{+}\exp\{i(k_z z - \omega t)\} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\alpha\beta} , \end{aligned} \tag{179}$$

where h^{+} is the amplitude of the oscillation. Spacetime stretches and shortens due to the tension created by the gravitational wave [13]. Considering a length L^i_0 along the $i = x, y$-axis, the gravitational wave causes the length to oscillate of a factor $\delta L^i(t)$. Therefore we have the total length as a function of the time as $L^i(t) = L^i_0 + \delta L^i(t)$. According to the equation of the geodesic deviation in the proper detector, the frame is [6]

$$\ddot{L}^i(t) = \delta\ddot{L}^i(t) = \frac{1}{2}\ddot{h}_{\alpha\beta}L^i(t). \tag{180}$$

Then, at $z = 0$

$$\begin{aligned} \delta L^x(t) &= \frac{h^{+}}{2}L_0\cos(\omega t), \\ \delta L^y(t) &= -\frac{h^{+}}{2}L_0\cos(\omega t). \end{aligned} \tag{181}$$

In this case, as the x-axis stretches, the y-axis contracts and vice versa [13]. Same for the $\times$-polarization case, but with a rotation of 45° of the axes.

7.2 The stochastic gravitational wave background

As we have already introduced, the Universe is permeated by a stochastic gravitational wave background. This gravitational wave signal arises from the incoherent sum of many weak, independent and unresolved sources [14, 110].

'Backgrounds' can also be generated by astrophysical sources, which cannot be resolved individually [111]. We like the differentiation proposed in *Grojean* [112], where the astrophysical one is the foreground, and the cosmological one is the background. Some backgrounds are stochastic by generation processes, whereas others are due to characteristics or limitations of the detector [113], resulting from the detector noise. By their nature, stochastic gravitational waves are inextinguishable from such noises. Hence, based on these, we can understand the complexity

[6] In Maggiore [109] Ch. 1.3, there is a derivation and explanation of this equation

and difficulties in searching for and observing cosmological gravitational waves [112]. Let us now examine which may be the possible sources of the stochastic gravitational wave background by generation process.

Slow-roll inflation

As we have seen in the previous section, during inflation, scalar, and tensor perturbations arise from the quantum fluctuations. The tensor perturbations are nothing but the primordial gravitational waves. As we have done for the scalar perturbation in Sec. 6, we can calculate tensor perturbations. We first decompose the tensor perturbation of the metric in terms of the helicity eigenstates as

$$h_{\alpha\beta} := 2C^{(t)}_{\alpha\beta} = \sum_{P=+,\times} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} h^P_{\mathbf{k}} \cdot e^P_{\alpha\beta}(\mathbf{k}), \tag{182}$$

with $P = \{+, \times\}$. Since the energy-momentum tensor provided by the inflation is diagonal, the tensor modes do not have any source, and therefore, the action becomes [78]

$$S^{(2)}_T = \frac{M^2_{pl}}{8} \int d\tau d^3x\, a^2(\tau) \left[(h'_{\alpha\beta})^2 - (\nabla h_{\alpha\beta})^2\right] = \sum_{P=+,\times} \int d\tau d^3\mathbf{k} \left(\frac{a(\tau)M_{pl}}{2}\right)^2 \left[(h^{P\prime}_{\mathbf{k}})^2 - \mathbf{k}^2(h^P_{\mathbf{k}})^2\right], \tag{183}$$

where

$$M^2_{pl} = \frac{1}{8\pi G}.$$

Now considering the transformation

$$v^P_{\mathbf{k}} = ah^P_{\mathbf{k}}, \tag{184}$$

which satisfies the equation of motion

$$v''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right) v_{\mathbf{k}} = 0, \tag{185}$$

where we omitted the polarisation index for the seek of clarity. Now we perform the standard quantisation (cf. Eq. 137) and consider the Bunch-Davies vacuum state defined by the condition in Eq. 143. As we have found for the scalar counterpart, assuming constant slow-roll parameter, we can express the solutions in terms of the Hankel functions, and according to the initial conditions, we find the following solution [86]

$$ah_{\mathbf{k}}(\tau) = v_k(\tau) = \sqrt{-\tau\frac{\pi}{2}} H^{(1)}_\mu(-k\tau), \quad \text{where} \quad \mu := \frac{3-\varepsilon}{2(1-\varepsilon)}. \tag{186}$$

Now the total tensor power spectrum is

$$P_T := 2(P_+ + P_\times), \tag{187}$$

and, therefore, the resulting dimensionless tensor power spectrum in the superhorizon limit $(k \ll aH)$ is

$$\Delta^2_T = \frac{k^3 P_T}{2\pi^2} = 2^{2\mu} \frac{\Gamma^2(\mu)}{\Gamma^2\left(\frac{3}{2}\right)} (1-\varepsilon)^{2\mu-1} \left(\frac{H}{2\pi}\right)^2. \tag{188}$$

(P)reheating

At the end of inflation, during the period of reheating, it est, when the inflaton oscillates about the true vacuum and decays into lighter fields (which correspond to the material that constitutes today's Universe) [13]. The production of lighter fields via parametric resonance (preheating process) corresponds to a non-perturbative, non-linear and out-of-equilibrium phenomenon [114]. The violent excitation is expected to produce large scalar metric perturbations [115, 116] and tensor metric perturbations (gravitational waves) [117, 118, 119, 120]. A formal treatment of this phase is beyond the scope of this brief introduction. Nevertheless, for more interested readers, we recommend the book by *Lemoine et al.* [121] in which *Kofman* explains this phase treated with different theories, as well as the generation of gravitational waves.

We consider a single-field slow-roll inflationary model (as described in Sec. 6), which is coupled with a single scalar matter field χ with a coupling $(\chi\phi)^2$. The two fields are perturbed up to the linear order as [55]

$$\phi(t, \mathbf{x}) = \phi(t) + \delta\phi(t, \mathbf{x}), \tag{189}$$

$$\chi(t, \mathbf{x}) = \delta\chi(t, \mathbf{x}). \tag{190}$$

The background solutions are

$$\begin{aligned} &\phi(t) = \Phi(t)\sin m_\phi t \quad \text{with} \quad m_\phi = \sqrt{\frac{d^2V}{d\phi^2}}, \\ &\chi(t) = 0 \end{aligned} \tag{191}$$

where m_ϕ is the oscillation frequency of the inflaton ϕ about the minimum of $V(\phi)$. At the linear order, the equations of motion yield [55, 121]

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \left(\frac{k^2}{a^2} + m_\phi^2\right)\delta\phi_{\mathbf{k}} = 0, \tag{192}$$

$$\delta\ddot{\chi}_{\mathbf{k}} + 3H\delta\dot{\chi}_{\mathbf{k}} + \left(\frac{k^2}{a^2} + g^2\phi^2(t)\right)\delta\chi_{\mathbf{k}} = 0. \tag{193}$$

From now on, we follow *Maggiore* [55], Eq. 193 can be written as

$$\frac{d^2}{dz^2}(\delta\chi_k) + 3\frac{H}{m_\phi}\frac{d}{dz}(\delta\chi_k) + [A(k) - 2q\cos(2z)]\delta\chi_k, \tag{194}$$

where $z = m_\phi t$, the resonance parameter $q = (g\Phi)^2/(4m_\phi^2)$ and $A(k) = k^2/(m_\phi a)^2 + 2q$. If q is large, a wide range of k can be in resonance, and one is in the broad resonance regime. In this case, energy is pumped efficiently by the inflaton field into the resonant modes $\delta\chi_k$; once the resonant modes reach an amplitude $\chi^2 \sim \Phi^2$, the perturbative approach breaks down and preheating ends. Therefore, the matter field χ dissipates its energy through interaction with other fields, eventually leading to thermalisation.

The numerical study shows that: initially, the χ field grows linearly, and this phase evolves into a highly non-linear phase (see, e.g. *Lemoine* [121]), characterised by the formation, expansion, and collision of bubble-like field inhomogeneities. Most gravitational waves are produced in this phase by the collision of these bubbles [55].

Phase transition

Hereafter, we will discuss the gravitational waves produced during phase transitions very synthetically. Firstly, phase transitions are a generic prediction of quantum field theories, and it is reasonable to expect that some of them occurred in the early Universe. The standard model of particle physics predicts that phase transitions occur at 150 MeV, QCD phase transition (quark-gluon confinement), and at 100 GeV, electroweak phase transition. Both are in the early phases of the radiation-dominated era [122].

If the phase transitions are of second order or smooth crossovers, they should not lead to observable gravitational waves [123]. However, first-order phase transitions are expected to produce gravitational waves via bubbles nucleation (see, e.g. [123, 124, 125, 126, 127]).

A concise explanation of the gravitational waves' production is given by *Hogan* [124]; he proposed considering relativistic matter that undergoes a first-order phase transition from a high-temperature phase H to a low-temperature phase L. After that, L-phase bubbles nucleate at isolated points in the plasma and rapidly expand until the mean pressure equilibrium of the two phases. Nucleation ceases when the bubbles have expanded sufficiently to compress the remaining H-phase and return to conditions where the two phases can coexist in equilibrium. The pressure waves of the expanding bubbles then produce gravitational waves. These produce an acoustic background noise that, in turn, generates a gravitational wave background.

Primordial fluctuation

As discussed in Sec. 6, during the slow-roll phase of inflation, scalar perturbations, as well as tensor perturbations arise. The formers are encoded by the comoving curvature perturbation ζ, and the latter, described previously in this section, by the primordial gravitational waves $h_{\alpha\beta}$. Hereafter, we are going to discuss the scalar perturbations.

Since the underlying model of inflation is not known, the scalar perturbations might deviate significantly from quasi-scale invariance at small scales [101]. First-order scalar perturbations directly induce gravitational waves at second and higher-order as they couple with the tensor modes. The counterpart of induced gravitational waves is the formation of primordial black holes. Primordial black holes are generated in those regions where the density contrast $\delta\rho/\rho$ is higher to a certain threshold [101, 106].

Once the primordial black holes form, they may also be sources for gravitational waves. They can generate gravitational waves in two ways: from a merging process and graviton emission [106]. The former is due to the formation of binaries due to three-body interaction with the nearest primordial black hole and the eventual merge of the black holes. The latter is the process of graviton emission by Hawking radiation/evaporation (for detailed discussions, see, e.g. [128, 129] for binaries and [130] for evaporation).

7.3 Induced Gravitational Waves

As introduced previously, gravitational waves can be induced by perturbations as well. Many papers, reviews, and books treat this kind of gravitational waves, for example [20, 21, 104, 106, 108, 55, 23]. In order to analyse the induced gravitational waves, we need to perturb the metric and the cosmological fluid quantities to the non-linear order (second or higher order of perturbation). In this section, we will continue the discussion of Sec 7.1. Therefore, we consider again small perturbations from a Minkowski spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + \frac{1}{2}h^{(2)}_{\mu\nu}, \tag{195}$$

where the indices (1) and (2) correspond to the first and second order. Then exactly as in Sec. 7.1, we can find the Ricci tensor and scalar, from which we can find the Einstein's field equations

$$\begin{aligned} R^{(1)}_{\mu\nu} - \left(\frac{1}{2}Rg_{\mu\nu}\right)^{(1)} &= 8\pi G T^{(1)}_{\mu\nu}, \\ R^{(2)}_{\mu\nu} - \left(\frac{1}{2}Rg_{\mu\nu}\right)^{(2)} &= 8\pi G T^{(2)}_{\mu\nu} \end{aligned} \tag{196}$$

Following the notation and explanation of *Jetzer* [108], the left-hand side of the second order Einstein's field equation can be written in terms of $t_{\mu\nu}$

$$t_{\mu\nu} := \frac{1}{8\pi G}\left[R^{(2)}_{\mu\nu} - \left(\frac{1}{2}Rg_{\mu\nu}\right)^{(2)}\right]. \tag{197}$$

Therefore, putting together the two Einstein's field equations 196, we find

$$R^{(1)}_{\mu\nu} - \left(\frac{1}{2}Rg_{\mu\nu}\right)^{(1)} = 8\pi G\left(T_{\mu\nu} + t_{\mu\nu}\right), \tag{198}$$

where $T_{\mu\nu} \equiv T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu}$. Following the procedure to get Eq. 170, we find

$$\Box h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} + t_{\mu\nu}\right). \tag{199}$$

The right-hand side of Eq. 199 is called source term. This equation shows that gravitational waves linear in $h_{\mu\nu}$ can be induced by a contribution from higher-order perturbations of the metric; in the specific case of the second order. At the linear order, scalar, vector, and tensor modes, evolution is governed by uncoupled equations of motion. The evolution becomes more complicated, at higher order. Since the scalar, vector and tensor modes mix. For example, combinations of scalar-scalar, scalar-tensor, and tensor-tensor perturbations, represent a source for gravitational waves [22, 131].

The initial part of the next chapter develops the Einstein's field equation, as described by Eq. 199, adopting the FLRW metric instead of Minkowski.

Part III
Scalar-Induced Gravitational Waves

This section aims to present scalar-induced gravitational waves. To do that, we discuss second-order relativistic perturbation theory, which is needed to find the evolution equations for gravitational waves. The linear order results are also significant, so we include them in Appendix A. Below are the results we have obtained. First, we present the evolution equations for gravitational waves, Sec. 8, without imposing any gauge conditions. In other words, we derive Einstein's field equations from the metric and the cosmological fluid. Once the evolution equations are obtained, we calculate the general solutions of the gravitational wave equation in the radiation and matter domains, Sec. 9. As we have seen in the theory part, we need to choose a gauge, i.e. a correspondence between the real and the background universe. Our choice is the conformal Newtonian gauge (also known as the zero-shear gauge), Sec. 10. Having obtained the wave equation solutions in our gauge choice, we can describe the power spectrum, Sec. 11. We have everything we need to find the relative energy density of gravitational waves, Sec. 12. The results are presented both analytically and numerically in this section.

8 Evolution Equations

The following section presents the main steps needed to obtain the Gravitational Wave equation. The real inhomogeneous Universe is defined as a composition of the FLRW metric (homogeneous and isotropic) and perturbations to the background metric.

$$
\begin{aligned}
g_{00} &= -a^2\left(1+2A\right), & g^{00} &= -\frac{1}{a^2}\left(-1+2A-4A^2+B_\alpha B^\alpha\right),\\
g_{0\alpha} &= -a^2 B_\alpha, & g^{0\alpha} &= \frac{1}{a^2}\left(-B^\alpha+2AB^\alpha+2B^\beta C^\alpha_\beta\right),\\
g_{\alpha\beta} &= a^2\left(\bar{g}_{\alpha\beta}+2C_{\alpha\beta}\right), & g^{\alpha\beta} &= \frac{1}{a^2}\left(\bar{g}^{\alpha\beta}-2C^{\alpha\beta}-B^\alpha B^\beta+4C^{\alpha\gamma}C^\beta_\gamma\right).
\end{aligned}
\tag{200}
$$

whereby $A \equiv A^{(1)} + \frac{1}{2}A^{(2)}$, $B_\alpha \equiv B^{(1)}_\alpha + \frac{1}{2}B^{(2)}_\alpha$, and $C_{\alpha\beta} \equiv C^{(1)}_{\alpha\beta} + \frac{1}{2}C^{(2)}_{\alpha\beta}$ are perturbations, up to the second order, from the homogeneous and isotropic Universe described by the FLRW metric.

From the metric Eq. 200 has been derived the Christoffel Symbols (connections), according to Eq. 2,

$$
\Gamma^\mu_{\nu\rho} := \frac{1}{2}g^{\mu\sigma}\left(\partial_\rho g_{\nu\sigma}+\partial_\nu g_{\rho\sigma}-\partial_\sigma g_{\nu\rho}\right). \tag{201}
$$

According to Eq. 3, we can find the Riemann tensor

$$
R^\mu_{\nu\rho\sigma} := \partial_\rho\Gamma^\mu_{\nu\sigma}-\partial_\sigma\Gamma^\mu_{\nu\rho}+\Gamma^\varepsilon_{\nu\sigma}\Gamma^\mu_{\rho\varepsilon}-\Gamma^\varepsilon_{\nu\rho}\Gamma^\mu_{\sigma\varepsilon} \tag{202}
$$

Contracting the Riemann Tensor, one gets the Ricci tensor, and by contracting it again, one obtains the Ricci scalar, according to Eq. 4:

$$
R_{\mu\nu} := R^\rho_{\mu\rho\nu}, \qquad R := R^\mu_\mu, \tag{203}
$$

the second-order expansion of Christoffel symbols, Riemann tensor, Ricci tensor and Ricci scalar can be found in Appendix B. Now, it is possible to define the Einstein tensor, defined in the theory part as:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{204}$$

The Einstein's Field equations (EFE) 205 have been computed, where the convention $c = 1$ has been used.

$$G^{\mu}_{\nu} = 8\pi G T^{\mu}_{\nu} \tag{205}$$

The various components of the Einstein and Energy-Momentum Tensor have been expanded up to the second order in perturbation, obtaining the following results (Eqs. 206, 207, 208, 213, 214 and 215). For the Einstein Tensor:

$$\begin{aligned}
G^0_0 =& \Lambda - 3\left(\frac{\mathcal{H}}{a}\right)^2 + \frac{1}{a^2}^{(1)}\left[2\mathcal{H}\left(C^{\alpha}_{\alpha}\right)' + 6\mathcal{H}^2 A - 2\mathcal{H}B^{\alpha}_{;\alpha} - C^{\alpha\beta}_{;\alpha\beta} + C^{\alpha;\beta}_{\alpha;\beta}\right] + \frac{1}{a^2}^{(2)}\left[\frac{1}{2}(C_{\alpha\beta})'(C^{\alpha\beta})'\right. \\
& - \frac{1}{2}(C^{\alpha}_{\alpha})'(C^{\beta}_{\beta})' + 4\mathcal{H}C^{\alpha\beta}(C_{\alpha\beta})' + 4\mathcal{H}A(C^{\alpha}_{\alpha})' - \mathcal{H}(C^{\alpha}_{\alpha})' - 12\mathcal{H}^2A^2 + 3\mathcal{H}^2 A + 3\mathcal{H}^2 B_{\alpha}B^{\alpha} \\
& + 2\mathcal{H}B^{\alpha}A_{;\alpha} - \left(C^{\beta}_{\beta}\right)' B^{\alpha}_{;\alpha} + 4\mathcal{H}AB^{\alpha}_{;\alpha} - \mathcal{H}B^{\alpha}_{;\alpha} - 2\mathcal{H}B^{\alpha}C^{\beta}_{\beta;\alpha} - B^{\alpha}\left(C^{\beta}_{\beta;\alpha}\right)' - \frac{1}{2}B^{\alpha}_{;\alpha}B^{\beta}_{;\beta} \\
& + 4\mathcal{H}B^{\alpha}C^{\beta}_{\alpha;\beta} + B^{\alpha}\left(C^{\beta}_{\alpha;\beta}\right)' - \frac{1}{2}B^{\alpha}B^{\beta}_{;\alpha\beta} - 2C^{\alpha\beta}C^{\gamma}_{\gamma;\alpha\beta} - \frac{1}{2}C^{\alpha\beta}_{;\alpha\beta} + \frac{1}{2}B^{\alpha}B^{;\beta}_{\alpha;\beta} + \frac{1}{2}C^{\alpha;\beta}_{\alpha;\beta} \\
& + (C_{\alpha\beta})'B^{\alpha}_{;\beta} + 4\mathcal{H}C_{\alpha\beta}B^{\alpha;\beta} + \frac{1}{4}B_{\beta;\alpha}B^{\alpha;\beta} + \frac{1}{4}B_{\alpha;\beta}B^{\beta;\alpha} + \frac{1}{2}C^{\gamma}_{\gamma;\beta}C^{\alpha;\beta}_{\alpha} + 2C^{\alpha\beta}_{;\alpha}C^{\gamma}_{\beta;\gamma} \\
& \left. -2C^{\alpha;\beta}_{\alpha}C^{\gamma}_{\beta;\gamma} + 4C^{\alpha\beta}C^{\gamma}_{\alpha;\beta\gamma} - 2C^{\alpha\beta}C^{;\gamma}_{\alpha\beta;\gamma} + C_{\alpha\gamma;\beta}C^{\alpha\beta;\gamma} - \frac{3}{2}C_{\alpha\beta;\gamma}C^{\alpha\beta;\gamma}\right],
\end{aligned} \tag{206}$$

$$\begin{aligned}
G^0_{\beta} =& \frac{1}{a^2}^{(1)}\left[\left(C^{\alpha}_{\beta;\alpha}\right)' - \frac{1}{2}B^{;\alpha}_{\beta;\alpha} + \frac{1}{2}B^{\alpha}_{;\beta\alpha} - 2\mathcal{H}A_{;\beta} + \left(C^{\alpha}_{\alpha;\beta}\right)'\right] + \frac{1}{a^2}^{(2)}\left[\left(C^{\alpha}_{\beta}\right)' C^{\gamma}_{\gamma;\alpha} + 2AC^{\alpha}_{\beta;\alpha}\right. \\
& - \frac{1}{2}\left(C^{\alpha}_{\beta;\alpha}\right)' + AB^{;\alpha}_{\beta;\alpha} - \frac{1}{4}B^{;\alpha}_{\beta;\alpha} + \frac{1}{4}B^{\alpha}_{;\beta\alpha} + B^{\alpha}C^{\gamma}_{\gamma;\beta\alpha} + (C_{\beta\alpha})'A^{;\alpha} + \frac{1}{2}B_{\beta;\alpha}A^{;\alpha} \\
& - \frac{1}{2}C^{\gamma}_{\gamma;\alpha}B^{;\alpha}_{\beta} - (C^{\alpha}_{\alpha})'A_{;\beta} + 8\mathcal{H}AA_{;\beta} - B^{\alpha}_{;\alpha}A_{;\beta} - \mathcal{H}A_{;\beta} - 2\mathcal{H}B^{\alpha}B_{\alpha;\beta} + \frac{1}{2}A^{;\alpha}B_{\alpha;\beta} \\
& + \frac{1}{2}C^{\gamma}_{\gamma;\alpha}B^{\alpha}_{;\beta} - (C^{\alpha\gamma})'\left(C_{\alpha\gamma;\beta}\right) - 2C^{\alpha\gamma}(C_{\alpha\gamma;\beta})' - 2A\left(C^{\alpha}_{\alpha}\right)'_{;\beta} + \frac{1}{2}(C^{\alpha}_{\alpha})^{\gamma}_{;\beta} + 2(C^{\alpha}_{\beta})'C^{\gamma}_{\alpha;\gamma} \\
& + B^{;\alpha}_{\beta}C^{\gamma}_{\alpha;\gamma} - B^{\alpha}_{;\beta}C^{\gamma}_{\alpha;\gamma} + 2C^{\alpha\gamma}\left(C_{\beta\alpha}\right)'_{;\gamma} - B^{\alpha}C^{\gamma}_{\beta;\alpha\gamma} - B^{\alpha}C^{\gamma}_{\alpha;\beta\gamma} + B^{\alpha}C^{;\gamma}_{\beta\alpha;\gamma} - C_{\beta\gamma;\alpha}B^{\alpha;\gamma} \\
& \left. + C_{\beta\alpha;\gamma}B^{\alpha;\gamma} + C_{\alpha\gamma}B^{;\alpha\gamma}_{\beta} - C_{\alpha\gamma}B^{\alpha;\gamma}_{;\beta}\right],
\end{aligned} \tag{207}$$

$$
\begin{aligned}
G^\alpha_\beta =&\Lambda\bar{g}^\alpha_\beta - \frac{1}{a^2}\bar{g}^\alpha_\beta\left[\mathcal{H}^2+2\mathcal{H}'\right] + \frac{1}{a^2}^{(1)}\Big[(C^\alpha_\beta)'' - \bar{g}^\alpha_\beta(C^\gamma_\gamma)'' + 2\mathcal{H}(C^\alpha_\beta)' + 2\mathcal{H}\bar{g}^\alpha_\beta A' - 2\mathcal{H}\bar{g}^\alpha_\beta(C^\gamma_\gamma)'\\
&+ 2\mathcal{H}^2\bar{g}^\alpha_\beta A + 4\mathcal{H}'\bar{g}^\alpha_\beta A - \mathcal{H}B^{;\alpha}_\beta + \frac{1}{2}(B^{;\alpha}_\beta)' + \mathcal{H}B^\alpha_{;\beta} + \frac{1}{2}(B^\alpha_{;\beta})' - A^{;\alpha}_{;\beta} - C^{\gamma;\beta}_{\gamma;\alpha} - 2\mathcal{H}\bar{g}^\alpha_\beta B^\gamma_{;\gamma}\\
&-\bar{g}^\alpha_\beta(B^\gamma_{;\gamma})' + C^{\beta;\gamma}_{\gamma;\alpha} + C^{;\beta\gamma}_{\alpha\gamma} + \bar{g}^\alpha_\beta A^{;\gamma}_{;\gamma} - C^{\beta;\gamma}_{\alpha;\gamma} - \bar{g}^\alpha_\beta C^{\gamma\delta}_{;\gamma\delta} + \bar{g}^\alpha_\beta C^{\gamma;\delta}_{\gamma;\delta}\Big] + \frac{1}{a^2}^{(2)}\Big[-A'(C^\alpha_\beta)'\\
&- 2\left(C^{\alpha\gamma}\right)'\left(C_{\beta\gamma}\right)' + \left(C^\alpha_\beta\right)'\left(C^\gamma_\gamma\right)' - 2A\left(C^\alpha_\beta\right)'' - 2C^{\alpha\gamma}C''_{\beta\gamma} + \frac{1}{2}\left(C^\alpha_\beta\right)'' + \bar{g}^\alpha_\beta A\left(C^\gamma_\gamma\right)'\\
&+ \frac{3}{2}\bar{g}^\alpha_\beta\left(C_{\gamma\delta}\right)'\left(C^{\gamma\delta}\right)' - \frac{1}{2}\bar{g}^\alpha_\beta\left(C^\gamma_\gamma\right)'\left(C^\delta_\delta\right)' + 2\bar{g}^\alpha_\beta C^{\gamma\delta}\left(C_{\gamma\delta}\right)'' + 2\bar{g}^\alpha_\beta A\left(C^\gamma_\gamma\right)'' - \frac{1}{2}\bar{g}^\alpha_\beta\left(C^\gamma_\gamma\right)''\\
&- 4\mathcal{H}A\left(C^\alpha_\beta\right)' - 4\mathcal{H}C^{\alpha\gamma}\left(C_{\beta\gamma}\right)' + \mathcal{H}\left(C^\alpha_\beta\right)' - 8\mathcal{H}\bar{g}^\alpha_\beta AA' + \mathcal{H}\bar{g}^\alpha_\beta A' + 2H\bar{g}^\alpha_\beta B^\gamma(B_\gamma)'\\
&+ 4\mathcal{H}\bar{g}^\alpha_\beta C^{\gamma\delta}(C_{\gamma\delta})' + 4\mathcal{H}\bar{g}^\alpha_\beta A(C^\gamma_\gamma)' - \mathcal{H}\bar{g}^\alpha_\beta(C^\gamma_\gamma)' - 4\mathcal{H}^2\bar{g}^\alpha_\beta A^2 + \mathcal{H}^2\bar{g}^\alpha_\beta A + \mathcal{H}^2\bar{g}^\alpha_\beta B_\gamma B^\gamma\\
&- 8\mathcal{H}'g^\alpha_\beta A^2 + 2\mathcal{H}'\bar{g}^\alpha_\beta A + 2\mathcal{H}'\bar{g}^\alpha_\beta B_\gamma B^\gamma - \frac{1}{2}A'B^{;\alpha}_\beta + \frac{1}{2}\left(C^\gamma_\gamma\right)'B^{;\alpha}_\beta - 2\mathcal{H}AB^{;\alpha}_\beta - A\left(B^{;\alpha}_\beta\right)'\\
&+ \frac{1}{2}\mathcal{H}B^{;\alpha}_\beta + \frac{1}{4}\left(B^{;\alpha}_\beta\right)' - (B^\gamma)'C^{;\alpha}_{\beta\gamma} - 2\mathcal{H}B^\gamma C^{;\alpha}_{\beta\gamma} - B^\gamma\left(C^{;\alpha}_{\beta\gamma}\right)' - 2\mathcal{H}B^\alpha A_{;\beta} + A^{;\alpha}A_{;\beta}\\
&- \frac{1}{2}A'B^\alpha_{;\beta} + \frac{1}{2}\left(C^\gamma_\gamma\right)'B^\alpha_{;\beta} - 2\mathcal{H}AB^\alpha_{;\beta} - \frac{1}{2}B^{\gamma;\alpha}B_{\gamma;\beta} - 2\mathcal{H}C^\alpha_\gamma B^\gamma_{;\beta} - A\left(B^\alpha_{;\beta}\right)' - C^\alpha_\gamma\left(B^\gamma_{;\beta}\right)'\\
&+ \frac{1}{2}\mathcal{H}B^\alpha_{;\beta} + \frac{1}{4}\left(B^\alpha_{;\beta}\right)' - (B^{;\gamma})'C^\alpha_{\gamma;\beta} - 2\mathcal{H}B^\gamma C^\alpha_{\gamma;\beta} + C^{\gamma\delta;\alpha}C_{\gamma\delta;\beta} - B^\gamma\left(C^\alpha_{\gamma;\beta}\right)' + B^\alpha\left(C^\gamma_{\gamma;\beta}\right)'\\
&+ 2AA^{;\alpha}_{;\beta} - \frac{1}{2}A^{;\alpha}_{;\beta} - B^\gamma B^{;\alpha}_{\gamma;\beta} + 2C^{\gamma\delta}C^{;\alpha}_{\gamma\delta;\beta} - \frac{1}{2}C^{\gamma;\alpha}_{\gamma;\beta} + 2\mathcal{H}\bar{g}^\alpha_\beta B^\gamma A_{;\gamma} + \left(C^\alpha_\beta\right)'B^\beta_{;\beta} + \bar{g}^\alpha_\beta A'B^\gamma_{;\gamma}\\
&- \bar{g}^\alpha_\beta\left(C^\delta_\delta\right)'B^\gamma_{;\gamma} + 4\mathcal{H}\bar{g}^\alpha_\beta AB^\gamma_{;\gamma} + \frac{1}{2}B^{;\alpha}_\beta B^\gamma_{;\gamma} + \frac{1}{2}B^\alpha_{;\beta}B^\gamma_{;\gamma} + 2\bar{g}^\alpha_\beta A\left(B^\gamma_{;\gamma}\right)' - \mathcal{H}\bar{g}^\alpha_\beta B^\gamma_{;\gamma} - \frac{1}{2}\bar{g}^\alpha_\beta\left(B^\gamma_{;\gamma}\right)'\\
&+ (B^\gamma)'C^\alpha_{\beta;\gamma} + 2\mathcal{H}B^\gamma C^\alpha_{\beta;\gamma} - \bar{g}^\alpha_\beta\left(B^\gamma\right)'C^\delta_{\delta;\gamma} - 2\mathcal{H}\bar{g}^\alpha_\beta B^\gamma C^\delta_{\delta;\gamma} + C^{\gamma;\alpha}_\beta C^\delta_{\delta;\gamma} + C^{\alpha\gamma}_{;\beta}C^\delta_{\delta;\gamma}\\
&+ 2B^\gamma\left(C^\alpha_{\beta;\gamma}\right)' - B^\alpha\left(C^\gamma_{\beta;\gamma}\right)' - 2\bar{g}^\alpha_\beta B^\gamma\left(C^\gamma_{\delta;\gamma}\right)' + \frac{1}{2}B^\gamma B^{;\alpha}_{\beta;\gamma} + \frac{1}{2}C^{\gamma;\alpha}_{\beta;\gamma} + \frac{1}{2}B^\gamma B^\alpha_{;\beta\gamma}\\
&+ \frac{1}{2}B^\alpha B^\gamma_{;\beta\gamma} + 2C^{\alpha\gamma}C^\delta_{\delta;\beta\gamma} + \frac{1}{2}C^{\alpha;\gamma}_{;\beta\gamma} - 2\bar{g}^\alpha_\beta AA^{;\gamma}_{;\gamma} + \frac{1}{2}\bar{g}^\alpha_\beta A^{;\gamma}_{;\gamma} - \frac{1}{2}B^\alpha B^{;\gamma}_{\beta;\gamma} - \frac{1}{2}C^{\alpha;\gamma}_{\beta;\gamma} + C^{;\alpha}_{\beta\gamma}A^{;\alpha}\\
&+ C^\alpha_{\gamma;\beta}A^{;\gamma} - \bar{g}^\alpha_\beta A_{;\gamma}A^{;\gamma} - C^\alpha_{\beta;\gamma}A^{;\gamma} + \bar{g}^\alpha_\beta C^\delta_{\delta;\gamma}A^{;\gamma} - (C_{\beta\gamma})'B^{\alpha;\gamma} - \frac{1}{2}B_{\beta;\gamma}B^{\alpha;\gamma} - \left(C^\alpha_\gamma\right)'B^{;\gamma}_\beta\\
&- 2\mathcal{H}C^\alpha_\gamma B^{;\gamma}_\beta - C^\alpha_\gamma\left(B^{;\gamma}_\beta\right)' - C^\delta_{\delta;\gamma}C^{\alpha;\gamma}_\beta + 2C^\alpha_\gamma A^{;\gamma}_{;\beta} - \frac{1}{2}\bar{g}^\alpha_\beta B^\gamma_{;\gamma}B^\delta_{;\delta} + 2\bar{g}^\alpha_\beta\left(B^\gamma\right)'\left(C^\delta_{\gamma;\delta}\right)\\
&+ 4\mathcal{H}\bar{g}^\alpha_\beta B^\gamma C^\delta_{\gamma;\delta} - 2C^{\gamma;\alpha}_\beta C^\delta_{\gamma;\delta} - 2C^{\alpha\gamma}_{;\beta}C^\delta_{\gamma;\delta} - 2\bar{g}^\alpha_\beta A^{;\gamma}C^\delta_{\gamma;\delta} + 2C^{\alpha;\gamma}_\beta C^\delta_{\gamma;\delta} + 2\bar{g}^\alpha_\beta B^\gamma\left(C^\delta_{\gamma;\delta}\right)'\\
&- 2C^{\gamma\delta}C^{;\alpha}_{\beta\gamma;\delta} - 2C^{\gamma\delta}C^\alpha_{\gamma;\beta\delta} - 2C^{\alpha\gamma}C^\delta_{\gamma;\beta\delta} - \bar{g}^\alpha_\beta B^\gamma B^\delta_{;\gamma\delta} + 2C^{\gamma\delta}C^\alpha_{\beta;\gamma\delta} - 2C^{\alpha\gamma}C^\delta_{\beta;\gamma\delta}
\end{aligned}
$$

$$
\begin{aligned}
&-2\bar{g}^{\alpha}_{\beta}C^{\gamma\delta}C^{\varepsilon}_{\varepsilon;\gamma\delta}-\frac{1}{2}\bar{g}^{\alpha}_{\beta}C^{\gamma\delta}_{;\gamma\delta}+\bar{g}^{\alpha}_{\beta}B^{\gamma}B^{;\delta}_{\gamma;\delta}+2C^{\alpha\gamma}C^{;\delta}_{\beta\gamma;\delta}+\frac{1}{2}\bar{g}^{\alpha}_{\beta}C^{\gamma;\delta}_{\gamma;\delta}+\bar{g}^{\alpha}_{\beta}\left(C_{\gamma\delta}\right)'B^{\gamma;\delta}\\
&+4\mathcal{H}\bar{g}^{\alpha}_{\beta}C_{\gamma\delta}B^{\gamma;\delta}-\frac{1}{4}\bar{g}^{\alpha}_{\beta}B_{\delta;\gamma}B^{\gamma;\delta}+\frac{3}{4}\bar{g}^{\alpha}_{\beta}B_{\gamma;\delta}B^{\gamma;\delta}+2\bar{g}^{\alpha}_{\beta}C_{\gamma\delta}\left(B^{\gamma;\delta}\right)'-2C_{\beta\delta;\gamma}C^{\alpha\gamma;\delta}\\
&+2C_{\beta\gamma;\delta}C^{\alpha\gamma;\delta}+\frac{1}{2}\bar{g}^{\alpha}_{\beta}C^{\varepsilon}_{\varepsilon;\delta}C^{\gamma;\delta}_{\gamma}-2\bar{g}^{\alpha}_{\beta}C_{\gamma\delta}A^{;\gamma\delta}+2\bar{g}^{\alpha}_{\beta}C^{\gamma\delta}_{;\gamma}C^{\varepsilon}_{\delta;\varepsilon}-2\bar{g}^{\alpha}_{\beta}C^{\gamma;\delta}_{\gamma}C^{\varepsilon}_{\delta;\varepsilon}\\
&+4\bar{g}^{-\alpha}_{\beta}C^{\gamma\delta}C^{\varepsilon}_{\gamma;\delta\varepsilon}-2\bar{g}^{\alpha}_{\beta}C^{\gamma\delta}C^{;\varepsilon}_{\gamma\delta;\varepsilon}+\bar{g}^{\alpha}_{\beta}C_{\gamma\varepsilon;\delta}C^{\gamma\delta;\varepsilon}-\frac{3}{2}\bar{g}^{\alpha}_{\beta}C_{\gamma\delta;\varepsilon}c^{\gamma\delta;\varepsilon}\Bigg].
\end{aligned}
\tag{208}
$$

The Energy-Momentum Tensor can be decomposed into fluid quantities in the following way:

$$
T_{\mu\nu}\equiv\rho u_{\mu}u_{\nu}+p\mathcal{H}_{\mu\nu}+q_{\mu}u_{\nu}+q_{\nu}u_{\mu}+\pi_{\mu\nu},
\tag{209}
$$

where $\mathcal{H}_{\mu\nu}$ is the projection tensor defined as

$$
\mathcal{H}_{\mu\nu}=g_{\mu\nu}+u_{\mu}u_{\nu},
\tag{210}
$$

and the variables ρ, p, q_{μ} and $\pi_{\mu\nu}$ are respectively the energy density, the isotropic pressure, the energy flux and the anisotropic pressure measured by the observer in the u_{μ}-frame. These fluid quantities have been decomposed into background and perturbations

$$
\rho=\bar{\rho}+\delta\rho,\qquad p=\bar{p}+\delta p,\qquad q_{\alpha}=aQ_{\alpha},\qquad \pi_{\alpha\beta}=a^2\Pi_{\alpha\beta},
\tag{211}
$$

where $\delta\rho$, δp, Q_{α} and $\Pi_{\alpha\beta}$ are perturbed up to the second order. To compute the Energy-Momentum Tensor we need to define the frame four-vector u_{μ} as

$$
\begin{aligned}
&u^0\equiv\frac{1}{a^2}\left(1-A+\frac{3}{2}A^2+\frac{1}{2}U^{\alpha}U_{\alpha}-U^{\alpha}B_{\alpha}\right),\qquad u^{\alpha}\equiv\frac{1}{a}U^{\alpha},\\
&u_0\equiv-\frac{1}{a^2}\left(1+A-\frac{1}{2}A^2+\frac{1}{2}U^{\alpha}U_{\alpha}\right),\qquad u_{\alpha}\equiv\frac{1}{a}\left(U_{\alpha}-B_{\alpha}+AB_{\alpha}+2U^{\beta}C_{\alpha\beta}\right).
\end{aligned}
\tag{212}
$$

Thus the components of the Energy-Momentum Tensor are:

$$
T^0_0=-\rho-\left(\bar{\rho}+\bar{p}\right)U^{\alpha}\left(U_{\alpha}-B_{\alpha}\right)-Q^{\alpha}\left(2U_{\alpha}-B_{\alpha}\right),
\tag{213}
$$

$$
\begin{aligned}
T^0_{\beta}=&\left(\bar{\rho}+\bar{p}\right)\left(U_{\beta}-B_{\beta}+2AB_{\beta}+2C_{\beta\alpha}U^{\alpha}-AU_{\beta}\right)+(1-A)Q_{\beta}+\left(\delta\rho+\delta p\right)\left(U_{\beta}-B_{\beta}\right)+\\
&\left(U^{\alpha}-B^{\alpha}\right)\Pi_{\beta\alpha},
\end{aligned}
\tag{214}
$$

$$
T^{\alpha}_{\beta}=p\delta^{\alpha}_{\beta}+\Pi^{\alpha}_{\beta}-2C^{\alpha\gamma}\Pi_{\beta\gamma}+\left(\bar{\rho}+\bar{p}\right)U^{\alpha}\left(U_{\beta}-B_{\beta}\right)+Q_{\beta}U^{\alpha}+Q^{\alpha}\left(U_{\beta}-B_{\beta}\right).
\tag{215}
$$

The gravitational waves are tensor perturbations, which means we can extract the tensor modes by applying an operator to the obtained results. According to Hwang et al. [132], transverse-tracefree (TT) projection of the pure space component of a tensor $S_{\alpha\beta}$ is:

$$
\begin{aligned}
\mathcal{P}^{\gamma\delta}_{\alpha\beta}S_{\gamma\delta}\equiv&S_{\alpha\beta}-\frac{1}{3}\bar{g}_{\alpha\beta}S^{\gamma}_{\gamma}+\frac{1}{2}\left(\nabla_{\alpha}\nabla_{\beta}-\frac{1}{3}\bar{g}_{\alpha\beta}\Delta\right)\left(\Delta+3K\right)^{-1}\left[S^{\gamma}_{\gamma}-3\Delta^{-1}\left(S^{\gamma\delta}_{|\gamma\delta}\right)\right]\\
&-2\nabla_{(\alpha}\left(\Delta+2K\right)^{-1}\left[S^{\gamma}_{\beta)|\gamma}-\nabla_{\beta)}\Delta^{-1}\left(S^{\gamma\delta}_{|\gamma\delta}\right)\right].
\end{aligned}
\tag{216}
$$

We are considering a flat space, therefore $K = 0$, $\bar{g}^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$, and we are looking at the pure spatial component of the tensor, thus the covariant derivative ; is replaced with |: $\nabla_{\alpha} T \equiv T_{|\alpha}$. We can simplify the expression in Eq. 216 creating a tracefree tensor $T_{\alpha\beta}$ and setting the condition for flat space.
Let $\bar{S}_{\alpha\beta} = S_{\alpha\beta} - \frac{1}{3}\bar{g}_{\alpha\beta} S^{\gamma}_{\gamma}$ and because the 3 metric $\bar{S}^{\alpha}_{\beta} = g^{\alpha\gamma}\bar{S}_{\gamma\beta}$

$$ {}^{(t)}S^{\alpha}_{\beta} = \bar{g}^{\alpha\varepsilon}\mathcal{P}^{\gamma\delta}_{\varepsilon\beta}\bar{S}_{\gamma\delta} = \bar{S}^{\alpha}_{\beta} + \frac{1}{2}\Delta^{-2}\left(\nabla^{\alpha}\nabla_{\beta} + \delta^{\alpha}_{\beta}\Delta\right)\bar{S}^{|\gamma\delta}_{\gamma\delta} - \Delta^{-1}\left(\nabla^{\alpha}\bar{S}^{|\gamma}_{\beta\gamma} + \nabla_{\beta}\bar{S}^{\alpha|\gamma}_{\gamma}\right). \tag{217} $$

Firstly, we need to calculate the tracefree Einstein (Eq. 218) and Energy-Momentum (Eq. 220) tensors and then applying on them the tensor mode operator Eq. 217. Then we can solve Eq. 217 for both tensors and rewrite the solution of the Einstein Field Equation according to the results obtained.

The notation in the last equality of Eq. 218 is needed to simplify the results after the application of the operator defined in Eq. 217, because: ${}^{(1)(t)}\bar{G}^{\alpha}_{\beta}$ corresponds to the differential form of the gravitational wave equation (LHS), ${}^{(2)(t)}\bar{G}^{\alpha}_{\beta} = 0$ as it is composed only of single perturbation terms and ${}^{(3)(t)}\bar{G}^{\alpha}_{\beta}$ is composed on quadratic terms which will summed with the Energy-Momentum tensor ${}^{(t)}\bar{T}^{\alpha}_{\beta}$ (Eq. 221) in the RHS of the gravitational wave equation.

$$
\begin{aligned}
a^2\bar{G}^{\alpha}_{\beta} = & \Big\{\Big[\left(C^{\alpha}_{\beta}\right)'' + 2\mathcal{H}\left(C^{\alpha}_{\beta}\right)' - \Delta C^{\alpha}_{\beta}\Big] - \frac{1}{3}\delta^{\alpha}_{\beta}\Big[\left(C^{\gamma}_{\gamma}\right)'' + 2\mathcal{H}\left(C^{\gamma}_{\gamma}\right)' - \Delta C^{\gamma}_{\gamma}\Big]\Big\} \\
& + \Big\{\Big(\mathcal{H}B^{|\alpha}_{\beta} + \frac{1}{2}\left(B^{|\alpha}_{\beta}\right)' + \mathcal{H}B^{\alpha}_{|\beta} + \frac{1}{2}\left(B^{\alpha}_{|\beta}\right)' - A^{|\alpha}_{|\beta} - C^{\gamma|\alpha}_{\gamma|\beta} + C^{\gamma|\alpha}_{\beta|\gamma} + C^{\alpha\gamma}_{|\beta\gamma}\Big) - \frac{1}{3}\delta^{\alpha}_{\beta}\Big(2\mathcal{H}B^{\gamma}_{|\gamma} \\
& + \left(B^{\gamma}_{|\gamma}\right)' - \Delta A + 2C^{\gamma\delta}_{|\gamma\delta} - \Delta C^{\gamma}_{\gamma}\Big)\Big\} \\
& + \Big\{\Big[-A'\left(C^{\alpha}_{\beta}\right)' - 2A\left(C^{\alpha}_{\beta}\right)'' - 4\mathcal{H}A\left(C^{\alpha}_{\beta}\right)' - \frac{1}{2}A'B^{\alpha}{}_{|\beta} - \frac{1}{2}A'B^{|\alpha}{}_{\beta} - 2\mathcal{H}AB^{\alpha}{}_{|\beta} - 2\mathcal{H}AB^{|\alpha}{}_{\beta} \\
& - A\left(B^{\alpha}{}_{|\beta}\right)' - A\left(B^{|\alpha}{}_{\beta}\right)' + A^{|\alpha}A_{|\beta} + 2AA^{|\alpha}_{|\beta} + C^{\alpha}_{\gamma}{}_{|\beta}A^{|\gamma} + C^{|\alpha}_{\beta\gamma}A^{|\gamma} - C^{\alpha}_{\beta|\gamma}A^{|\gamma} + 2C^{\alpha}_{\gamma}A^{|\gamma}_{|\beta}\Big] \\
& + \Big[-2\mathcal{H}B^{\alpha}A_{|\beta} - B^{\alpha}\left(C^{\gamma}_{\beta|\gamma}\right)' + B^{\alpha}\left(C^{\gamma}_{\gamma|\beta}\right)' + \frac{1}{2}B^{\alpha}B^{\gamma}_{|\gamma\beta} - \frac{1}{2}B^{\alpha}\Delta B_{\beta} - B^{\gamma}B^{|\alpha}_{\gamma|\beta} - B^{\gamma}\left(C^{|\alpha}_{\gamma\beta}\right)' \\
& + \left(C^{\gamma}_{\gamma}\right)' B^{\alpha}{}_{|\beta} + \left(C^{\gamma}_{\gamma}\right)' B^{|\alpha}{}_{\beta} - \left(B^{\gamma}\right)' C^{\alpha}_{\gamma|\beta} - \left(B^{\gamma}\right)' C^{|\alpha}_{\gamma\beta} - \frac{1}{2}\left(B^{\alpha|\gamma}B_{\beta|\gamma} + B^{\gamma|\alpha}B_{\gamma|\beta}\right) \\
& - 2\mathcal{H}B^{\gamma}C^{\alpha}_{\gamma|\beta} - 2\mathcal{H}B^{\gamma}C^{|\alpha}_{\gamma\beta} - 2\mathcal{H}C^{\alpha}_{\gamma}B^{\gamma}{}_{|\beta} - 2\mathcal{H}C^{\alpha}_{\gamma}B^{|\gamma}{}_{\beta} - C^{\alpha}_{\gamma}\left(B^{\gamma}{}_{|\beta}\right)' - C^{\alpha}_{\gamma}\left(B^{|\gamma}{}_{\beta}\right)' \\
& + \frac{1}{2}B^{\alpha}{}_{|\beta}B^{\gamma}{}_{|\gamma} + \frac{1}{2}B^{|\alpha}{}_{\beta}B^{\gamma}{}_{|\gamma} + 2B^{\gamma}\left(C^{\alpha}_{\beta|\gamma}\right)' + \frac{1}{2}B^{\gamma}B^{\alpha}{}_{|\beta\gamma} + \frac{1}{2}B^{\gamma}B^{|\alpha}_{\beta|\gamma} - B^{\gamma}\left(C^{\alpha}_{\gamma|\beta}\right)' \\
& + \left(C^{\alpha}_{\beta}\right)' B^{\gamma}_{|\gamma} + \left(B^{\gamma}\right)' C^{\alpha}_{\beta|\gamma} + 2\mathcal{H}B^{\gamma}C^{\alpha}_{\beta|\gamma} - \left(C_{\beta\gamma}\right)' B^{\alpha|\gamma} - \left(C^{\alpha}_{\gamma}\right)' B^{|\gamma}_{\beta}\Big] + \Big[-2\left(C^{\alpha\gamma}\right)'\left(C_{\beta\gamma}\right)' \\
& + \left(C^{\alpha}_{\beta}\right)'\left(C^{\gamma}_{\gamma}\right)' - 2C^{\alpha\gamma}\left(C_{\beta\gamma}\right)'' - 4\mathcal{H}C^{\alpha\gamma}\left(C_{\beta\gamma}\right)' + C^{\gamma\delta|\alpha}C_{\gamma\delta|\beta} + 2C^{\gamma\delta}C^{|\alpha}_{\gamma\delta|\beta} + C^{\gamma\alpha}{}_{|\beta}C^{\delta}_{\delta|\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&+C^{\gamma|\alpha}_{\beta}C^{\delta}_{\delta|\gamma}+2C^{\alpha\gamma}C^{\delta}_{\delta|\beta\gamma}-C^{\delta}_{\delta|\gamma}C^{\alpha|\gamma}_{\beta}-2C^{\alpha\gamma}_{|\beta}C^{\delta}_{\gamma|\delta}+2C^{\alpha|\gamma}_{\beta}C^{\delta}_{\gamma|\delta}-2C^{\gamma\delta}C^{\alpha}_{\gamma|\beta\delta}-2C^{\gamma\delta}C^{|\alpha}_{\gamma\beta|\delta}\\
&-4C^{\alpha\gamma}C^{\delta}_{(\beta|\gamma)\delta}-2C_{\beta\delta|\gamma}C^{\alpha\gamma|\delta}+2C^{\gamma\delta}C^{\alpha}_{\beta|\gamma\delta}+2C^{\alpha\gamma}\Delta C_{\beta\gamma}-2C_{\beta\delta|\gamma}C^{\alpha\gamma|\delta}+2C_{\beta\gamma|\delta}C^{\alpha\gamma|\delta}\Big]\\
&-\frac{1}{3}\delta^{\alpha}_{\beta}\left(\left[-A'\left(C^{\gamma}_{\gamma}\right)'-2A\left(C^{\gamma}_{\gamma}\right)''-4\mathcal{H}A\left(C^{\gamma}_{\gamma}\right)'-2\mathcal{H}B^{\gamma}A_{|\gamma}-A'B^{\gamma}_{|\gamma}-4\mathcal{H}AB^{\gamma}_{|\gamma}\right.\right.\\
&\left.-2A\left(B^{\gamma}_{|\gamma}\right)'+2A\Delta A+A_{|\gamma}A^{|\gamma}-C^{\delta}_{\delta|\gamma}A^{|\gamma}+2C_{\gamma\delta}A^{|\gamma\delta}+2A^{|\gamma}C^{\delta}_{\gamma|\delta}\right]+\left[2\left(C^{\delta}_{\delta}\right)'B^{\gamma}_{|\gamma}+\left(B^{\gamma}\right)'C^{\delta}_{\delta|\gamma}\right.\\
&+2\mathcal{H}B^{\gamma}C^{\delta}_{\delta|\gamma}+3B^{\gamma}\left(C^{\delta}_{\delta|\gamma}\right)'+B^{\gamma}_{|\gamma}B^{\delta}_{|\delta}-2\left(B^{\gamma}\right)'C^{\delta}_{\gamma|\delta}-4\mathcal{H}B^{\gamma}C^{\delta}_{\gamma|\delta}-3B^{\gamma}\left(C^{\delta}_{\gamma|\delta}\right)'\\
&\left.+\frac{3}{2}B^{\gamma}B^{\delta}_{|\gamma\delta}-\frac{3}{2}B^{\gamma}\Delta B_{\gamma}-2\left(C_{\gamma\delta}\right)'B^{\gamma|\delta}-4\mathcal{H}C_{\gamma\delta}B^{\gamma|\delta}-B_{\gamma|\delta}B^{\gamma|\delta}-2C_{\gamma\delta}\left(B^{\gamma|\delta}\right)'\right]\\
&+\left[-2\left(C_{\gamma\delta}\right)'\left(C^{\gamma\delta}\right)'+\left(C^{\gamma}_{\gamma}\right)'\left(C^{\delta}_{\delta}\right)'-2C^{\gamma\delta}\left(C_{\gamma\delta}\right)''-4\mathcal{H}C^{\gamma\delta}\left(C_{\gamma\delta}\right)'+4C^{\gamma\delta}C^{\varepsilon}_{\varepsilon|\gamma\delta}-C^{\varepsilon}_{\varepsilon|\delta}C^{\gamma|\delta}_{\gamma}\right.\\
&\left.\left.-4C^{\gamma\delta}_{|\gamma}C^{\varepsilon}_{\delta|\varepsilon}+4C^{\gamma|\delta}_{\gamma}C^{\varepsilon}_{\delta|\varepsilon}-8C^{\gamma\delta}C^{\varepsilon}_{\gamma|\delta\varepsilon}+4C^{\gamma\delta}\Delta C_{\gamma\delta}-2C_{\gamma\varepsilon|\delta}C^{\gamma\delta|\varepsilon}+3C_{\gamma\delta|\varepsilon}C^{\gamma\delta|\varepsilon}\right]\right)\Big\}\\
:=&\left\{{}^{(a)}\bar{G}^{\alpha}_{\beta}\right\}+\left\{{}^{(b)}\bar{G}^{\alpha}_{\beta}\right\}+\left\{{}^{(c)}\bar{G}^{\alpha}_{\beta}\right\},
\end{aligned}
\tag{218}
$$

where for simplicity, we consider ${}^{(c)}\bar{G}^{\alpha}_{\beta}:=({}^{(a)}P^{\alpha}_{\beta})-\frac{1}{3}\delta^{\alpha}_{\beta}({}^{(b)}P^{\gamma}_{\gamma})$. Then

$$
\begin{aligned}
{}^{(a)(t)}\bar{G}^{\alpha}_{\beta}=&\frac{1}{a^2(\eta)}\left\{\left(\frac{\partial^2}{\partial t^2}+2\mathcal{H}\frac{\partial}{\partial t}-\Delta\right){}^{(t)}C^{\alpha}_{\beta}\right\},\\
{}^{(b)(t)}\bar{G}^{\alpha}_{\beta}=&0,\\
{}^{(c)(t)}\bar{G}^{\alpha}_{\beta}=&\frac{1}{a^2(\eta)}\left\{({}^{(a)}P^{\alpha}_{\beta})+\frac{1}{2}\delta^{\alpha}_{\beta}({}^{(b)}P^{\varepsilon}_{\varepsilon})+\Delta^{-1}\left[\frac{1}{2}({}^{(b)}P^{\varepsilon|\alpha}_{\varepsilon|\beta})+\frac{1}{2}\delta^{\alpha}_{\beta}({}^{(a)}P^{|\gamma\delta}_{\gamma\delta})-({}^{(a)}P^{|\alpha\gamma}_{\beta\gamma})\right.\right.\\
&\left.\left.-({}^{(a)}P^{\alpha|\gamma}_{\gamma|\beta})\right]+\frac{1}{2}\Delta^{-2}({}^{(a)}P^{|\gamma\delta\alpha}_{\gamma\delta|\beta})\right\},
\end{aligned}
\tag{219}
$$

$$
\begin{aligned}
\bar{T}^{\alpha}_{\beta}=&\Big\{\Pi^{\alpha}_{\beta}-2C^{\alpha\varepsilon}\Pi_{\beta\varepsilon}+(\bar{\rho}+\bar{p})U^{\alpha}\left(U_{\beta}-B_{\beta}\right)+Q_{\beta}U^{\alpha}+Q^{\alpha}\left(U_{\beta}-B_{\beta}\right)\\
&-\frac{1}{3}\delta^{\alpha}_{\beta}\left[(\bar{\rho}+\bar{p})U^{\varepsilon}\left(U_{\varepsilon}-B_{\varepsilon}\right)+Q_{\varepsilon}U^{\varepsilon}+Q^{\varepsilon}\left(U_{\varepsilon}-B_{\varepsilon}\right)\right]\Big\},
\end{aligned}
\tag{220}
$$

$$
\begin{aligned}
{}^{(t)}T^{\alpha}_{\beta}=&\Big\{\Pi^{\alpha}_{\beta}-2C^{\alpha\varepsilon}\Pi_{\beta\varepsilon}+(\bar{\rho}+\bar{p})U^{\alpha}\left(U_{\beta}-B_{\beta}\right)+Q_{\beta}U^{\alpha}+Q^{\alpha}\left(U_{\beta}-B_{\beta}\right)-\frac{1}{2}\delta^{\alpha}_{\beta}\left[(\bar{\rho}+\bar{p})U^{\varepsilon}\left(U_{\varepsilon}-B_{\varepsilon}\right)\right.\\
&\left.+Q_{\varepsilon}U^{\varepsilon}+Q^{\varepsilon}\left(U_{\varepsilon}-B_{\varepsilon}\right)\right]\Big\}+\Delta^{-1}\left\{\frac{1}{2}\left[(\bar{\rho}+\bar{p})U^{\varepsilon}\left(U_{\varepsilon}-B_{\varepsilon}\right)+Q_{\varepsilon}U^{\varepsilon}+Q^{\varepsilon}\left(U_{\varepsilon}-B_{\varepsilon}\right)\right]^{|\alpha}_{|\beta}\right.\\
&+\frac{1}{2}\delta^{\alpha}_{\beta}\left[\left(\Pi_{\gamma\delta}-2C^{\varepsilon}_{\gamma}\Pi_{\delta\varepsilon}\right)^{|\gamma\delta}+\left((\bar{\rho}+\bar{p})U_{\gamma}\left(U_{\delta}-B_{\delta}\right)+Q_{\delta}U_{\gamma}+Q_{\gamma}\left(U_{\delta}-B_{\delta}\right)\right)^{|\gamma\delta}\right]
\end{aligned}
$$

$$-\left(\left(\Pi_{\beta\gamma}-2C^{\varepsilon}_{\beta}\Pi_{\gamma\varepsilon}\right)^{|\gamma\alpha}+\left(\Pi^{\alpha}_{\gamma}-2C^{\alpha\varepsilon}\Pi_{\gamma\varepsilon}\right)^{|\gamma}_{|\beta}+[(\bar{\rho}+\bar{\rho})U_{\beta}\left(U_{\gamma}-B_{\gamma}\right)+Q_{\gamma}U_{\beta}$$

$$+Q_{\beta}\left(U_{\gamma}-B_{\gamma}\right)]^{|\gamma\alpha}+[(\bar{\rho}+\bar{\rho})U^{\alpha}\left(U_{\gamma}-B_{\gamma}\right)+Q_{\gamma}U^{\alpha}+Q^{\alpha}\left(U_{\gamma}-B_{\gamma}\right)]^{|\gamma}_{|\beta}\right)\Big\}$$

$$+\frac{1}{2}\Delta^{-2}\left\{\left(\Pi_{\gamma\delta}-2C^{\varepsilon}_{\gamma}\Pi_{\delta\varepsilon}\right)^{|\gamma\delta\alpha}_{|\beta}+[(\bar{\rho}+\bar{p})U_{\gamma}\left(U_{\delta}-B_{\delta}\right)+Q_{\delta}U_{\gamma}+Q_{\gamma}\left(U_{\delta}-B_{\delta}\right)]^{|\gamma\delta\alpha}_{|\beta}\right\}. \tag{221}$$

Rewriting the Einstein's Field Equations 205 according to the results Eq. 219 and Eq. 221, we obtain the Gravitational Wave Equation 222:

$$\left(\frac{\partial^2}{\partial\eta^2}+2\mathcal{H}\frac{\partial}{\partial\eta}-\Delta\right){}^{(t)}C^{\alpha}_{\beta}={}^{(t)}S^{\alpha}_{\beta}, \tag{222}$$

where the tensor ${}^{(t)}S^{\alpha}_{\beta}$ is the quadratic source term of gravitational waves and is defined as:

$${}^{(t)}S^{\alpha}_{\beta}\equiv 8\pi a^2(\eta)G^{(t)}T^{\alpha}_{\beta}-a^2(\eta)^{(c)(t)}\bar{G}^{\alpha}_{\beta}. \tag{223}$$

We consider only the scalar perturbations as sources of tensor modes, as in [132]. From now on, we replace all the covariant derivative with the partial derivative (denoted with a comma), since we are working on a flat space.

9 Fourier Analysis

According to *Maggiore* [55], to solve the wave equation 222, it is convenient to work in the momentum space. Therefore, the Fourier counterparts of ${}^{(t)}C_{\alpha\beta} := h_{\alpha\beta}$ and $S_{\alpha\beta}$ (where for clarity we omit the index ${}^{(t)}$) are:

$$\begin{aligned} h_{\alpha\beta}\left(\eta, \mathbf{x}\right) &= \int \frac{d^3\mathbf{k}}{\left(2\pi\right)^3} e^{i\mathbf{kx}} h_{\alpha\beta}\left(\eta, \mathbf{k}\right), \\ S_{\alpha\beta}\left(\eta, \mathbf{x}\right) &= \int \frac{d^3\mathbf{k}}{\left(2\pi\right)^3} e^{i\mathbf{kx}} S_{\alpha\beta}\left(\eta, \mathbf{k}\right). \end{aligned} \tag{224}$$

One can expand $h_{\alpha\beta}\left(\eta, \mathbf{k}\right)$ and $S_{\alpha\beta}\left(\eta, \mathbf{k}\right)$ in the basis of the polarization tensors $e^P_{\alpha\beta}$, with $P = \{+, \times\}$

$$\begin{aligned} h_{\alpha\beta}\left(\eta, \mathbf{k}\right) &= \sum_{P=+,\times} e^P_{\alpha\beta}(\hat{\mathbf{k}}) h_P\left(\eta, \mathbf{k}\right), \\ S_{\alpha\beta}\left(\eta, \mathbf{k}\right) &= \sum_{P=+,\times} e^P_{\alpha\beta}(\hat{\mathbf{k}}) S_P\left(\eta, \mathbf{k}\right). \end{aligned} \tag{225}$$

We omit the polarization symbol P, as the polarization modes must have the same statistical properties in a parity-preserving universe [132]. Therefore we consider only a polarization mode. In the Fourier space and using the identities in 225, the wave equation in Eq. 222 becomes

$$\left(\frac{\partial^2}{\partial\eta^2} + 2\mathcal{H}\frac{\partial}{\partial\eta} + k^2\right) h\left(\eta, \mathbf{k}\right) = S\left(\eta, \mathbf{k}\right). \tag{226}$$

Introducing a field v such that $h\left(\eta, \mathbf{k}\right) = \frac{1}{a(\eta)} v\left(\eta, \mathbf{k}\right)$, then Eq. 226 becomes

$$v''\left(\eta, \mathbf{k}\right) + \left(k^2 - \frac{a''(\eta)}{a(\eta)}\right) v\left(\eta, \mathbf{k}\right) = a(\eta) S\left(\eta, \mathbf{k}\right) \tag{227}$$

where the prime indicates a time derivative based on the conformal time η. The solution of Eq. 227 is different in the RDE and MDE, as in the first one $a(\eta) \propto \eta$ and in the other one $a(\eta) \propto \eta^2$. Therefore, the homogeneous solutions in the two eras differ, but the particular solutions have the same form. For this reason, we show the particular solution, and then at the end, we give the results for the two eras.

The particular solution can be found by Green's function method. Let us fix the wave number $\mathbf{k}$ and find the solutions of Eq. 227. Consequently, let us define $v_1\left(\eta, \mathbf{k}\right) \equiv v_1(\eta)$ and $v_2\left(\eta, \mathbf{k}\right) \equiv v_2(\eta)$ be solutions of the homogeneous differential equation 227. Then the Green's function $G\left(\eta, \tilde{\eta}\right)$ is

$$G\left(\eta, \tilde{\eta}\right) = \frac{v_1(\tilde{\eta}) v_2(\eta) - v_1(\eta) v_2(\tilde{\eta})}{W(\tilde{\eta})}, \tag{228}$$

where $W(\tilde{\eta})$ is the Wronskian, which is defined as

$$W(\tilde{\eta}) := v_1'(\tilde{\eta}) v_2(\tilde{\eta}) - v_1(\tilde{\eta}) v_2'(\tilde{\eta}). \tag{229}$$

Then the general solution of Eq. 227 is

$$v\left(\eta, \mathbf{k}\right) = v_{hom}\left(\eta, \mathbf{k}\right) + \int_0^\eta d\tilde{\eta}\;\; G(\eta, \tilde{\eta}, \mathbf{k}) a(\tilde{\eta}) S(\tilde{\eta}, \mathbf{k}), \tag{230}$$

where the homogeneous solution is a linear combination of $v_1(\eta, \mathbf{k})$ and $v_2(\eta, \mathbf{k})$. Then, using the definition of $v(\eta, \mathbf{k})$, we find the general solution of Eq. 226

$$h(\eta, \mathbf{k}) = \frac{1}{a(\eta)} \left[v_{hom}(\eta, \mathbf{k}) + \int_0^\eta d\tilde{\eta} \;\; G(\eta, \tilde{\eta}, \mathbf{k}) a(\tilde{\eta}) S(\tilde{\eta}, \mathbf{k}) \right] \tag{231}$$

In **radiation dominated era**, with $a(\eta) \propto \eta$, the homogeneous Eq. 227 reduces to

$$v''(\eta, \mathbf{k}) + k^2 v(\eta, \mathbf{k}) = 0. \tag{232}$$

Using the Ansatz of plane waves, the two solutions are

$$v_1(\eta, \mathbf{k}) = c_1 \sin(k\eta), \qquad v_2(\eta, \mathbf{k}) = c_2 \cos(k\eta), \tag{233}$$

where $k = |\mathbf{k}|$, and the corresponding Green's function is

$$G(\eta, \tilde{\eta}, \mathbf{k}) = \sin[k(\tilde{\eta} - \eta)]. \tag{234}$$

The solution of Eq. 231, for a given k-mode in radiation dominated era, becomes

$$h_{\mathbf{k}}(\eta) = \frac{1}{a(\eta)} \left[c_1 \sin(k\eta) + c_2 \cos(k\eta) + \int_0^\eta d\tilde{\eta} \;\; \sin[k(\tilde{\eta} - \eta)] a(\tilde{\eta}) S(\tilde{\eta}, \mathbf{k}) \right]. \tag{235}$$

In **matter dominated era**, where $a(\eta) \propto \eta^2$, the homogeneous Eq. 227 reduces to

$$\begin{aligned} v''(\eta, \mathbf{k}) + \left(k^2 - \frac{2}{\eta^2}\right) v(\eta, \mathbf{k}) &= 0 \\ x^2 \frac{d^2}{dx^2} v(x) + \left(x^2 - 2\right) v(x) &= 0 \end{aligned} \tag{236}$$

where $x = k\eta$. The second equality in Eq. 236 can be solved with the Frobenius method, where the Ansatz is a power series. The two solutions are then expressed in terms of the spherical Bessel functions

$$\begin{aligned} v_1(x) &= \sqrt{\frac{2}{\pi}} x j_1(x) = \sqrt{\frac{2}{\pi}} x \left(\frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \right), \\ v_2(x) &= \sqrt{\frac{2}{\pi}} x y_1(x) = -\sqrt{\frac{2}{\pi}} x \left(\frac{\cos(x)}{x^2} + \frac{\sin(x)}{x} \right). \end{aligned} \tag{237}$$

Therefore, the corresponding Green's function is

$$G(\eta, \tilde{\eta}, \mathbf{k}) = k\eta\tilde{\eta} \left[j_1(k\tilde{\eta}) y_1(k\eta) - j_1(k\eta) y_1(k\tilde{\eta}) \right], \tag{238}$$

and the solution of Eq. 231, for a given k-mode in matter-dominated era becomes

$$h_{\mathbf{k}}(\eta) = \frac{k\eta}{a(\eta)} \left[c_1 j_1(k\eta) + c_2 y_1(k\eta) + \int_0^\eta d\tilde{\eta} \;\; \tilde{\eta} \left[j_1(k\tilde{\eta}) y_1(k\eta) - j_1(k\eta) y_1(k\tilde{\eta}) \right] a(\tilde{\eta}) S(\tilde{\eta}, \mathbf{k}) \right]. \tag{239}$$

10 Gauge transformation

Recalling the decomposition in scalar, vector, and tensor, Eq. 77, we get the following gauge transformation of the perturbative quantities of the metric and cosmological fluid quantities. We start defining the vector field ξ^μ as follows:

$$\xi^\mu := (T, \mathcal{L}^\alpha) \,. \tag{240}$$

Then, the metric perturbation quantities are:

$$\begin{aligned}\widetilde{A} =& A - (T' + \mathcal{H}T) - A'T - 2A\,(T' + \mathcal{H}T) + \frac{3}{2}T'^2 + TT'' + \frac{1}{2}\left(2\mathcal{H}^2 + \mathcal{H}'\right)T^2 \\ & + 3\mathcal{H}T'T - A_{,\alpha}\mathcal{L}^\alpha - B_\alpha\mathcal{L}'^\alpha + T_{,\alpha}\mathcal{L}'^\alpha + \mathcal{L}^\alpha\left(T'_{,\alpha} + \mathcal{H}T_{,\alpha}\right) - \frac{1}{2}\mathcal{L}'^\alpha\mathcal{L}'_\alpha,\end{aligned} \tag{241}$$

$$\begin{aligned}\widetilde{B}_\alpha =& B_\alpha - T_{,\alpha} + \mathcal{L}'_\alpha - 2AT_{,\alpha} - \left(B'_\alpha + 2\mathcal{H}B_\alpha\right)T - B_\alpha T' + 2T'T_{,\alpha} + T\left(T'_{,\alpha} + 2\mathcal{H}T_{,\alpha}\right) \\ & - B_{\alpha,\beta}\mathcal{L}^\beta - B_\beta\mathcal{L}^\beta{}_{,\alpha} + 2C_{\alpha\beta}\mathcal{L}'^\beta - \mathcal{L}'_\alpha T' + T_{,\beta}\mathcal{L}^\beta_{,\alpha} + \mathcal{L}^\beta T_{,\alpha\beta} - T\left(\mathcal{L}''_\alpha + 2\mathcal{H}\mathcal{L}'_\alpha\right) \\ & - 2\mathcal{L}_{(\alpha,\beta)}\mathcal{L}^\beta - \mathcal{L}^\beta\mathcal{L}'_{\alpha,\beta},\end{aligned} \tag{242}$$

$$\begin{aligned}\widetilde{C}_{\alpha\beta} =& C_{\alpha\beta} - \mathcal{H}T\delta_{\alpha\beta} - \mathcal{L}_{(\alpha,\beta)} + B_{(\alpha}T_{,\beta)} - \left(C'_{\alpha\beta} + 2\mathcal{H}C_{\alpha\beta}\right)T + \frac{1}{2}\left(2\mathcal{H}^2 + \mathcal{H}'\right)T^2\delta_{\alpha\beta} - \frac{1}{2}T_{,\alpha}T_{,\beta} \\ & + \mathcal{H}TT'\delta_{\alpha\beta} - C_{\alpha\beta,\gamma}\mathcal{L}^\gamma - 2C_{\gamma(\alpha}\mathcal{L}^\gamma_{,\beta)} + \mathcal{L}'_{(\alpha}T_{,\beta)} + \mathcal{H}\mathcal{L}^\gamma T_{,\gamma}\delta_{\alpha\beta} + 2\mathcal{H}\mathcal{L}_{(\alpha,\beta)}T + \mathcal{L}'_{(\alpha,\beta)}T \\ & + \mathcal{L}^\gamma{}_{,(\alpha}\mathcal{L}_{\beta),\gamma} + \mathcal{L}^\gamma\mathcal{L}_{(\alpha,\beta)\gamma} + \frac{1}{2}\mathcal{L}^\gamma{}_{,\alpha}\mathcal{L}_{\gamma,\beta}.\end{aligned} \tag{243}$$

To get the cosmological fluid quantities, we use Eq. 77 and the definition of the quantities. Consequently, we find:

$$\begin{aligned}\tilde{u}^0 =& \frac{1}{a}\Big[1 - A\mathcal{H}T + T' + \frac{3}{2}A^2 - B^\alpha U_\alpha + \frac{1}{2}U^\alpha U_\alpha + A'T - \mathcal{H}AT + U^\alpha T_{,\alpha} \\ & + \frac{1}{2}\left(\mathcal{H}^2 - \mathcal{H}'\right)T^2 - AT' - TT'' + \mathcal{L}^\alpha A_{,\alpha} - \mathcal{H}\mathcal{L}^\alpha T_{,\alpha} - \mathcal{L}^\alpha T'_{,\alpha}\Big], \\ \tilde{u}^\alpha =& \frac{1}{a}\Big[U^\alpha + (\mathcal{L}^\alpha)' - A\,(\mathcal{L}^\alpha)' + \mathcal{H}\,(\mathcal{L}^\alpha)'\,T - (\mathcal{L}^\alpha)''\,T + \mathcal{H}U^\alpha T - (u^\alpha)'\,T + U^\beta\mathcal{L}^\alpha_{,\beta} \\ & - \mathcal{L}^\beta\,(\mathcal{L}^\alpha)'_{,\beta} - \mathcal{L}^\beta U^\alpha_{,\beta}\Big],\end{aligned} \tag{244}$$

$$\widetilde{\delta\rho} = \delta\rho - (\bar{\rho} + \delta\rho)'\,T - \delta\rho_{,\alpha}\mathcal{L}^\alpha + \bar{\rho}'\,(TT' + T_{,\alpha}\mathcal{L}^\alpha) + \frac{1}{2}\bar{\rho}''TT, \tag{245}$$

$$\widetilde{\delta p} = \delta p - (\bar{p} + \delta p)'\,T - \delta p_{,\alpha}\mathcal{L}^\alpha + \bar{p}'\,(TT' + T_{,\alpha}\mathcal{L}^\alpha) + \frac{1}{2}\bar{p}''TT, \tag{246}$$

$$\widetilde{Q}_\alpha = Q_\alpha - \left(\mathcal{H}Q_\alpha + Q'{}_\alpha\right)T - \mathcal{L}^\beta Q_{\alpha,\beta} - Q^\beta\mathcal{L}_{\beta,\alpha}, \tag{247}$$

$$\widetilde{\Pi}_{\alpha\beta} = \Pi_{\alpha\beta} - \left(\mathcal{H}\Pi_{\alpha\beta} + \Pi'_{\alpha\beta}\right) T - 2\mathcal{L}^{\gamma}_{,(\alpha}\Pi_{\beta)\gamma} - \mathcal{L}^{\gamma}\Pi_{\alpha\beta,\gamma} + \frac{2}{3}C^{\gamma\delta}\Pi_{\gamma\delta}\delta_{\alpha\beta}. \tag{248}$$

To decompose the gauge-transformed metric perturbations into scalars, vectors, and tensors; we need to separate $\mathcal{L}^{\alpha}$ in scalar and vector part

$$\mathcal{L}^{\alpha} := L^{,\alpha} + L^{\alpha} \tag{249}$$

So, perturbations defined in Eq. 77 gauge transform as:

$$\widetilde{\alpha} = \alpha - \frac{1}{a}(aT)' + A_{\xi}, \tag{250}$$

$$\widetilde{\beta} = \beta - T + L' + \Delta^{-1}\nabla^{\alpha}B_{\xi\alpha}, \tag{251}$$

$$\widetilde{\gamma} = \gamma - L + \frac{1}{2}\Delta^{-1}\left(3\Delta^{-1}\nabla^{\alpha}\nabla^{\beta}C_{\xi\alpha\beta} - C^{\alpha}_{\xi\alpha}\right), \tag{252}$$

$$\widetilde{\varphi} = \varphi + \mathcal{H}T + \left[\frac{1}{3}C^{\alpha}_{\xi\alpha} - \frac{1}{6}\Delta^{-1}\left(3\Delta^{-1}\nabla^{\alpha}\nabla^{\beta}C_{\xi\alpha\beta} - C^{\alpha}_{\xi\alpha}\right)\right], \tag{253}$$

$$\widetilde{B}^{(v)}_{\alpha} = \left(B^{(n)}_{\alpha} + L^{(v)}_{\alpha}\right) + \left(B_{\xi\alpha} - \nabla_{\alpha}\Delta^{-1}\nabla^{\beta}B_{\xi\beta}\right), \tag{254}$$

$$\widetilde{C}^{(v)}_{\alpha} = C^{(v)}_{\alpha} - \mathcal{L}^{(v)}_{\alpha} + \left[2\Delta^{-1}\left(\nabla^{\beta}C_{\{\alpha\beta} - \nabla_{\alpha}\Delta^{-1}\nabla^{\beta}\nabla^{\gamma}C_{\xi\beta\gamma}\right)\right], \tag{255}$$

$$\widetilde{C}^{(t)}_{\alpha\beta} = \bar{C}^{(t)}_{\alpha\beta} + \left[\bar{C}_{\xi\alpha\beta} + \frac{1}{2}\Delta^{-1}\left(\nabla_{\alpha}\nabla_{\beta} + \delta_{\alpha\beta}\Delta\right)\bar{C}^{|\gamma\delta}_{\xi\gamma\delta} + \Delta^{-1}\left(2\nabla^{(\alpha}\bar{C}^{|\gamma}_{\xi\beta)\gamma}\right)\right], \tag{256}$$

$$\tilde{U}^{0} = \frac{1}{a}\left[1 - A\mathcal{H}T + T' + U^{0}_{\xi}\right], \tag{257}$$

$$\tilde{U} = \frac{1}{a}\left[U + L' + U_{\xi}\right], \tag{258}$$

$$\tilde{U}^{(v)\alpha} = \frac{1}{a}\left[U^{(v)\alpha} + (L^{\alpha})' + U^{\alpha}_{\xi}\right], \tag{259}$$

which are expressed as in *Noh, and Hwang* [18], i.e. A_{ξ} represent the quadratic parts of Eq. 241 and similarly for all other variables. This representation allows us to simplify the calculation. Looking back at Eq. 222, we note that: the source term is a polynomial composed of second-order monomials. These are a combination of two quantities, except for the anisotropic stress. As we are interested in the second-order contribution, the individual perturbations must be of the first order. For this reason, the representation is helpful. On the other hand, the LHS of the equation is composed of a single term and the anisotropic stress in the source term, which is why it is also necessary to have a second-order description. We recall that a barred $\bar{C}$ is the tracefree tensor C.

One can express the gauge transformation of Eq. 222 as follows:

$$\left(\frac{\partial^2}{\partial\eta^2} + 2\mathcal{H}\frac{\partial}{\partial\eta} - \Delta\right){}^{(t)}\widetilde{C}_{\alpha\beta} = \widetilde{S}_{\alpha\beta}, \tag{260}$$

where $\widetilde{S}_{\alpha\beta} = 8\pi a^2(\eta)G^{(t)}\widetilde{T}_{\alpha\beta} - a^2(\eta)^{(c)(t)}\widetilde{\widetilde{G}}_{\alpha\beta}$.

10.1 Conformal Newtonian Gauge

Previously we have calculated the general solution of the gravitational wave equation Eq. 231, where Eq. 235 and Eq. 239 are the solutions in the RDE and MDE, respectively. Both solutions depend on the source term $S(\eta, \mathbf{k})$. We aim to fix the gauge condition according to the Conformal Newtonian gauge (cf. Sec. 4.3.2) to fix the redundant degrees of freedom.

Let us recall the Conformal Newtonian gauge conditions, i.e. spatial conditions $\beta = 0 = \gamma$ and temporal conditions, $\chi = a(\beta + \gamma') = 0$. These conditions completely fix $\xi^\mu = (0, 0)$. As previously explained, we define $\psi := \alpha$ and $\phi := \varphi$. Important to remember that we are only considering scalars as sources of tensor modes. It means that the source term consists only of scalar perturbations. Therefore, the source term in Eq. 260 becomes:

$$
\begin{aligned}
S_{\alpha\beta} =& - \Big[2\psi\psi_{,\alpha\beta} + \psi_{,\alpha}\psi_{,\beta} + 2\psi_{,(\alpha}\phi_{,\beta)} + 2\phi\psi_{,\alpha\beta} + 4\phi\phi_{,\alpha\beta} + 3\phi_{,\alpha}\phi_{,\beta} - \frac{1}{3}\delta_{\alpha\beta}(2\psi\Delta\psi + \psi_{,\gamma}\psi^{,\gamma} \\
&+ 2\psi_{,\gamma}\phi^{,\gamma} + 2\phi\Delta\psi + 4\phi\Delta\phi + 3\phi_{,\gamma}\phi^{,\gamma})\Big] + 8\pi a^2(\eta)G\Big[\Pi_{,\alpha\beta} - 2\phi\Pi_{,\alpha\beta} + 3(\bar{\rho} + \bar{p})U_{,\alpha}U_{,\beta} \\
&- \frac{1}{3}\delta_{\alpha\beta}\left(\Pi^{,\gamma}_{,\gamma} - 2\phi\Pi^{,\gamma}_{,\gamma} + 3(\bar{\rho} + \bar{p})U_{,\gamma}U^{,\gamma}\right)\Big],
\end{aligned}
\tag{261}
$$

where we have used the definition of [18] Eq. 175, i.e. $Q_{,\alpha} \equiv -(\bar{\rho} + \bar{p})U_{,\alpha}$ and $U_\alpha = -U_{,\alpha}$. Then, using the following background and first-order relations (cf. Sec. 3.2 for the background evolution, and Appendix A for the generic evolution equations at the linear order)

$$\kappa = \frac{3}{a}(\mathcal{H}\psi - \phi'), \tag{262}$$

$$\kappa = 12\pi G(\bar{\rho} + \bar{p})aU, \tag{263}$$

$$\phi + \psi = -8\pi a^2(\eta)G\Pi, \tag{264}$$

and the transverse traceless condition, we can rewrite Eq. 261 as

$$
\begin{aligned}
S_{\alpha\beta} =& - \Big[2\psi\psi_{,\alpha\beta} + \psi_{,\alpha}\psi_{,\beta} + 2\psi_{,(\alpha}\phi_{,\beta)} + 2\phi\psi_{,\alpha\beta} + 4\phi\phi_{,\alpha\beta} + 3\phi_{,\alpha}\phi_{,\beta} - \frac{1}{3}\delta_{\alpha\beta}(2\psi\Delta\psi \\
&+ \psi_{,\gamma}\psi^{,\gamma} + 2\psi_{,\gamma}\phi^{,\gamma} + 2\phi\Delta\psi + 4\phi\Delta\phi + 3\phi_{,\gamma}\phi^{,\gamma})\Big] + \Big[2\phi(\phi + \psi)_{,\alpha\beta} \\
&+ 24\pi a^2(\eta)G(\bar{\rho} + \bar{p})U_{,\alpha}U_{,\beta} - \frac{1}{3}\delta_{\alpha\beta}\left(2\phi(\phi + \psi)^{,\gamma}_{,\gamma} + 24\pi a^2(\eta)G(\bar{\rho} + \bar{p})U_{,\gamma}U^{,\gamma}\right)\Big],
\end{aligned}
\tag{265}
$$

In the Fourier Space, this equation of motion corresponds to Eq. 226. The source term of Eq. 226 is a convolution of two first-order scalar perturbations at different wave numbers [20]. Recalling $S(\eta, \mathbf{k}) = e^{\alpha\beta}(\hat{k})S_{\alpha\beta}(\eta, \mathbf{k})$, in the momentum space Eq. 265 yields

$$
\begin{aligned}
S(\eta, \mathbf{k}) = \int \frac{d^3q}{(2\pi)^3}\, \left[e^{\alpha\beta}(\hat{k})q_\alpha q_\beta\right] \cdot \Big[&\psi(\eta, \mathbf{q})\psi(\eta, \mathbf{k} - \mathbf{q}) - 2\psi(\eta, \mathbf{q})\phi(\eta, \mathbf{k} - \mathbf{q}) \\
&- \phi(\eta, \mathbf{q})\phi(\eta, \mathbf{k} - \mathbf{q}) + 24\pi a^2(\eta)G(\bar{\rho} + \bar{p})U(\eta, \mathbf{q})U(\eta, \mathbf{k} - \mathbf{q})\Big].
\end{aligned}
\tag{266}
$$

Using the background and first-order relations, we can express U in terms of ϕ, and Eq. 266 becomes

$$\begin{aligned} S(\eta,\mathbf{k}) = \frac{1}{a^2(1+w)} \int \frac{d^3q}{(2\pi)^3} \left[e^{\alpha\beta}(\hat{k})q_\alpha q_\beta\right] \cdot \Big[&(5+w)\psi(\eta,\mathbf{q})\psi(\eta,\mathbf{k}-\mathbf{q}) \\ &- (1+w)\big(2\psi(\eta,\mathbf{q})\phi(\eta,\mathbf{k}-\mathbf{q}) + \phi(\eta,\mathbf{q})\phi(\eta,\mathbf{k}-\mathbf{q})\big) \\ &+ \frac{4}{\mathcal{H}}\left(\psi(\eta,\mathbf{q})\phi'(\eta,\mathbf{k}-\mathbf{q}) + \phi'(\eta,\mathbf{q})\psi(\eta,\mathbf{k}-\mathbf{q})\right) \\ &+ \frac{4}{\mathcal{H}^2}\phi'(\eta,\mathbf{q})\phi'(\eta,\mathbf{k}-\mathbf{q})\Big], \end{aligned} \tag{267}$$

where $w = \bar{p}/\bar{\rho}$. We split the first order perturbation into transfer functions $\Phi(k\eta)$, $\Psi(k\eta)$ and primordial fluctuation $\zeta(\mathbf{k})$

$$\phi(\eta,\mathbf{k}) = \Phi(k\eta)\zeta(\mathbf{k}), \tag{268}$$

$$\psi(\eta,\mathbf{k}) = \Psi(k\eta)\zeta(\mathbf{k}). \tag{269}$$

Replacing these into Eq. 267, we find

$$S(\eta,\mathbf{k}) = \int \frac{d^3q}{(2\pi)^3} \left[e^{\alpha\beta}(\hat{k})q_\alpha q_\beta\right] \cdot f(\eta,\mathbf{k},\mathbf{q})\zeta(\mathbf{q})\zeta(\mathbf{k}-\mathbf{q}), \tag{270}$$

where

$$\begin{aligned} f(\eta,\mathbf{k},\mathbf{q}) \equiv &\frac{1}{a^2(1+w)}\Big[(5+w)\Psi(|\mathbf{q}|\eta)\Psi(|\mathbf{k}-\mathbf{q}|\eta) - (1+w)\big(2\Psi(|\mathbf{q}|\eta)\Phi(|\mathbf{k}-\mathbf{q}|\eta) \\ &+ \Phi(|\mathbf{q}|\eta)\Phi(|\mathbf{k}-\mathbf{q}|\eta)\big) + \frac{4}{\mathcal{H}}\left(\Psi(|\mathbf{q}|\eta)\Phi'(|\mathbf{k}-\mathbf{q}|\eta) + \Phi'(|\mathbf{q}|\eta)\Psi(|\mathbf{k}-\mathbf{q}|\eta)\right) \\ &+ \frac{4}{\mathcal{H}^2}\Phi'(|\mathbf{q}|\eta)\Phi'(|\mathbf{k}-\mathbf{q}|\eta)\Big]. \end{aligned} \tag{271}$$

Eq 270 corresponds to the source of gravitational waves generated by the primordial fluctuation $\zeta(\mathbf{k})$. As we will discuss in the next subsection, the formulation used, allows us to describe the power spectrum of gravitational waves in terms of the primordial one.

11 Power Spectrum

Let us consider the power spectrum of scalar-induced gravitational waves. Using the definition of power spectrum, defined by the two-point correlation function. Then the two-point correlation function is

$$\langle\, h_{\mathbf{k}}(\eta)h_{\mathbf{k}'}(\eta)\,\rangle = \frac{1}{a^2(\eta)}\int_0^{\eta} d\widetilde{\eta}_2 \int_0^{\eta} d\widetilde{\eta}_1\; G_{\mathbf{k}}(\eta,\widetilde{\eta}_1)G_{\mathbf{k}'}(\eta,\widetilde{\eta}_2)a(\widetilde{\eta}_1)a(\widetilde{\eta}_2)\,\langle\, S_{\mathbf{k}}(\widetilde{\eta}_1)S_{\mathbf{k}'}(\widetilde{\eta}_2)\,\rangle. \tag{272}$$

The power spectrum of tensor metric perturbations is

$$\langle\, h_{\mathbf{k}}(\eta)h_{\mathbf{k}'}(\eta)\,\rangle \equiv (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')P_h(k,\eta), \tag{273}$$

and the power spectrum of primordial fluctuations is defined as

$$\langle\, \zeta(\mathbf{k})\zeta(\mathbf{k}')\,\rangle \equiv (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')P_\zeta(k). \tag{274}$$

According to Equation 270, the two-point correlation function for the source terms can be written in terms of primordial fluctuations:

$$\langle\, S_{\mathbf{k}}(\widetilde{\eta}_1)S_{\mathbf{k}'}(\widetilde{\eta}_2)\,\rangle = \int \frac{d^3\mathbf{q}}{(2\pi)^3} e(\mathbf{k},\mathbf{q})f(\widetilde{\eta}_1,\mathbf{k},\mathbf{q}) \int \frac{d^3\mathbf{q}'}{(2\pi)^3} e(\mathbf{k}',\mathbf{q}')f(\widetilde{\eta}_2,\mathbf{k}',\mathbf{q}')\,\langle\, \zeta(\mathbf{q})\zeta(\mathbf{k}-\mathbf{q})\zeta(\mathbf{q}')\zeta(\mathbf{k}'-\mathbf{q}')\,\rangle, \tag{275}$$

where $e(\mathbf{k},\mathbf{q}) \equiv e^{\alpha\beta}(\hat{k})q_\alpha q_\beta$.
According to the Wick's Theorem, one can split higher-order point correlation function as a sum over all permutation of field pairs product; in our case means

$$\begin{aligned}\langle\, \zeta(\mathbf{q})\zeta(\mathbf{k}-\mathbf{q})\zeta(\mathbf{q}')\zeta(\mathbf{k}'-\mathbf{q}')\,\rangle =& \langle\, \zeta(\mathbf{q})\zeta(\mathbf{k}-\mathbf{q})\,\rangle\langle\, \zeta(\mathbf{q}')\zeta(\mathbf{k}'-\mathbf{q}')\,\rangle + \langle\, \zeta(\mathbf{q}')\zeta(\mathbf{k}-\mathbf{q})\,\rangle\langle\, \zeta(\mathbf{q})\zeta(\mathbf{k}'-\mathbf{q}')\,\rangle \\ &+ \langle\, \zeta(\mathbf{k}-\mathbf{q})\zeta(\mathbf{k}'-\mathbf{q}')\,\rangle\langle\, \zeta(\mathbf{q})\zeta(\mathbf{q}')\,\rangle.\end{aligned} \tag{276}$$

Inserting in Eq. 275 the definition of primordial power spectrum Eq. 274 and splitting the four-point correlation function of primordial fluctuation as in Eq. 276 we get

$$\langle\, S_{\mathbf{k}}(\widetilde{\eta}_1)S_{\mathbf{k}'}(\widetilde{\eta}_2)\,\rangle = \delta(\mathbf{k}+\mathbf{k}')\int d^3q\; (e(\mathbf{k},\mathbf{q}))^2 P_\zeta(|\mathbf{q}|)P_\zeta(|\mathbf{k}-\mathbf{q}|)f(\widetilde{\eta}_1,\mathbf{k},\mathbf{q})\cdot[f(\widetilde{\eta}_2,\mathbf{k},\mathbf{q}) + f(\widetilde{\eta}_2,\mathbf{k},\mathbf{k}-\mathbf{q})] \tag{277}$$

12 Relative Energy Density

In the following subsections, we will present an analytical and numerical approach to determine the relative energy density of gravitational waves Ω_h. The first approach is based on the research conducted by *Baumann et al.* [20] on gravitational waves. This study made some simplifications to allow the analytical calculation of the relative energy density. The second approach is based on the semi-analytical approach proposed by *Ananda et al.* [133] and by *Kohri, and Terada* [134].

Both approaches incorporate considerations aimed at reducing complexity. First, let us define the Bardeen potentials as linear combinations of scalar perturbation, which are gauge invariant

$$\Phi(\eta, \mathbf{k}) = \phi + \mathcal{H}\left(\beta - \gamma'\right) = \phi + \frac{\mathcal{H}}{a}\chi, \tag{278}$$

$$\Psi(\eta, \mathbf{k}) = \psi - \frac{1}{a}\left[a\left(\beta - \gamma'\right)\right]' = \psi - \chi, \tag{279}$$

where in the conformal Newtonian gauge $\chi = 0$. According to our gauge choice, Bardeen's potentials are simply $\Phi = \phi$ and $\Psi = \psi$. Considering the separation of the two metric perturbations given in Eq. 268 and Eq. 269, the Bardeen potentials can be separated, as well as transfer functions and primordial fluctuation.

According to *Dodelson* [43], we know that the two gravitational potentials Φ and Ψ, defined in Eq. 278 and respectively Eq. 279, are equal and opposite unless the photons' quadrupole moment Θ_2 and neutrinos quadrupole moment $\mathcal{N}_2$ are negligible small. The relation between these quantities is

$$(\Phi(\eta, \mathbf{k}) + \Psi(\eta, \mathbf{k})) = -\frac{32\pi G a^2}{k^2}\left(\rho_\gamma \Theta_2 + \rho_\nu \mathcal{N}_2\right), \tag{280}$$

where ρ_γ and ρ_ν are, respectively, the energy density of photons and energy density of neutrinos. Typically, they are negligibly small; therefore, we can assume they are zero, thus $\Psi \approx -\Phi$. Therefore, from the first order relation 264, we find that the scalar component of the anisotropic stress is zero.

In addition to that, the energy-momentum tensor Eqs. 213-215 are given in the general u_α-frame. In this case, the tensor contains ten independent parameters: one of energy density, one of isotropic pressure, five of anisotropic stress, and the remaining three can be assigned to velocity U_α or flow Q_α. In order to fix the redundant degrees of freedom without any loss of generality, we can choose a frame. The two most commonly used frames are the normal and energy frames. Choosing the energy frame, we get $Q_\alpha = 0$ [18], and therefore, we are left with a perfect fluid governed by the energy density and the isotropic pressure.

In this case, the function f in Eq. 271 is expressed solely by the potential Φ as

$$\begin{aligned} f(\eta, \mathbf{k}, \mathbf{q}) \equiv \frac{2}{3a^2(\eta)(1+w)}\Big[&(5+3w)\Phi(|\mathbf{q}|\eta)\Phi(|\mathbf{k}-\mathbf{q}|\eta) + \frac{2}{\mathcal{H}}\Big(\Phi'(|\mathbf{q}|\eta)\Phi(|\mathbf{k}-\mathbf{q}|\eta) \\ &+ \Phi(|\mathbf{q}|\eta)\Phi'(|\mathbf{k}-\mathbf{q}|\eta)\Big) + \frac{2}{\mathcal{H}^2}\Phi'(|\mathbf{q}|\eta)\Phi'(|\mathbf{k}-\mathbf{q}|\eta)\Big]. \end{aligned} \tag{281}$$

Following *Mukhanov* [70], in the presence of scalar perturbations only, the evolution of the

Bardeen potential Φ for a perfect fluid is governed by

$$\Delta\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \delta\rho \tag{282}$$

$$(\Phi' + \mathcal{H}\Phi)_{,\alpha} = 4\pi G a^2 (\bar{\rho} + \bar{p}) v \tag{283}$$

$$\Phi'' + \mathcal{H}(2\Phi + \Phi') + (\mathcal{H}^2 + 2\mathcal{H}')\Phi = -4\pi G a^2 \delta p. \tag{284}$$

Then, we define the pressure as a function of the density energy and the entropy $\delta p = c_s^2\delta\rho + \tau\delta S$. According to this definition, combining Eqs. 282 and 284, we get the equation of motion for the potential Φ

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' + \left(\mathcal{H}^2(1 + 3c_s^2) + 2\mathcal{H}'\right)\Phi = -4\pi G a^2 \tau\delta S. \tag{285}$$

Considering adiabatic perturbations, such that $\delta S = 0$, the equation of state can be expressed as $p = \omega\rho$, where: for the RD epoch $\omega = 1/3$ and for the MD epoch $\omega = 0$, i.e. $p = 0$.

Let us now consider the two eras separately:
In a Universe dominated by relativistic matter, the scale factor $a(\eta) \propto \eta^{2/(1+3\omega)}$, thus the equation of motion governing the Bardeen potential in the Fourier space yields:

$$\Phi'' + \frac{6(1+\omega)}{(1+3\omega)\eta}\Phi' + k^2\omega\Phi = 0 \tag{286}$$

This equation can be solved using the Frobenius method, and its exact solution is:

$$\Phi(\eta, \mathbf{k}) = (\sqrt{\omega}k\eta)^{-\alpha}\left[c_1(k)J_\alpha(\sqrt{\omega}k\eta) + c_2(k)Y_\alpha(\sqrt{\omega}k\eta)\right], \tag{287}$$

where J_α is the first kind Bessel function, Y_α is the second kind Bessel function and $\alpha \equiv \frac{5+3\omega}{2(1+3\omega)}$.
In RD epoch the scale factor is $a \propto \eta$, $\alpha = 3/2$ and therefore the solution becomes

$$\Phi(\eta, \mathbf{k}) = \sqrt{\frac{2}{\pi}}\left(\frac{k\eta}{\sqrt{3}}\right)^{-3/2}\left[c_1(k)j_1\left(\frac{k\eta}{\sqrt{3}}\right) + c_2(k)y_1\left(\frac{k\eta}{\sqrt{3}}\right)\right], \tag{288}$$

where j_1 and y_1 are the spherical Bessel functions. In order to find the coefficient, we consider the potential at early times, just after inflation end, $\eta \to 0$.

$$\begin{aligned}
\lim_{\eta\to 0}\Phi(\eta, \mathbf{k}) &= \lim_{x\to 0}\sqrt{\frac{2}{\pi}}x^{-2}\left[c_1(k)\left(\frac{\sin x}{x} - \cos x\right) - c_2(k)\left(\frac{\cos x}{x} - \sin x\right)\right] \\
&= \sqrt{\frac{2}{\pi}}\frac{c_1(k)}{3} - \sqrt{\frac{2}{\pi}}c_2(k)\lim_{x\to 0}\left(\frac{1}{x^3} + \frac{1}{x} - \frac{x}{8}\right) \\
&= \sqrt{\frac{2}{\pi}}\frac{c_1(k)}{3} \stackrel{!}{=} \zeta(k).
\end{aligned} \tag{289}$$

where $x \equiv k\eta/\sqrt{3}$; the second equality is found by expanding the trigonometric functions around zero, and the third equality is obtained by setting $c_2(k) = 0$. According to *Maggiore* [55], we must set $c_2(k) = 0$ because if this mode is present at the epoch in which we set the initial condition and is of comparable size to $c_1(k)$, it means that towards the end of the inflation/early in RD epoch this mode must have been extraordinarily large, concerning $c_1(k)$. It makes more sense to assume that the two modes have similar sizes once they emerge from inflation, as there is no evidence of physical mechanisms that would explain a substantial difference between the two modes. Therefore, by the time we set the initial conditions, $c_2(k)$ has to vanish.

Therefore, the Bardeen potential in the RD epoch is

$$\Phi(\sqrt{3}x) = \frac{1}{x^2}\left[\frac{\sin x}{x} - \cos x\right], \tag{290}$$

from this solution, we need to consider the difference between superhorizon modes ($k\eta \ll 1$) and subhorizon modes($k\eta > 1$). Taylor expanding the previous equation for $x \to 0$, we see that superhorizon modes freeze out; on the other hand, subhorizon modes decay and oscillate, i.e.

$$\Phi(k\eta) = \begin{cases} 1 & , k\eta \ll 1, \quad \eta < \eta_{eq} \\ -\frac{\cos(k\eta)}{(k\eta)^2} & , k\eta > 1, \quad \eta < \eta_{eq} \end{cases}. \tag{291}$$

As the decay of the subhorizon modes dominates the oscillation, we can ignore the oscillation and merge the two expressions, and get an expression valid for sub- and superhorizon modes:

$$\Phi(k\eta) = \frac{1}{1 + (k\eta)^2}. \tag{292}$$

In a universe dominated by non-relativistic matter, like in the MD epoch, we have $a \propto \eta^2$, and the equation of motion for the Bardeen potential becomes

$$\Phi'' + \frac{6}{\eta}\Phi' = 0. \tag{293}$$

The exact solution of this differential equation is

$$\Phi = c_1(k) + \frac{c_2(k)}{\eta^5}, \tag{294}$$

and as before the decaying mode $c_2(k) = 0$ vanishes. This means that the potential during the MD phase freezes on all scales, and it has the same constant value arising at η_{eq}, the time at which the transition from the RD era to the MD era takes place.

To summarise:

$$\Phi(k\eta) = \begin{cases} \frac{1}{1+(k\eta)^2} & , \eta < \eta_{eq} \text{ (RDE)} \\ \frac{1}{1+(k\eta_{eq})^2} & , \eta > \eta_{eq} \text{ (MDE)} \end{cases}. \tag{295}$$

The source term of the equation of motion for gravitational waves is fully determined by the potential we have treated so far. Therefore, one can find the solution for the tensor modes replacing the gravitational potential as given in Eq. 295 in the source term definition, and, then solve Eq. 235 and 239 for a description of gravitational waves in RD epoch and, respectively, in MD epoch. Compare *Gong* [22] for second-order analytic solutions.

12.1 Analytical Approach

Hereafter, we will provide a purely analytical solution of the power spectrum and the relative energy density of scalar-induced gravitational waves. To calculate those, we can follow the description proposed by *Baumann et al.* [20]. The idea is to assume that tensor modes (gravitational waves) production is instantaneous once the mode $\mathbf{k}$ enters the horizon. The evolution of the source term determines the evolution of gravitational waves, and they redshift as long as their magnitude is greater than S/k^2. Once the magnitude reaches the value of S/k^2 during MDE, it freezes out at a constant value. Then we define the transfer function $T(\eta, k)$ for gravitational waves, as

$$h(\eta, \mathbf{k}) = T(\eta, k) h^{(i)}(\eta_i, \mathbf{k}), \tag{296}$$

where η_i is the exact moment when tensor modes are generated after horizon crossing, and thus we denote $h^{(i)}$ as the value of tensor modes at η_i.

We estimate $h^{(i)}$ from the equation of motion for gravitational waves, neglecting the time derivatives in the equation of motion, and we get the following relation:

$$h^{(i)}(\mathbf{k}) \sim \frac{S^{(i)}}{k^2} \sim \frac{1}{k^2} \int d^3\mathbf{q} \;\; e(\mathbf{k}, \mathbf{q}) \Phi(q\eta_i) \Phi(|\mathbf{k} - \mathbf{q}|\eta_i) \zeta(\mathbf{q}) \zeta(\mathbf{k} - \mathbf{q}), \tag{297}$$

where we define

$$e(\mathbf{k}, \mathbf{q}) = q^2(1 - \mu^2) \qquad \text{with} \qquad \mu \equiv \frac{\mathbf{k} \cdot \mathbf{q}}{kq}. \tag{298}$$

Then the two-point correlation function for gravitational waves of this kind becomes:

$$\langle\, h^{(i)}_{\mathbf{k}}(\eta) h^{(i)}_{\mathbf{k}'}(\eta) \,\rangle \sim \frac{1}{k^4} \delta(\mathbf{k} + \mathbf{k}') \int d^3\mathbf{q} \left[(1 - \mu^2)\right]^2 \Phi^2(q\eta_i) \Phi^2(|\mathbf{k} - \mathbf{q}|\eta_i) \frac{\Delta^2_\zeta(q)}{q^3} \frac{\Delta^2_\zeta(|\mathbf{k} - \mathbf{q}|)}{|\mathbf{k} - \mathbf{q}|^3} \tag{299}$$

The primordial Power Spectrum is given as follows (observational parametrisation)

$$\Delta^2_\zeta(k) = \frac{4}{9} A^2_\zeta(k_0) \left(\frac{k}{k_0}\right)^{n_s - 1}, \tag{300}$$

where $A^2_\zeta \approx (2.101^{+0.031}_{-0.034}) \times 10^{-9}$ and $n_s \approx 1$. Thus, we can consider a scale-invariant power spectrum for an analytical approach. Then the power spectrum of gravitational waves, which enter the horizon during the RD era, is

$$\begin{aligned} P^{(i)}_h(k) &= \frac{k^3}{2\pi^2} \langle\, |h^{(i)}_{\mathbf{k}}(\eta)^2| \,\rangle \\ &\sim \frac{A^4_\zeta}{k} \int d^3\mathbf{q} \;\; q(1 - \mu^2)^2 \frac{1}{\left[1 + (\frac{q}{k})^2\right]^2} \frac{1}{\left[1 + \left(\frac{|\mathbf{k} - \mathbf{q}|}{k}\right)^2\right]^2} \frac{1}{|\mathbf{k} - \mathbf{q}|^3} \\ &= A^4_\zeta \int_0^\infty dx \int_{-1}^1 d\mu \;\; (1 - \mu^2)^2 \frac{x^3}{(1 + x^2)^2 \, (2 - 2x\mu + x^2)^2 \, (1 - 2x\mu + x^2)^{3/2}} \\ &\propto A^4_\zeta, \end{aligned} \tag{301}$$

where we have defined $x = q/k$. For gravitational waves crossing the horizon during the MD era, the power spectrum also depends on the scale:

$$
\begin{aligned}
P_h^{(i)}(k) &= \frac{k^3}{2\pi^2}\langle\, |h_{\mathbf{k}}^{(i)}(\eta)^2|\,\rangle \\
&\sim \frac{A_\zeta^4}{k}\int d^3\mathbf{q}\;\; q(1-\mu^2)^2 \frac{1}{\left[1+\left(\frac{q}{k_{eq}}\right)^2\right]^2}\frac{1}{\left[1+\left(\frac{|\mathbf{k}-\mathbf{q}|}{k_{eq}}\right)^2\right]^2}\frac{1}{|\mathbf{k}-\mathbf{q}|^3} \\
&= A_\zeta^4\int_0^\infty dx\int_{-1}^{1} d\mu\;\; (1-\mu^2)^2\frac{x^3}{\left[1+(xy)^2\right]^2\left[1+y^2(1-2x\mu+x^2)\right]^2(1-2x\mu+x^2)^{3/2}} \\
&\propto \frac{k_{eq}}{k}A_\zeta^4,
\end{aligned}
\tag{302}
$$

where $x = q/k$ and $y = k/k_{eq} \ll 1$, and for the proportionality we have to neglect the μ-term as proposed by *Baumann et al.* [20], such that we get a solvable integral. Therefore, the power spectrum of gravitational waves, when they enter the horizon, can be summarised as follows:

$$
P_h^{(i)}(k) \propto A_\zeta^4 \begin{cases} 1 & , k > k_{eq} \quad \text{(RDE)} \\ \frac{k_{eq}}{k} & , k < k_{eq} \quad \text{(MDE)} \end{cases} . \tag{303}
$$

To characterise the gravitational waves' evolution, we need to compute the transfer function $T(\eta, k)$. To do this, we need to estimate the time evolution of the source term.

During the RD epoch, the source term decays as

$$
\frac{S^{(f)}}{S^{(i)}} = \left(\frac{a_k}{a_{eq}}\right)^{\gamma(k)}, \tag{304}
$$

where according to *Baumann et al.* [20] numerical calculation gives an upper bound of $\gamma \approx 3$. While the tensor modes decay as a^{-1}, as shown in Eq. 231. Combining these considerations, we get

$$
\frac{h^{(f)}}{h^{(i)}} = \frac{a_k}{a_k^*} \approx \frac{S^{(f)}}{S^{(i)}} = \left(\frac{a_k}{a_{eq}}\right)^{\gamma(k)}. \tag{305}
$$

For a given η, it could be found a large k, which never settled down, by substitution $a_k^* = a(\eta)$ we get the critical wave number as:

$$
k_c(\eta) = \left(\frac{a(\eta)}{a_{eq}}\right)^{1/(\gamma-1)} k_{eq}. \tag{306}
$$

Modes with a wave number greater than this simply redshift as a^{-1}. On the other hand, modes which enter the horizon during the MD era have constant source terms. To summarise, the transfer function is

$$
T(\eta, k) = \begin{cases} \frac{a_{eq}}{a(\eta)}\frac{k_{eq}}{k} & , k > k_c(\eta) \\ \left(\frac{k}{k_{eq}}\right)^{-\gamma(k)} & , k_{eq} < k < k_c(\eta) \\ 1 & , k < k_{eq} \end{cases} . \tag{307}
$$

In order to compute the energy density of scalar-induced gravitational waves, we can use the definitions in *Maggiore*'s books [109, 55], using the conformal time η they state:

$$\rho_h = \frac{1}{32\pi a^2(\eta)G}\langle\, h'_{\alpha\beta}(\eta,x)h^{\alpha\beta\prime}(\eta,x)\,\rangle. \tag{308}$$

In the Fourier space, we adopt the separation in transfer function and primordial fluctuation, and using the primordial power spectrum, we get

$$\rho_h = \frac{1}{32\pi a^2(\eta)G}P_h^{(i)}|T'(\eta,k)|^2. \tag{309}$$

In the literature, one finds this expression without the time derivative of the transfer function. In agreement with [135], this approximates modes well inside the horizon. So we can define the relative energy density as:

$$\Omega_h(\eta,k) = \frac{\rho_h}{\rho_c} = \frac{1}{12\mathcal{H}^2(\eta)}k^2|T(\eta,k)|^2P_h^{(i)}(k). \tag{310}$$

Up to this point, we discussed only one polarisation. Considering both polarisations, the relative energy density is two times greater. To conclude, by combining Eq. 310 with Eq. 307 and Eq. 303, we find the following result

$$\Omega_h(\eta,k) \propto \frac{1}{6\mathcal{H}^2}A_\zeta^4(k_*)\left(\frac{k}{k_*}\right)^{n_s-1}\begin{cases}\left(\frac{a_{eq}}{a(\eta)}k_{eq}\right)^2 & ,k>k_c\ \text{(RDE)}\\ \left(\frac{k}{k_{eq}}\right)^{-2\gamma}k^2 & ,k_{eq}<k<k_c\ \text{(RDE)}\\ k_{eq}k & ,k<k_{eq}\ \text{(MDE)}\end{cases} \tag{311}$$

12.2 Numerical Approach

This subsection aims to provide a more accurate solution for the relative energy density of scalar-induced gravitational waves in a Universe containing only matter. Let us consider the tensor mode h as expressed in Eq. 296; i.e. we decompose the mode in its primordial part, depending only on the wave number, and the transfer function, depending on the wave number and time.

$$h(\eta,\mathbf{k}) = T(\eta,k)h^{(i)}(\eta_i,\mathbf{k}),$$

where $h(\eta,\mathbf{k})$ is the inhomogeneous part Eq. 239, the transfer function $T(\eta,k)$ is defined as

$$\begin{aligned} T(\eta,k) &= \frac{10}{3}\int_0^\eta d\tilde{\eta}\,\frac{k\eta\tilde{\eta}}{a(\eta)}\left[j_1(k\tilde{\eta})y_1(k\eta)-j_1(k\eta)y_1(k\tilde{\eta})\right]a(\tilde{\eta})\\ &= \frac{10}{3k^2}\Big(1-3\frac{j_1(k\eta)}{k\eta}\Big). \end{aligned} \tag{312}$$

Then, from Eq. 309, we can write the relative energy density as

$$\Omega_h(\eta,k) = \frac{\rho_h}{\rho_c} = \frac{1}{12\pi^2\mathcal{H}^2(\eta)}|T'(\eta,k)|^2P_h^{(i)}(k), \tag{313}$$

which differ from Eq. 310 in that, in this case, we consider the time derivative of the transfer function.

As in the previous subsection, let us consider firstly the transfer function. As discussed previously, the Bardeen potential in the MD universe is constant (see Eq. 294). Therefore, Eq. 281 is also constant $f = 10/3 \cdot \Phi \cdot \Phi$. The time component of Eq. 313 is

$$\frac{|T'(\eta,k)|^2}{\mathcal{H}^2(\eta)} = \frac{25}{k^{10}\eta^6}\left[\left(k^2\eta^2 - 3\right)\sin k\eta + 3k\eta\cos k\eta\right]^2. \tag{314}$$

While the time component of Eq. 310 is

$$\frac{|kT(\eta,k)|^2}{\mathcal{H}^2(\eta)} = \frac{25}{9k^8\eta^4}\left[k^3\eta^3 - 3\sin k\eta + 3k\eta\cos k\eta\right]^2. \tag{315}$$

The pure wave number dependent term of Eq. 310 and Eq. 313 is the same for both cases and corresponds to

$$P_h^{(i)}(k) = k^3 \int d^3\,\mathbf{q}(e(\mathbf{k},\mathbf{q}))^2 \left(\Phi(|\mathbf{q}|\eta)\Phi(|\mathbf{k}-\mathbf{q}|\eta)\right)^2 P_\zeta(|\mathbf{q}|)P_\zeta(|\mathbf{k}-\mathbf{q}|). \tag{316}$$

The quantity $e^P(\mathbf{k},\mathbf{q}) = e^{\alpha\beta(P)}(\hat{\mathbf{k}})q_\alpha q_\beta$ is equal to $\frac{1}{\sqrt{2}}q^2\sin^2\theta\cos 2\varphi$ for $P = +$ polarization and $\frac{1}{\sqrt{2}}q^2\sin^2\theta\sin 2\varphi$ for $P = \times$ polarization, where (q,θ,φ) is the coordinate of $\mathbf{q}$ in a spherical coordinate system. According to *Baumann et al.* [20] the angular coordinates can be described by the $\mu = \frac{\mathbf{k}\cdot\mathbf{q}}{kq}$, such that we can get the relation given in Eq. 298. Changing the coordinate system with the one described by q and μ we find

$$P_h^{(i)}(k) = k^3\int_0^\infty dq \int_{-1}^1 d\mu\, q^6(1-\mu^2)^2\left(\Phi(|\mathbf{q}|\eta)\Phi(|\mathbf{k}-\mathbf{q}|\eta)\right)^2 P_\zeta(|\mathbf{q}|)P_\zeta(|\mathbf{k}-\mathbf{q}|). \tag{317}$$

It turns out useful to change the variable into $u \equiv \frac{|\mathbf{k}-\mathbf{q}|}{k}$ and $v \equiv \frac{q}{k}$, after this change of variables, the previous equation can be written as

$$P_h^{(i)}(k) = k^3\int_0^\infty dv\int_{|1-v|}^{1+v} du\, k(k^2vu)^3\left[\frac{4v^2-(1+v^2-u^2)^2}{4vu}\right]^2\left(\Phi(v\eta)\Phi(u\eta)\right)^2 P_\zeta(kv)P_\zeta(ku). \tag{318}$$

Considering the dimensionless power spectrum, we find

$$P_h^{(i)}(k) = k^4\int_0^\infty dv\int_{|1-v|}^{1+v} du\,\left[\frac{4v^2-(1+v^2-u^2)^2}{4vu}\right]^2\left(\Phi(v\eta)\Phi(u\eta)\right)^2 \Delta_\zeta^2(kv)\Delta_\zeta^2(ku), \tag{319}$$

where the dimensionless power spectrum is defined as

$$\Delta_\zeta(k)^2 = A_\zeta^2\cdot\left(\frac{k}{k_*}\right)^{n_s-1}. \tag{320}$$

Therefore, the relative energy density of gravitational waves, according to Eq. 313, is

$$\begin{aligned}\Omega_h(\eta,k) =& \frac{25}{12\pi^2}\frac{1}{k^6\eta^6}\left[\left(k^2\eta^2-3\right)\sin k\eta + 3k\eta\cos k\eta\right]^2 \\ &\cdot\int_0^\infty dv\int_{|1-v|}^{1+v} du\,\left[\frac{4v^2-(1+v^2-u^2)^2}{4vu}\right]^2\left(\Phi(v\eta)\Phi(u\eta)\right)^2\Delta_\zeta^2(kv)\Delta_\zeta^2(ku),\end{aligned} \tag{321}$$

and, according to Eq. 310, is

$$\Omega_h(\eta,k) = \frac{25}{108\pi^2 k^4 \eta^4}\left[k^3\eta^3 - 3\sin k\eta + 3k\eta\cos k\eta\right]^2 \cdot \int_0^\infty dv \int_{|1-v|}^{1+v} du \left[\frac{4v^2-(1+v^2-u^2)^2}{4vu}\right]^2 \left(\Phi(v\eta)\Phi(u\eta)\right)^2 \Delta_\zeta^2(kv)\Delta_\zeta^2(ku). \tag{322}$$

Since we want to compute the relative energy density at the present day, $\eta=\eta_0$ is fixed, these equations are fully determined by the wave number. To simplify the analysis of the function, let us consider the time-dependent t^2 and time-independent I terms separately, such that it can be written as $\Omega_h \equiv t^2 \cdot I$.

Let us focus on the time independent term, the integral. In order to compute Eq. 321 and Eq. 322 numerically, we need to check that the integrand is continuous and differentiable everywhere and does not diverge on the whole integration space. Fig. 2 shows the integrand of Eq. 321 and Eq. 322 for different wavenumber values.

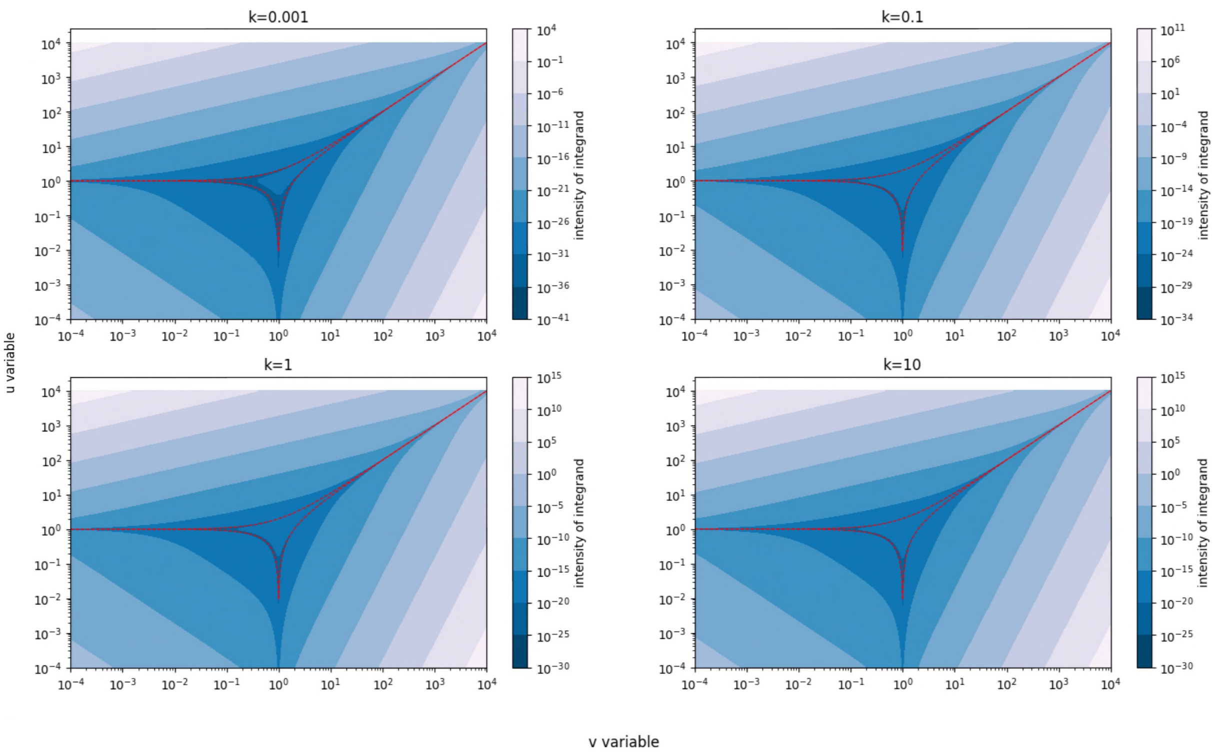

Figure 2: *The graphs show the value of the integrand for four different wave numbers over a (u,v)-space, spanning $[10^{-4},10^4]\times[10^{-4},10^4]$. One can observe that as the coordinate approaches $(u,v)\to(0,0)$, the value of the integrand diverges due to the denominator contribution. Similarly, the integrand diverges for high values of v and/or u, as the numerator dominates the function in these regions. The region within the dashed red lines corresponds to the integration space of the integral, i.e. $(0,\infty)\times[|1-v|,1+v]$. The integrand values within the region are continuous and finite, which permits us to compute the integration without any trouble.*

The region within the red dashed lines corresponds to the integration's space, i.e. v variable span $(0,\infty)$ and u variable span $[|1-v|, 1+v]$. We can observe that the integrand is defined everywhere and does not diverge inside the integration's region. Therefore, we should not run into any problems by performing numerical integration.

The integration in Eq. 321 and Eq. 322 returns an almost constant solution over the wave number space, as shown in Fig. 3. A result of this kind is expected because, as shown in Fig. 2, the integrand function in the integration domain for the four cases is almost the same. It means the solution is almost scale-invariant. From this result, we expect that the transfer function mainly characterises the relative energy density behaviour during the epoch we are considering. As we are only considering a Universe containing matter, the solution should be the transfer function squared, cf. Fig. 4, scaled by factor $\sim 10^{-15}$, the solution of the integration shown in Fig. 3.

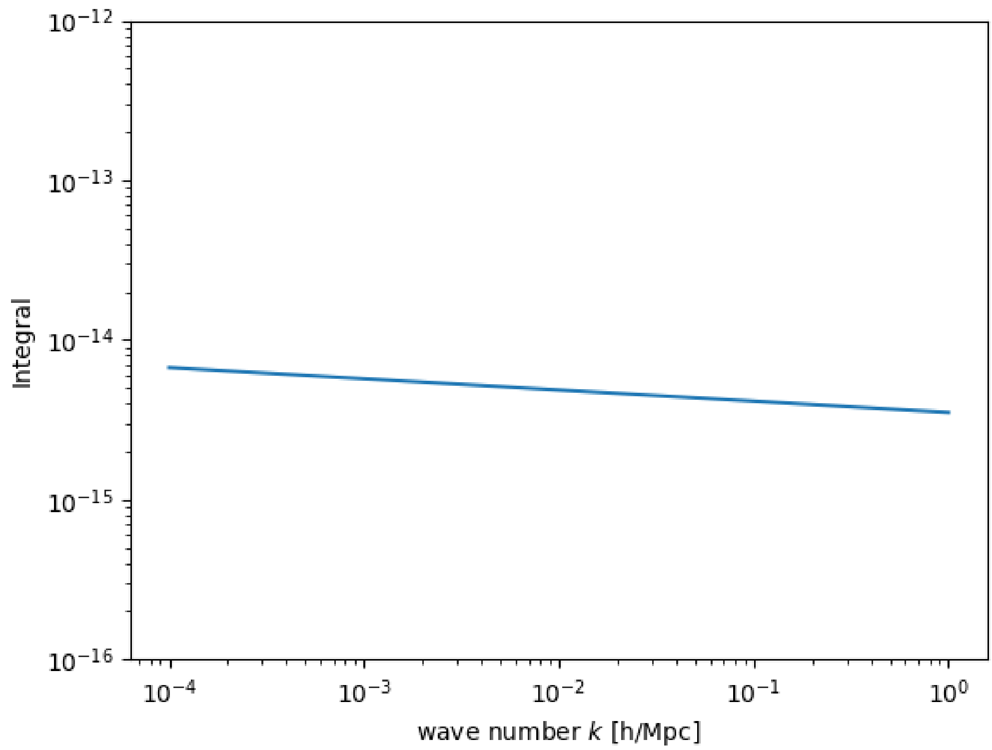

Figure 3: *The result of the integration in Eq. 321 and Eq. 322 within the region shown in Fig. 2. Which shows an almost scale invariant behaviour along the wave number values k.*

Consider the time evolution of the relative energy density for a fixed mode k, which is fully determined by the time-dependent term $t^2(x)$, where $x = k\eta$ (see Fig. 4). In Fig. 4, we represent the two different time-dependent terms of Eq. 321 and Eq. 322.

For the first equation, the solution is represented with the solid blue function in Fig. 4. We can observe that, in the superhorizon regime, $x \ll 1$, the leading term of the Taylor expansion is proportional to $\eta^4 \propto a^2$, which enhances the relative energy density. This leading term is valid only during an early period $x \to 0$. Once the mode enters the horizon $(x > 0)$, it starts to oscillate, and it is damped according to the dominant term $\eta^{-2} \propto a^{-1}$. While for the second equation, the solid black function in Fig. 4, in the superhorizon regime the dominant term of the Taylor expansion is proportional to $\eta^6 \propto a^3$ and the subhorizon behaviour is proportional to $\eta^2 \propto a$.

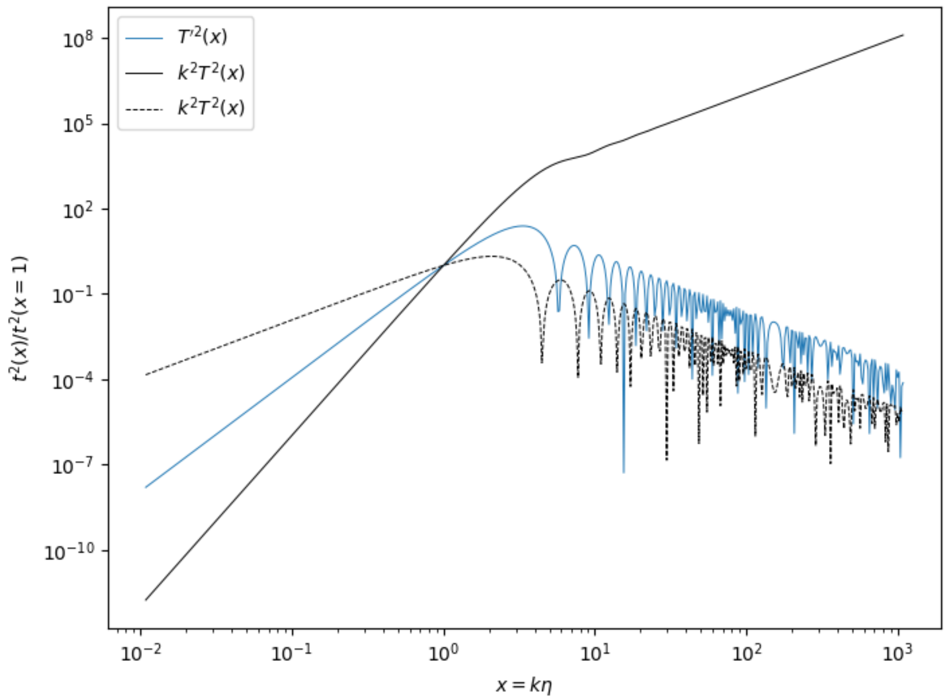

Figure 4: *Time evolution of the relative energy density for a Universe containing only matter. The plotted function is scaled by the transfer function evaluated at the horizon entering a given mode k. The blue function corresponds to the case $T'^2(x)$, while the black one to the case $k^2T^2(x)$. (blue) It shows two different regimes; the (early times) superhorizon behaviour is dominated by the leading term of the Taylor expansion $\propto a^2$, while the (late time) subhorizon is dominated by a damped oscillation, which, ignoring the oscillation, is proportional to $\propto a^{-1}$. (solid black) Similarly, there are two regimes, the superhorizon regime is dominated by the leading term of the Taylor expansion and is $\propto a^3$, while the subhorizon one is $\propto a$. (dashed black) This represents the case in which one uses the approximation $k^2T^2(x)$, but excludes the non-oscillating term in Eq. 312. Excluding this term, one obtains the same behaviour as the case $T'^2(x)$, once the mode enters the horizon $\propto a^{-1}$ and the superhorizon regimes is dominated by the leading term of the Taylor expansion and is $\propto a$.*

It is important to point out that adopting the prescription employed in Eq. 321 for the energy density of scalar-induced gravitational waves, we observe that inside the horizon, they do redshift as any other radiation fluid a^{-4},

$$\rho_h = \frac{1}{32\pi G a^2}\langle h'^2(\eta, k)\rangle \propto \frac{1}{a^2(\eta)}\langle T'^2(\eta, k)\rangle \propto \frac{1}{a^4(\eta)}, \tag{323}$$

in agreement with the results obtained by *Sipp, and Schäfer* [23]. On the other hand, the relation $T'^2 \sim k^2 T^2$ leads to a different result: inside the horizon, the second-order gravitational waves do not redshift as radiation

$$\rho_h = \frac{1}{32\pi G a^2}\langle h'^2(\eta,k)\rangle \propto \frac{1}{a^2}\langle T'^2(\eta,k)\rangle \simeq \frac{k^2}{a^2(\eta)}\langle T^2(\eta,k)\rangle \propto \frac{1}{a^2(\eta)}. \tag{324}$$

According to the literature, this approximation is valid only deep inside the horizon $k \gg \mathcal{H}$; and it is proposed e.g. in *Watanabe, and Komatsu* [135], *Baumann et al.* [20], *Hwang et al.* [21], and many others.

This result for the approximation is also different compared to the one obtained in the analytical section (Sec. 12.1), where the spectrum of scalar-induced gravitational waves do redshift for modes deep inside the horizon. Such modes are already inside the horizon during the radiation-dominated era. Therefore, they decay accordingly. Thus, this difference may be attributed to the fact that in the previous section, we considered the epoch dominated by radiation.

It is necessary to understand why these two approaches lead to different results during a matter-dominated period. *Hwang et al.* [21] showed that the power spectra of scalar-induced gravitational waves strongly depend on the choice of gauge. When the source term vanishes, this dependence drops, and the gravitational waves are gauge invariant. In this case, we say that the gravitational waves are free. The source term vanishes, for example, in a Universe dominated by radiation and with the source, the gauge dependence vanishes as well, according to *Inomata, and Terada* [136]. But this is not the case in a Universe dominated by dark matter or dark energy. In these cases, the Bardeen potentials are constant in time, and therefore, the source does not vanish and remains active.

In Eq. 312, we have two terms, a non-oscillating and time independent, and an oscillating one. The first one is associated with the gauge choice, and it is not to be considered as a gravitational wave because is not oscillating [136]. *Ali et al.* [137] obtained the same result for all the seven gauges considered, and they conclude that gravitational waves induced by scalars arise from the oscillating terms and behave as radiation in a matter-dominated epoch (for all the seven gauges), and the constant term does not represent a wave and, therefore, should not contribute to the relative energy density.

In Fig. 4, we show a third function, which represents the time-dependent part of Eq. 322, without the non-oscillating term,

$$\begin{aligned}\Omega_h(\eta,k) =& \frac{25}{12\pi^2 k^4\eta^4}\left[-\sin k\eta + k\eta\cos k\eta\right]^2 \\ &\cdot \int_0^\infty dv \int_{|1-v|}^{1+v} du \left[\frac{4v^2-(1+v^2-u^2)^2}{4vu}\right]^2 (\Phi(v\eta)\Phi(u\eta))^2 \Delta_\zeta^2(kv)\Delta_\zeta^2(ku).\end{aligned} \tag{325}$$

As we can observe, deep inside the horizon, Eq. 325 behaves like the time-dependent part of Eq. 321. The two transfer functions do redshift as $a^{-1}(\eta)$ and are shifted by $\pi/2$, as the dominant terms are $\propto \sin(x)/x^2$ and $\propto \cos(x)/x^2$, respectively for Eq. 321 and the modified Eq. 325.

The superhorizon behaviour of the dashed-black function is dominated by the leading term of the Taylor expansion, which is proportional to $\eta^2 \propto a$. The difference in the order of magnitude between the blue and dashed-black functions depends on the leading term during the superhorizon period. Namely, since the variation of the time-dependent part of Eq. 321

$\propto a^2$, while the one of the modified Eq. 325 $\propto a$, the two functions peak at different values, and therefore, we observe this difference. Excluding the non-oscillating term of the transfer function T as expressed in Eq. 312, we find that inside the horizon, the scalar-induced gravitational waves do redshift like radiation (confirming the result obtained in Eq. 323).

Thus in the conformal Newtonian gauge, the time derivative solves the problem of gauge artefacts arising from the approximation, as the tensor perturbations associated with them are constant in time. In addition to this, according to *Sipp, and Schäfer* [23], it must be used when the source is active, since only free gravitational waves follow the relation $h' \sim kh$, and thus $T'^2 \sim k^2T^2$. Another possibility is to exclude the non-oscillating term of the transfer function manually, as it does not represent a wave.

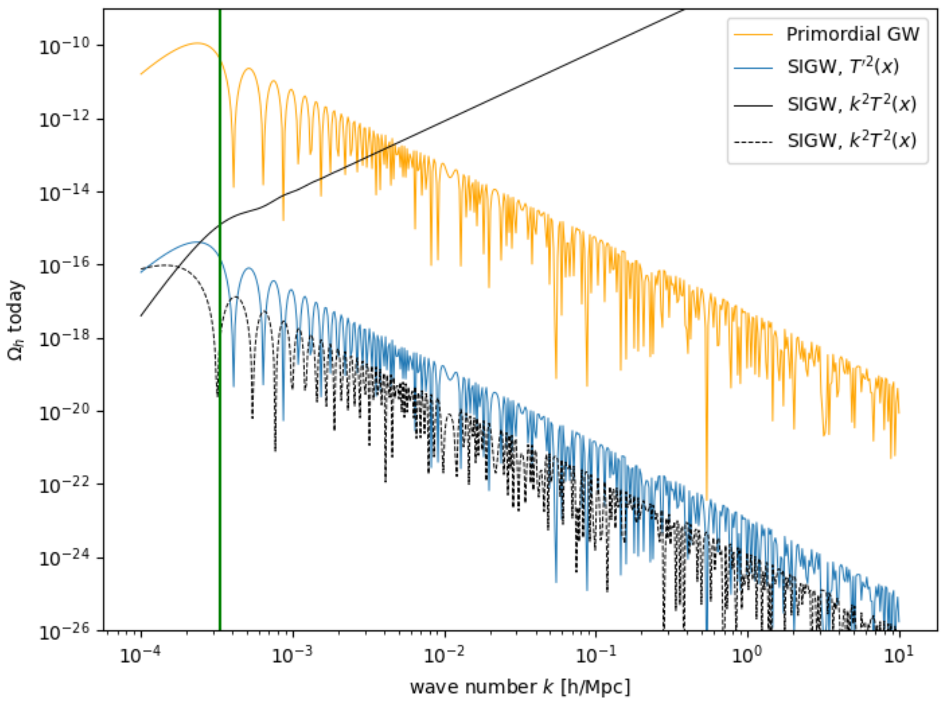

Figure 5: *The relative energy density of gravitational waves for the tensor modes induced by the scalars according to Eq. 321 (blue: SIGW, $T'^2(x)$), according to Eq. 322 (solid black: SIGW, $k^2T^2(x)$), Eq. 322 excluding the non-oscillating term (dashed black: SIGW $k^2T^2(x)$), in the conformal Newtonian gauge, and that of the primordial gravitational waves (orange: Primordial GW) are represented with respect to wavenumber, for a Universe containing only matter. Assuming constant potential, $\Phi(k\eta) = 3/5$ from* Hwang et al.*[21]. To conclude the vertical green line represents the horizon size at the present time.*

The relative energy density Ω_h for a Universe containing only matter at present can be recovered by numerical solution of Eq. 321 and Eq. 322, see Fig. 5. This figure shows the behaviour of the relative energy density adopting a constant Bardeen potential Φ as proposed by *Hwang et al.* [21]. As expected from the previous considerations, the scalar-induced gravitational wave spectra have the same behaviour as the corresponding transfer function solutions. However, they are scaled by a factor 10^{-15} due to the integral term of Eq. 321 and Eq. 322. In the same figure, we show the primordial gravitational waves spectrum. In order to obtain this figure, we considered an optimistic primordial gravitational waves background (ratio $r = A_s/A_T = 0.1$).

Assuming a Universe containing only matter and a constant Bardeen potential, the spectrum of first-order primordial gravitational waves (orange function in Fig. 5, arising from the inflationary tensor fluctuations, behave like the scalar-induced gravitational wave spectrum, for Eq. 321 (blue function in the same figure). This is the case, since the primordial tensor power spectrum and the integral of Eq. 321 are almost scale invariant and are both multiplied by the same transfer function. On the other hand, the spectrum of gravitational waves (solid black function in the figure) computed with Eq. 322 follows its transfer function behaviour. As well as the spectrum (dashed black in the figure) computed with the modified approximation, expressed by Eq. 325, has the same behaviour of its transfer function.

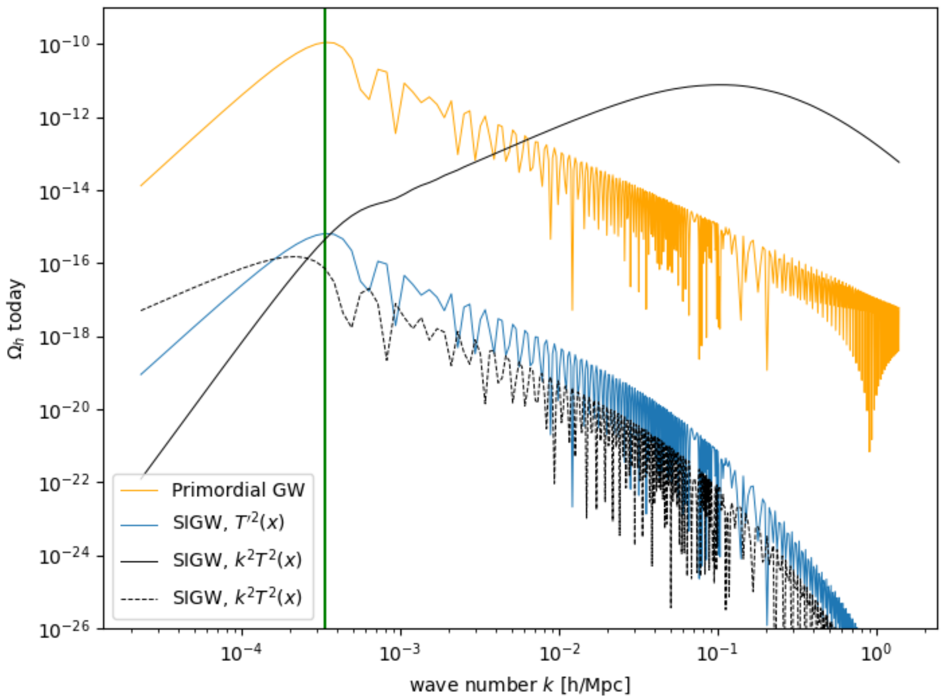

Figure 6: *The relative energy density of gravitational waves for the tensor modes induced by the scalars according to Eq. 321 (blue: SIGW, $T'^2(x)$), according to Eq. 322 (black: SIGW, $k^2T^2(x)$), Eq. 322 excluding the non-oscillating term (dashed black: SIGW $k^2T^2(x)$), in the conformal Newtonian gauge, and the primordial gravitational waves (orange: Primordial GW) are represented with respect to wavenumber, for a Universe containing only matter. Potential from CLASS, using $\Omega_r = 4.15 \times 10^{-5}$ and $\Omega_m = 1 - \Omega_r$. To conclude the vertical green line represents the horizon size at the present time.*

For consistency, we have computed numerically Eq. 321 and Eq. 322 using the Bardeen potential obtained from CLASS [138] instead of fixing them to the constant value as proposed in *Hwang et al.* [21]. Fig. 6 shows today's relative energy density spectrum in the case of a

Universe containing only matter using the Bardeen potential Φ recovered from CLASS, where we have used $\Omega_r = 4.15 \times 10^{-5}$ and $\Omega_m = 1 - \Omega_r$.

The differences in the primordial gravitational waves spectrum between Fig. 5 and 6 are to be attributed to the choice of wave numbers. The biggest discrepancy between the two graphs lies in the spectrum of induced gravitational waves. That is the result of the potential. Using the potential obtained from CLASS, the relative energy density decays faster for small scales. Adopting the Bardeen potential of CLASS, we can observe that the solid black function, which corresponds to the relative energy density described by Eq. 322, for small scales do redshift as the spectra arising from Eq. 321 and Eq. 325. This difference is explained by the fact that CLASS obtains the potential, taking into account the period in which there was radiation domination. Therefore, for all three cases of scalar-induced gravitational wave spectra, the small scales redshift (as expected by analytical calculation, cf. Eq. 311), while only the spectrum arising from Eq. 321 and Eq. 325 do redshift for intermediate scales.

Baumann et al. [20] results show a peak at the comoving-horizon scale at matter-radiation equality of the energy spectrum of scalar-induced gravitational waves, which exceed the one of primordial gravitational waves. In the case described by Eq. 322, we observe an excess of the scalar-induced gravitational waves, but at smaller scales. This shift can be due to the model parameters we have chosen; they considered a ΛCDM model, taking into account all the components, our model is considering almost only matter. In the other case, i.e. the one described by Eq. 321 we do not observe this surplus, confirming the analysis proposed by *Sipp, and Schäfer* [23].

Part IV
Conclusions & Outlook

During the inflationary period, primordial scalar and tensor fluctuations arise due to quantum fluctuations. To characterize the primordial gravitational waves observed today, which are the temporal evolution of primordial tensor fluctuations, we need to characterize other possible sources of gravitational waves. These form the stochastic background of gravitational waves. Primordial scalar fluctuations are possible sources of the background, as they may induce second-order contribution to the tensor perturbations' spectrum.

We have introduced the concept of inflationary quantum fluctuations in the theory section, showing how the quantum fluctuations produce scalar and tensor perturbations (respectively, in Sec. 6.2 and Sec. 7.2). Scalar-induced gravitational waves arise directly by any scalar perturbation of the metric and matter component of the Universe, therefore, we need only to consider relativistic perturbation theory (cf. Sec. 4), to compute them.

In this thesis, we have computed the scalar-induced spectrum of gravitational waves induced by the early Universe and evolved it using Einstein's field equations (theoretically described in Sec. 2.2). The perturbation of the metric has been performed on a Robertson-Walker background (see Sec. 3.1). To do that, we have performed second-order perturbation of the metric and of the fluid quantities, to obtain Einstein's field equations, initially without fixing any gauge condition (see Sec. 8).

The gravitational waves' evolution is given by the pure spatial components of the equations. The wave equation, in the cosmological case, takes the form of Eq. 222, which corresponds to a modified wave equation. Namely, in addition to the d'Alembert operator, we get a term depending on the expansion of the Universe and a source term arising from the perturbation perspective. In this preliminary consideration, we decided to maintain the calculation as general as possible, without choosing any gauge condition. The results obtained confirm the ones by *Noh, and Hwang* [18].

Once we obtain these results, the demand to fix the free parameter rises. Therefore, we showed the gauge transformation of all the quantities playing a role in Einstein's field equations. This general transformation permits us to describe the solution for any gauge. All subsequent calculations are been performed adopting the Conformal Newtonian gauge conditions. This gauge fixes completely the free parameter of the geometric term of the Einstein's field equations. According to this choice, it was possible to find the solutions to the equation of motion for gravitational waves in the radiation- and matter-dominated era and to express the solutions in terms of second-order contribution of scalar perturbations, incorporated in the source term.

Up to this point, except for the choice of Conformal Newtonian gauge, we considered the most general description of the phenomenon, considering the general u_α frame for the energy-momentum tensor. As we have more free parameters than the independent ones of the tensor, we decided to choose the energy frame. In this frame, the energy flux is set to zero. The obtained gravitational wave equation confirms the results shown in *Baumann et al.* [20], *Hwang et al.* [21], *Sipp, and Schäfer* [23], and many other of the literature.

In order to calculate the spectrum of scalar-induced gravitational waves, we used two gauge invariant potentials, the Bardeen potentials. Setting the quadrupole moments of photon and neutrino anisotropies to zero, as they are assumed to be negligibly small, the two Bardeen potentials result to be equal and opposite in sign. According to these considerations, we calculated the spectrum of gravitational waves induced by scalars in two ways. The first is a purely analytical approach, following the procedure proposed by *Baumann et al.* [20]. The second is a numerical approach, employing the potentials obtained from CLASS [138], and assuming a constant potential, as proposed in *Hwang et al.* [21]. Analytically, we considered the Universe in the period of radiation domination and matter domination, while numerically we considered a Universe containing only matter.

From the analytical results, we can observe that the gravitational waves induced by the scalars do not undergo redshift for large and intermediate scales, while they do undergo redshift for small scales. This result comes from approximating the time derivative of the tensor transfer function with $h' \sim kh$. According to *Hwang et al.* [21] this approximation is appropriate in subhorizon scales. Note that this is used in most of the literature.

In the numerical approach, we developed the results both by considering the approximation and by solving the time derivative directly. Contrary to what was expected, the two numerical procedures show different results. The procedure with the approximation, Eq. 322, confirms the results obtained analytically, as shown in Fig. 6. This is due to the fact that we adopt the same approximation for the analytical and numerical approach. On the other hand, the time-derivative procedure, Eq. 321, shows that even at intermediate scales the scalar-induced gravitational waves undergo a redshift. Based on the consideration of *Inomata, and Terada* [136], and *Ali et al.* [137], we have considered a modified approximation, where we excluded the non-oscillating term of Eq. 322 obtaining Eq. 325. The results of this equation confirm the ones found with Eq. 322.

According to *Sipp, and Schäfer* [23], for a matter-dominated Universe, this approximation cannot be adopted since it holds only for free gravitational waves. In other words, since during the matter domination, the Bardeen potential remains constant, the source term is active, and the gravitational waves are not free. Another essential difference between the two procedures is that the scalar-induced gravitational waves' spectrum resulting from the time derivative does not exceed the primordial one (cf. *Sipp, and Schäfer* [23], for similar results). Whereas the spectrum recovered from the approximation exceeds it.

As expected the gravitational waves induced by scalar perturbations do redshift as a radiation fluid a^{-4} Eq. 323, while using the approximation proposed in the literature, they do not behaves like radiation, cf Eq. 324. The modified approximation confirms the result obtained in Eq. 323.

Our considerations are based on a model of the Universe which contains no other component than matter. Therefore, for a more comprehensive analysis, it is necessary to consider a Planck-like Universe [56], i.e. to consider all energy components. In addition, it is certainly interesting to consider sources of cosmological gravitational waves other than scalar-induced gravitational waves. We leave these analyses for future works on these topics.

In this thesis, we have only considered the Conformal Newtonian gauge. An analysis of the results obtained in multiple gauges will enrich the discussion. These considerations will also be explored in more detail in future works.

In conclusion, the main question that emerges from the results is why the approximation proposed in the literature, e.g. *Baumann et al.* [20]; and *Hwang et al.* [21], differs significantly from the results obtained using the time derivative of the transfer function. Therefore, it is more than necessary a closer examination of the reasons which lead to this significant discrepancy of the relative energy density of scalar-induced gravitational waves.

Part V
Appendix

A Linear Order Einstein's Field Equations

As already introduced, at the linear order the scalar, vector, and tensor modes do not mix. Therefore, the Einstein's field equations can be decomposed into the evolution equations of the three kind of perturbations separately [34].

The scalar perturbations evolution equations are:

$$G^0_0: \qquad H\kappa + \frac{\Delta + 3K}{a^2}\varphi = -4\pi G\delta\rho, \tag{326}$$

$$G^0_\alpha: \qquad \kappa + \frac{\Delta + 3K}{a^2}\chi = 12\pi G(\rho + p)av, \tag{327}$$

$$G^\alpha_\alpha - G^0_0: \qquad \dot{\kappa} + 2H\kappa + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\alpha = 4\pi G(\delta\rho + 3\delta p), \tag{328}$$

$$G^\alpha_\beta - \frac{1}{3}\delta^\alpha_\beta G^\gamma_\gamma: \qquad \dot{\chi} + H\chi - \varphi - \alpha = 8\pi G\Pi, \tag{329}$$

$$T^\nu_{0;\nu}: \qquad \delta\dot{\rho} + 3H(\delta\rho + \delta p) - (\rho + p)\left(\kappa - 3H\alpha + \frac{1}{a}\Delta v\right) = 0, \tag{330}$$

$$T^\nu_{\alpha;\nu}: \qquad \frac{\left[a^4(\rho + p)v\right]^{\cdot}}{a^4(\rho + p)} - \frac{1}{a}\alpha - \frac{1}{a(\rho + p)}\left(\delta p + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi\right) = 0. \tag{331}$$

In the previous relations, we used the following definitions

$$\kappa := 3H\alpha - 3\dot{\varphi} - \frac{\Delta}{a^2}\chi, \tag{332}$$

$$\chi := a\beta + a\gamma', \tag{333}$$

$$\Pi_{\alpha\beta} := \frac{1}{a^2}\left(\Pi_{,\alpha|\beta} - \frac{1}{3}\bar{g}_{\alpha\beta}\Delta\Pi\right) + \frac{1}{a}\Pi_{(\alpha|\beta)} + \Pi^{(t)}_{\alpha\beta}. \tag{334}$$

The vector perturbations evolution yields:

$$G^0_\alpha: \qquad \frac{\Delta + 2K}{2a^2}\Psi^{(v)}_\alpha + 8\pi G(\rho + p)v^{(v)}_\alpha = 0, \tag{335}$$

$$G^\alpha_\beta: \qquad \dot{\Psi}^{(v)}_\alpha + 2H\Psi^{(v)}_\alpha = 8\pi G\Pi^{(v)}_\alpha, \tag{336}$$

$$T^\nu_{\alpha;\nu}: \qquad \frac{\left[a^4(\rho + p)v^{(v)}_\alpha\right]^{\cdot}}{a^4(\rho + p)} + \frac{\Delta + 2K}{2a^2}\frac{\Pi^{(v)}_\alpha}{\rho + p} = 0. \tag{337}$$

While the tensor perturbations at the linear order evolve according to:

$$G^\alpha_\beta : \qquad \ddot{C}^{(t)\alpha}_\beta + 3H\dot{C} + \frac{\alpha}{\beta} - \frac{\Delta - 2K}{a^2} C^{(t)\alpha}_\beta = 8\pi G \Pi^{(t)\alpha}_\beta \tag{338}$$

For our purposes, we are interested in the scalars evolution in a spatially flat Universe $K = 0$, and choosing the conformal Newtonian gauge (zero-shear gauge): $\chi = 0$, $\beta = 0 = \gamma$.

B Second Order Differential Geometry

B.1 Christoffel Symbols

From the definition in Eq. 2, we recover all the Christoffel symbols up to second-order in perturbation

$$\begin{aligned}
\Gamma^0_{00} =& \mathcal{H} + A' - 2AA' - A_{,\alpha}B^\alpha + B_\alpha \left(B^{\alpha\prime} + \frac{a'}{a} B^\alpha \right), \\
\Gamma^0_{0\alpha} =& A_{,\alpha} - \mathcal{H}B_\alpha - 2AA_{,\alpha} + 2\mathcal{H}AB_\alpha - B_\beta C^{\beta\prime}_\alpha + B^\beta B_{[\beta|\alpha]}, \\
\Gamma^\alpha_{00} =& A^{|\alpha} - B^{\alpha\prime} - \mathcal{H}B^\alpha + A'B^\alpha - 2A_{,\beta}C^{\alpha\beta} + 2C^\alpha_\beta \left(B^{\beta\prime} + \mathcal{H}B^\beta \right), \\
\Gamma^0_{\alpha\beta} =& \mathcal{H}(1 - 2A)\bar{g}_{\alpha\beta} + B_{(\alpha|\beta)} + C'_{\alpha\beta} + 2\mathcal{H}C_{\alpha\beta} + \mathcal{H}\bar{g}_{\alpha\beta} \left(4A^2 - B_\gamma B^\gamma \right) - \\
& - 2A \left(B_{(\alpha|\beta)} + C'_{\alpha\beta} + 2\mathcal{H}C_{\alpha\beta} \right) - B_\gamma \left(2C^\gamma_{(\alpha|\beta)} - C^{|\gamma}_{\alpha\beta} \right), \\
\Gamma^\alpha_{0\beta} =& \mathcal{H}\delta^\alpha_\beta + \frac{1}{2} \left(B^{|\alpha}_\beta - B^\alpha_{|\beta} \right) + C^{\alpha\prime}_\beta + B^\alpha \left(A_{,\beta} - \mathcal{H}B_\beta \right) + 2C^{\alpha\gamma} \left(B_{[\gamma|\beta]} - C'_{\gamma\beta} \right), \\
\Gamma^\alpha_{\beta\gamma} =& \bar{\Gamma}^\alpha_{\beta\gamma} + \mathcal{H}\bar{g}_{\beta\gamma}B^\alpha + 2C^\alpha_{(\beta|\gamma)} - C^{|\alpha}_{\beta\gamma} - 2C^\alpha_\delta \left(2C^\delta_{(\beta|\gamma)} - C^{|\delta}_{\beta\gamma} \right) - \\
& - 2\mathcal{H}\bar{g}_{\gamma\beta} \left(AB^\alpha + B^\delta C^\alpha_\delta \right) + B^\alpha \left(B_{(\beta|\gamma)} + C'_{\beta\gamma} + 2\mathcal{H}C_{\beta\gamma} \right),
\end{aligned} \tag{339}$$

B.2 Riemann Tensor

From the definition in Eq. 3, we recover the Riemann tensor up to second-order in perturbation. It is really useful to recall the (anti-)symmetries of the Riemann curvature tensor:

$$\text{Symmetry:} \quad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}, \tag{340}$$

$$\text{Antisymmetry:} \quad R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} \ \textbf{, and} \ R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}. \tag{341}$$

Therefore, we find

$$R_{0000} = 0 \tag{342}$$

$$R_{\alpha000} = 0 \tag{343}$$

$$R_{\alpha\beta00} = 0 \tag{344}$$

$$
\begin{aligned}
R_{\alpha 0\gamma 0} =& a^2\Big[-\bar{g}_{\alpha\gamma}\mathcal{H}' + {}^{(1)}(-C''_{\alpha\gamma} - C'{}_{\alpha\gamma}\mathcal{H} + A'\bar{g}_{\alpha\gamma}\mathcal{H} - 2C_{\alpha\gamma}\mathcal{H}' - \frac{1}{2}\mathcal{H}\nabla_\alpha B_\gamma \\
&- \frac{1}{2}\nabla_\alpha B'{}_\gamma - \frac{1}{2}\mathcal{H}\nabla_\gamma B_\alpha - \frac{1}{2}\nabla_\gamma B'{}_\alpha + \nabla_\gamma\nabla_\alpha A) + {}^{(2)}(A'C'{}_{\alpha\gamma} \\
&+ C'{}_\alpha{}^\beta C'{}_{\gamma\beta} - \frac{1}{2}C''{}_{\alpha\gamma} + 2A'C_{\alpha\gamma}\mathcal{H} - \frac{1}{2}C'{}_{\alpha\gamma}\mathcal{H} - 2AA'\bar{g}_{\alpha\gamma}\mathcal{H} \\
&+ \frac{1}{2}A'\bar{g}_{\alpha\gamma}\mathcal{H} + B^\beta B'{}_\beta\bar{g}_{\alpha\gamma}\mathcal{H} - B_\alpha B_\gamma\mathcal{H}^2 + B_\beta B^\beta\bar{g}_{\alpha\gamma}\mathcal{H}^2 \\
&- C_{\alpha\gamma}\mathcal{H}' + B_\gamma\mathcal{H}\nabla_\alpha A - \frac{1}{2}C'{}_{\gamma\beta}\nabla_\alpha B^\beta + \frac{1}{2}A'\nabla_\alpha B_\gamma - \frac{1}{4}\mathcal{H}\nabla_\alpha B_\gamma \\
&- \frac{1}{4}\nabla_\alpha B'{}_\gamma + B'^\beta\nabla_\alpha C_{\gamma\beta} + B^\beta\mathcal{H}\nabla_\alpha C_{\gamma\beta} - B^\beta\bar{g}_{\alpha\gamma}\mathcal{H}\nabla_\beta A - B'^\beta\nabla_\beta C_{\alpha\gamma} \\
&- B^\beta\mathcal{H}\nabla_\beta C_{\alpha\gamma} - \nabla_\alpha C_{\gamma\beta}\nabla^\beta A + \nabla_\beta C_{\alpha\gamma}\nabla^\beta A + \frac{1}{2}C'{}_{\gamma\beta}\nabla^\beta B_\alpha \\
&+ \frac{1}{4}\nabla_\beta B_\gamma\nabla^\beta B_\alpha + \frac{1}{2}C'{}_{\alpha\beta}\nabla^\beta B_\gamma - \frac{1}{4}\nabla_\alpha B_\beta\nabla^\beta B_\gamma + B_\alpha\mathcal{H}\nabla_\gamma A - \nabla_\alpha A\nabla_\gamma A \\
&+ \frac{1}{2}A'\nabla_\gamma B_\alpha + \frac{1}{4}\nabla_\alpha B^\beta\nabla_\gamma B_\beta - \frac{1}{4}\nabla^\beta B_\alpha\nabla_\gamma B_\beta - \frac{1}{2}C'{}_{\alpha\beta}\nabla_\gamma B^\beta - \frac{1}{4}\mathcal{H}\nabla_\gamma B_\alpha \\
&- \frac{1}{4}\nabla_\gamma B'{}_\alpha + B'^\beta\nabla_\gamma C_{\alpha\beta} + B^\beta\mathcal{H}\nabla_\gamma C_{\alpha\beta} - \nabla^\beta A\nabla_\gamma C_{\alpha\beta} + \frac{1}{2}\nabla_\gamma\nabla_\alpha A)(A'C'{}_{\alpha\gamma} \\
&+ C'{}_\alpha{}^\beta C'{}_{\gamma\beta} - \frac{1}{2}C''{}_{\alpha\gamma} + 2A'C_{\alpha\gamma}\mathcal{H} - \frac{1}{2}C'{}_{\alpha\gamma}\mathcal{H} - 2AA'\bar{g}_{\alpha\gamma}\mathcal{H} \\
&+ \frac{1}{2}A'\bar{g}_{\alpha\gamma}\mathcal{H} + B^\beta B'{}_\beta\bar{g}_{\alpha\gamma}\mathcal{H} - B_\alpha B_\gamma\mathcal{H}^2 + B_\beta B^\beta\bar{g}_{\alpha\gamma}\mathcal{H}^2 \\
&- C_{\alpha\gamma}\mathcal{H}' + B_\gamma\mathcal{H}\nabla_\alpha A - \frac{1}{2}C'{}_{\gamma\beta}\nabla_\alpha B^\beta + \frac{1}{2}A'\nabla_\alpha B_\gamma - \frac{1}{4}\mathcal{H}\nabla_\alpha B_\gamma \\
&- \frac{1}{4}\nabla_\alpha B'{}_\gamma + B'^\beta\nabla_\alpha C_{\gamma\beta} + B^\beta\mathcal{H}\nabla_\alpha C_{\gamma\beta} - B^\beta\bar{g}_{\alpha\gamma}\mathcal{H}\nabla_\beta A - B'^\beta\nabla_\beta C_{\alpha\gamma} \\
&- B^\beta\mathcal{H}\nabla_\beta C_{\alpha\gamma} - \nabla_\alpha C_{\gamma\beta}\nabla^\beta A + \nabla_\beta C_{\alpha\gamma}\nabla^\beta A + \frac{1}{2}C'{}_{\gamma\beta}\nabla^\beta B_\alpha \\
&+ \frac{1}{4}\nabla_\beta B_\gamma\nabla^\beta B_\alpha + \frac{1}{2}C'{}_{\alpha\beta}\nabla^\beta B_\gamma - \frac{1}{4}\nabla_\alpha B_\beta\nabla^\beta B_\gamma + B_\alpha\mathcal{H}\nabla_\gamma A - \nabla_\alpha A\nabla_\gamma A \\
&+ \frac{1}{2}A'\nabla_\gamma B_\alpha + \frac{1}{4}\nabla_\alpha B^\beta\nabla_\gamma B_\beta - \frac{1}{4}\nabla^\beta B_\alpha\nabla_\gamma B_\beta - \frac{1}{2}C'{}_{\alpha\beta}\nabla_\gamma B^\beta - \frac{1}{4}\mathcal{H}\nabla_\gamma B_\alpha \\
&- \frac{1}{4}\nabla_\gamma B'{}_\alpha + B'^\beta\nabla_\gamma C_{\alpha\beta} + B^\beta\mathcal{H}\nabla_\gamma C_{\alpha\beta} - \nabla^\beta A\nabla_\gamma C_{\alpha\beta} + \frac{1}{2}\nabla_\gamma\nabla_\alpha A)\Big]
\end{aligned}
\tag{345}
$$

$$
\begin{aligned}
R_{0\beta\gamma\delta} =& a^2\Big[{}^{(1)}\Big(B_\delta \bar{g}_{\beta\gamma}\mathcal{H}^2 - B_\gamma \bar{g}_{\beta\delta}\mathcal{H}^2 + \bar{g}_{\beta\delta}\mathcal{H}\nabla_\gamma A - \nabla_\gamma C'_{\beta\delta} - \frac{1}{2}\nabla_\gamma\nabla_\beta B_\delta - \bar{g}_{\beta\gamma}\mathcal{H}\nabla_\delta A \\
&+ \nabla_\delta C'_{\beta\gamma} + \frac{1}{2}\nabla_\delta\nabla_\beta B_\gamma\Big) + {}^{(2)}\Big(B_\delta C'_{\beta\gamma}\mathcal{H} - B_\gamma C'_{\beta\delta}\mathcal{H} + B^\alpha C'_{\delta\alpha}\bar{g}_{\beta\gamma}\mathcal{H} \\
&- B^\alpha C'_{\gamma\alpha}\bar{g}_{\beta\delta}\mathcal{H} + 2B_\delta C_{\beta\gamma}\mathcal{H}^2 - 2B_\gamma C_{\beta\delta}\mathcal{H}^2 - 2AB_\delta\bar{g}_{\beta\gamma}\mathcal{H}^2 + \frac{1}{2}B_\delta\bar{g}_{\beta\gamma}\mathcal{H}^2 \\
&+ 2AB_\gamma\bar{g}_{\beta\delta}\mathcal{H}^2 - \frac{1}{2}B_\gamma\bar{g}_{\beta\delta}\mathcal{H}^2 - \frac{1}{2}B^\alpha\bar{g}_{\beta\delta}\mathcal{H}\nabla_\alpha B_\gamma + \frac{1}{2}B^\alpha\bar{g}_{\beta\gamma}\mathcal{H}\nabla_\alpha B_\delta - C'_\delta{}^\alpha\nabla_\alpha C_{\beta\gamma} \\
&+ C'_\gamma{}^\alpha\nabla_\alpha C_{\beta\delta} + \frac{1}{2}\nabla_\alpha C_{\beta\delta}\nabla^\alpha B_\gamma - \frac{1}{2}\nabla_\alpha C_{\beta\gamma}\nabla^\alpha B_\delta + \frac{1}{2}B_\delta\mathcal{H}\nabla_\beta B_\gamma - \frac{1}{2}B_\gamma\mathcal{H}\nabla_\beta B_\delta \\
&+ C'_\delta{}^\alpha\nabla_\beta C_{\gamma\alpha} + \frac{1}{2}\nabla^\alpha B_\delta\nabla_\beta C_{\gamma\alpha} - C'_\gamma{}^\alpha\nabla_\beta C_{\delta\alpha} - \frac{1}{2}\nabla^\alpha B_\gamma\nabla_\beta C_{\delta\alpha} + C'_{\beta\delta}\nabla_\gamma A \\
&+ 2C_{\beta\delta}\mathcal{H}\nabla_\gamma A - 2A\bar{g}_{\beta\delta}\mathcal{H}\nabla_\gamma A + \frac{1}{2}\nabla_\beta B_\delta\nabla_\gamma A + \frac{1}{2}\bar{g}_{\beta\delta}\mathcal{H}\nabla_\gamma A + \frac{1}{2}B^\alpha\bar{g}_{\beta\delta}\mathcal{H}\nabla_\gamma B_\alpha \\
&- \frac{1}{2}\nabla_\alpha C_{\beta\delta}\nabla_\gamma B^\alpha + \frac{1}{2}\nabla_\beta C_{\delta\alpha}\nabla_\gamma B^\alpha + \frac{1}{2}B_\delta\mathcal{H}\nabla_\gamma B_\beta + C'_\delta{}^\alpha\nabla_\gamma C_{\beta\alpha} + \frac{1}{2}\nabla^\alpha B_\delta\nabla_\gamma C_{\beta\alpha} \\
&- \frac{1}{2}\nabla_\gamma C'_{\beta\delta} - \frac{1}{4}\nabla_\gamma\nabla_\beta B_\delta - C'_{\beta\gamma}\nabla_\delta A - 2C_{\beta\gamma}\mathcal{H}\nabla_\delta A + 2A\bar{g}_{\beta\gamma}\mathcal{H}\nabla_\delta A - \frac{1}{2}\nabla_\beta B_\gamma\nabla_\delta A \\
&- \frac{1}{2}\nabla_\gamma B_\beta\nabla_\delta A - \frac{1}{2}\bar{g}_{\beta\gamma}\mathcal{H}\nabla_\delta A - \frac{1}{2}B^\alpha\bar{g}_{\beta\gamma}\mathcal{H}\nabla_\delta B_\alpha + \frac{1}{2}\nabla_\alpha C_{\beta\gamma}\nabla_\delta B^\alpha \\
&- \frac{1}{2}\nabla_\beta C_{\gamma\alpha}\nabla_\delta B^\alpha - \frac{1}{2}\nabla_\gamma C_{\beta\alpha}\nabla_\delta B^\alpha - \frac{1}{2}B_\gamma\mathcal{H}\nabla_\delta B_\beta + \frac{1}{2}\nabla_\gamma A\nabla_\delta B_\beta - C'_\gamma{}^\alpha\nabla_\delta C_{\beta\alpha} \\
&- \frac{1}{2}\nabla^\alpha B_\gamma\nabla_\delta C_{\beta\alpha} + \frac{1}{2}\nabla_\gamma B^\alpha\nabla_\delta C_{\beta\alpha} + \frac{1}{2}\nabla_\delta C'_{\beta\gamma} + \frac{1}{4}\nabla_\delta\nabla_\beta B_\gamma\Big)\Big]
\end{aligned}
\tag{346}
$$

$$
\begin{aligned}
R_{\alpha\beta\gamma\delta} =& a^2\Big[-\bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\mathcal{H}^2 + \bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta}\mathcal{H}^2 + {}^{(1)}\Big(C'_{\beta\delta}\bar{g}_{\alpha\gamma}\mathcal{H} - C'_{\beta\gamma}\bar{g}_{\alpha\delta}\mathcal{H} - C'_{\alpha\delta}\bar{g}_{\beta\gamma}\mathcal{H} + C'_{\alpha\gamma}\bar{g}_{\beta\delta}\mathcal{H} \\
&+ 2C_{\beta\delta}\bar{g}_{\alpha\gamma}\mathcal{H}^2 - 2C_{\beta\gamma}\bar{g}_{\alpha\delta}\mathcal{H}^2 - 2C_{\alpha\delta}\bar{g}_{\beta\gamma}\mathcal{H}^2 + 2A\bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\mathcal{H}^2 + 2C_{\alpha\gamma}\bar{g}_{\beta\delta}\mathcal{H}^2 \\
&- 2A\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta}\mathcal{H}^2 + \frac{1}{2}\bar{g}_{\beta\delta}\mathcal{H}\nabla_\alpha B_\gamma - \frac{1}{2}\bar{g}_{\beta\gamma}\mathcal{H}\nabla_\alpha B_\delta - \frac{1}{2}\bar{g}_{\alpha\delta}\mathcal{H}\nabla_\beta B_\gamma + \frac{1}{2}\bar{g}_{\alpha\gamma}\mathcal{H}\nabla_\beta B_\delta \\
&+ \frac{1}{2}\bar{g}_{\beta\delta}\mathcal{H}\nabla_\gamma B_\alpha - \frac{1}{2}\bar{g}_{\alpha\delta}\mathcal{H}\nabla_\gamma B_\beta - \nabla_\gamma\nabla_\alpha C_{\beta\delta} + \nabla_\gamma\nabla_\beta C_{\alpha\delta} - \frac{1}{2}\bar{g}_{\beta\gamma}\mathcal{H}\nabla_\delta B_\alpha \\
&+ \frac{1}{2}\bar{g}_{\alpha\gamma}\mathcal{H}\nabla_\delta B_\beta + \nabla_\delta\nabla_\alpha C_{\beta\gamma} - \nabla_\delta\nabla_\beta C_{\alpha\gamma}\Big) + {}^{(2)}(-C'_{\alpha\delta}C'_{\beta\gamma} + C'_{\alpha\gamma}C'_{\beta\delta} \\
&+ 2C_{\beta\delta}C'_{\alpha\gamma}\mathcal{H} - 2C_{\beta\gamma}C'_{\alpha\delta}\mathcal{H} - 2C_{\alpha\delta}C'_{\beta\gamma}\mathcal{H} + 2C_{\alpha\gamma}C'_{\beta\delta}\mathcal{H} - 2AC'_{\beta\delta}\bar{g}_{\alpha\gamma}\mathcal{H} \\
&+ \frac{1}{2}C'_{\beta\delta}\bar{g}_{\alpha\gamma}\mathcal{H} + 2AC'_{\beta\gamma}\bar{g}_{\alpha\delta}\mathcal{H} - \frac{1}{2}C'_{\beta\gamma}\bar{g}_{\alpha\delta}\mathcal{H} + 2AC'_{\alpha\delta}\bar{g}_{\beta\gamma}\mathcal{H} - \frac{1}{2}C'_{\alpha\delta}\bar{g}_{\beta\gamma}\mathcal{H}
\end{aligned}
$$

$$
\begin{aligned}
&- 2AC'{}_{\alpha\gamma}\bar{g}_{\beta\delta}\mathcal{H} + \frac{1}{2}C'{}_{\alpha\gamma}\bar{g}_{\beta\delta}\mathcal{H} - 4C_{\alpha\delta}C_{\beta\gamma}\mathcal{H}^2 + 4C_{\alpha\gamma}C_{\beta\delta}\mathcal{H}^2 - 4AC_{\beta\delta}\bar{g}_{\alpha\gamma}\mathcal{H}^2 \\
&+ C_{\beta\delta}\bar{g}_{\alpha\gamma}\mathcal{H}^2 + 4AC_{\beta\gamma}\bar{g}_{\alpha\delta}\mathcal{H}^2 - C_{\beta\gamma}\bar{g}_{\alpha\delta}\mathcal{H}^2 + 4AC_{\alpha\delta}\bar{g}_{\beta\gamma}\mathcal{H}^2 - C_{\alpha\delta}\bar{g}_{\beta\gamma}\mathcal{H}^2 \\
&- 4A^2\bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\mathcal{H}^2 + A\bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\mathcal{H}^2 + B_\varepsilon B^\varepsilon\bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\mathcal{H}^2 - 4AC_{\alpha\gamma}\bar{g}_{\beta\delta}\mathcal{H}^2 + C_{\alpha\gamma}\bar{g}_{\beta\delta}\mathcal{H}^2 \\
&+ 4A^2\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta}\mathcal{H}^2 - A\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta}\mathcal{H}^2 - B_\varepsilon B^\varepsilon\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta}\mathcal{H}^2 + \frac{1}{2}C'{}_{\beta\delta}\nabla_\alpha B_\gamma + C_{\beta\delta}\mathcal{H}\nabla_\alpha B_\gamma \\
&- A\bar{g}_{\beta\delta}\mathcal{H}\nabla_\alpha B_\gamma - \frac{1}{2}C'{}_{\beta\gamma}\nabla_\alpha B_\delta - C_{\beta\gamma}\mathcal{H}\nabla_\alpha B_\delta + A\bar{g}_{\beta\gamma}\mathcal{H}\nabla_\alpha B_\delta + \frac{1}{4}\bar{g}_{\beta\delta}\mathcal{H}\nabla_\alpha B_\gamma \\
&- \frac{1}{4}\bar{g}_{\beta\gamma}\mathcal{H}\nabla_\alpha B_\delta - B^\varepsilon\bar{g}_{\beta\delta}\mathcal{H}\nabla_\alpha C_{\gamma\varepsilon} + B^\varepsilon\bar{g}_{\beta\gamma}\mathcal{H}\nabla_\alpha C_{\delta\varepsilon} - \frac{1}{2}C'{}_{\alpha\delta}\nabla_\beta B_\gamma - C_{\alpha\delta}\mathcal{H}\nabla_\beta B_\gamma \\
&+ A\bar{g}_{\alpha\delta}\mathcal{H}\nabla_\beta B_\gamma - \frac{1}{4}\nabla_\alpha B_\delta\nabla_\beta B_\gamma + \frac{1}{2}C'{}_{\alpha\gamma}\nabla_\beta B_\delta + C_{\alpha\gamma}\mathcal{H}\nabla_\beta B_\delta - A\bar{g}_{\alpha\gamma}\mathcal{H}\nabla_\beta B_\delta \\
&+ \frac{1}{4}\nabla_\alpha B_\gamma\nabla_\beta B_\delta - \frac{1}{4}\bar{g}_{\alpha\delta}\mathcal{H}\nabla_\beta B_\gamma + \frac{1}{4}\bar{g}_{\alpha\gamma}\mathcal{H}\nabla_\beta B_\delta + B^\varepsilon\bar{g}_{\alpha\delta}\mathcal{H}\nabla_\beta C_{\gamma\varepsilon} + \nabla_\alpha C_{\delta\varepsilon}\nabla_\beta C_\gamma{}^\varepsilon \\
&- B^\varepsilon\bar{g}_{\alpha\gamma}\mathcal{H}\nabla_\beta C_{\delta\varepsilon} - \nabla_\alpha C_\gamma{}^\varepsilon\nabla_\beta C_{\delta\varepsilon} + \frac{1}{2}C'{}_{\beta\delta}\nabla_\gamma B_\alpha + C_{\beta\delta}\mathcal{H}\nabla_\gamma B_\alpha - A\bar{g}_{\beta\delta}\mathcal{H}\nabla_\gamma B_\alpha \\
&+ \frac{1}{4}\nabla_\beta B_\delta\nabla_\gamma B_\alpha - \frac{1}{2}C'{}_{\alpha\delta}\nabla_\gamma B_\beta - C_{\alpha\delta}\mathcal{H}\nabla_\gamma B_\beta + A\bar{g}_{\alpha\delta}\mathcal{H}\nabla_\gamma B_\beta - \frac{1}{4}\nabla_\alpha B_\delta\nabla_\gamma B_\beta \\
&+ \frac{1}{4}\bar{g}_{\beta\delta}\mathcal{H}\nabla_\gamma B_\alpha - \frac{1}{4}\bar{g}_{\alpha\delta}\mathcal{H}\nabla_\gamma B_\beta - B^\varepsilon\bar{g}_{\beta\delta}\mathcal{H}\nabla_\gamma C_{\alpha\varepsilon} - \nabla_\beta C_{\delta\varepsilon}\nabla_\gamma C_\alpha{}^\varepsilon \\
&+ B^\varepsilon\bar{g}_{\alpha\delta}\mathcal{H}\nabla_\gamma C_{\beta\varepsilon} + \nabla_\alpha C_{\delta\varepsilon}\nabla_\gamma C_\beta{}^\varepsilon - \frac{1}{2}\nabla_\gamma\nabla_\alpha C_{\beta\delta} + \frac{1}{2}\nabla_\gamma\nabla_\beta C_{\alpha\delta} - \frac{1}{2}C'{}_{\beta\gamma}\nabla_\delta B_\alpha \\
&- C_{\beta\gamma}\mathcal{H}\nabla_\delta B_\alpha + A\bar{g}_{\beta\gamma}\mathcal{H}\nabla_\delta B_\alpha - \frac{1}{4}\nabla_\beta B_\gamma\nabla_\delta B_\alpha - \frac{1}{4}\nabla_\gamma B_\beta\nabla_\delta B_\alpha + \frac{1}{2}C'{}_{\alpha\gamma}\nabla_\delta B_\beta \\
&+ C_{\alpha\gamma}\mathcal{H}\nabla_\delta B_\beta - A\bar{g}_{\alpha\gamma}\mathcal{H}\nabla_\delta B_\beta + \frac{1}{4}\nabla_\alpha B_\gamma\nabla_\delta B_\beta + \frac{1}{4}\nabla_\gamma B_\alpha\nabla_\delta B_\beta - \frac{1}{4}\bar{g}_{\beta\gamma}\mathcal{H}\nabla_\delta B_\alpha \\
&+ \frac{1}{4}\bar{g}_{\alpha\gamma}\mathcal{H}\nabla_\delta B_\beta + B^\varepsilon\bar{g}_{\beta\gamma}\mathcal{H}\nabla_\delta C_{\alpha\varepsilon} + \nabla_\beta C_{\gamma\varepsilon}\nabla_\delta C_\alpha{}^\varepsilon + \nabla_\gamma C_{\beta\varepsilon}\nabla_\delta C_\alpha{}^\varepsilon - B^\varepsilon\bar{g}_{\alpha\gamma}\mathcal{H}\nabla_\delta C_{\beta\varepsilon} \\
&- \nabla_\gamma C_\alpha{}^\varepsilon\nabla_\delta C_{\beta\varepsilon} - \nabla_\alpha C_{\gamma\varepsilon}\nabla_\delta C_\beta{}^\varepsilon + \frac{1}{2}\nabla_\delta\nabla_\alpha C_{\beta\gamma} - \frac{1}{2}\nabla_\delta\nabla_\beta C_{\alpha\gamma} + B^\varepsilon\bar{g}_{\beta\delta}\mathcal{H}\nabla_\varepsilon C_{\alpha\gamma} \\
&- B^\varepsilon\bar{g}_{\beta\gamma}\mathcal{H}\nabla_\varepsilon C_{\alpha\delta} - B^\varepsilon\bar{g}_{\alpha\delta}\mathcal{H}\nabla_\varepsilon C_{\beta\gamma} - \nabla_\delta C_\alpha{}^\varepsilon\nabla_\varepsilon C_{\beta\gamma} + B^\varepsilon\bar{g}_{\alpha\gamma}\mathcal{H}\nabla_\varepsilon C_{\beta\delta} \\
&+ \nabla_\gamma C_\alpha{}^\varepsilon\nabla_\varepsilon C_{\beta\delta} + \nabla_\beta C_{\delta\varepsilon}\nabla^\varepsilon C_{\alpha\gamma} + \nabla_\delta C_{\beta\varepsilon}\nabla^\varepsilon C_{\alpha\gamma} - \nabla_\varepsilon C_{\beta\delta}\nabla^\varepsilon C_{\alpha\gamma} \\
&- \nabla_\beta C_{\gamma\varepsilon}\nabla^\varepsilon C_{\alpha\delta} - \nabla_\gamma C_{\beta\varepsilon}\nabla^\varepsilon C_{\alpha\delta} + \nabla_\varepsilon C_{\beta\gamma}\nabla^\varepsilon C_{\alpha\delta} - \nabla_\alpha C_{\delta\varepsilon}\nabla^\varepsilon C_{\beta\gamma} + \nabla_\alpha C_{\gamma\varepsilon}\nabla^\varepsilon C_{\beta\delta}\Big)\Big]
\end{aligned}
\tag{347}
$$

B.3 Ricci Tensor and Scalar

$$R_{00} = -3\mathcal{H}' + {}^{(1)}(-C''^{\alpha}{}_{\alpha} + 3A'\mathcal{H} - C'^{\alpha}{}_{\alpha}\mathcal{H} - \mathcal{H}\nabla_\alpha B^\alpha - \nabla_\alpha B'^\alpha + \nabla_\alpha\nabla^\alpha A) + {}^{(2)}(A'C'^{\alpha}{}_{\alpha}$$
$$+ C'_{\alpha\beta}C'^{\alpha\beta} + 2C^{\alpha\beta}C''_{\alpha\beta} - \frac{1}{2}C''^{\alpha}{}_{\alpha} - 6AA'\mathcal{H} + \frac{3}{2}A'\mathcal{H} + 3B^\alpha B'_\alpha\mathcal{H} + 2C^{\alpha\beta}C'_{\alpha\beta}\mathcal{H}$$
$$- \frac{1}{2}C'^{\alpha}{}_{\alpha}\mathcal{H} + 2B_\alpha B^\alpha\mathcal{H}^2 + B_\alpha B^\alpha\mathcal{H}' - B^\alpha\mathcal{H}\nabla_\alpha A + A'\nabla_\alpha B^\alpha - \frac{1}{2}\mathcal{H}\nabla_\alpha B^\alpha$$
$$- \frac{1}{2}\nabla_\alpha B'^\alpha - B'^\alpha\nabla_\alpha C^\beta{}_\beta - B^\alpha\mathcal{H}\nabla_\alpha C^\beta{}_\beta + \frac{1}{2}\nabla_\alpha\nabla^\alpha A - \nabla_\alpha A\nabla^\alpha A + \nabla_\alpha C^\beta{}_\beta\nabla^\alpha A$$
$$+ 2B'^\alpha\nabla_\beta C_\alpha{}^\beta + 2B^\alpha\mathcal{H}\nabla_\beta C_\alpha{}^\beta - 2\nabla^\alpha A\nabla_\beta C_\alpha{}^\beta + 2C_{\alpha\beta}\mathcal{H}\nabla^\beta B^\alpha - \frac{1}{2}\nabla_\alpha B_\beta\nabla^\beta B^\alpha$$
$$+ \frac{1}{2}\nabla_\beta B_\alpha\nabla^\beta B^\alpha + 2C_{\alpha\beta}\nabla^\beta B'^\alpha - 2C_{\alpha\beta}\nabla^\beta\nabla^\alpha A) \tag{348}$$

$$R_{0\alpha} = {}^{(1)}(-2B_\beta\mathcal{H}^2 - B_\beta\mathcal{H}' + \nabla_\alpha C'_\beta{}^\alpha + \frac{1}{2}\nabla_\alpha\nabla^\alpha B_\beta - \frac{1}{2}\nabla_\alpha\nabla_\beta B^\alpha + 2\mathcal{H}\nabla_\beta A - \nabla_\beta C'^{\alpha}{}_{\alpha})$$
$$+ {}^{(2)}(-B^\alpha C''_{\beta\alpha} + A'B_\beta\mathcal{H} - B_\beta C'^{\alpha}{}_{\alpha}\mathcal{H} - 2B^\alpha C'_{\beta\alpha}\mathcal{H} + 4AB_\beta\mathcal{H}^2 - B_\beta\mathcal{H}^2$$
$$+ 2AB_\beta\mathcal{H}' - \frac{1}{2}B_\beta\mathcal{H}' - B_\beta\mathcal{H}\nabla_\alpha B^\alpha - B^\alpha\mathcal{H}\nabla_\alpha B_\beta - \frac{1}{2}B^\alpha\nabla_\alpha B'_\beta + C'_\beta{}^\alpha\nabla_\alpha C^\gamma{}_\gamma$$
$$+ \frac{1}{2}\nabla_\alpha C'_\beta{}^\alpha + \frac{1}{4}\nabla_\alpha\nabla^\alpha B_\beta + B^\alpha\nabla_\alpha\nabla_\beta A - \frac{1}{4}\nabla_\alpha\nabla_\beta B^\alpha - C'_{\beta\alpha}\nabla^\alpha A$$
$$- \frac{1}{2}\nabla_\alpha B_\beta\nabla^\alpha A + \frac{1}{2}\nabla_\alpha C^\gamma{}_\gamma\nabla^\alpha B_\beta + C'^{\alpha}{}_{\alpha}\nabla_\beta A - 4A\mathcal{H}\nabla_\beta A + \nabla_\alpha B^\alpha\nabla_\beta A + \mathcal{H}\nabla_\beta A$$
$$+ B^\alpha\mathcal{H}\nabla_\beta B_\alpha - \frac{1}{2}\nabla^\alpha A\nabla_\beta B_\alpha - \frac{1}{2}\nabla_\alpha C^\gamma{}_\gamma\nabla_\beta B^\alpha - \frac{1}{2}B^\alpha\nabla_\beta B'_\alpha + C'^{\alpha\gamma}\nabla_\beta C_{\alpha\gamma}$$
$$+ 2C^{\alpha\gamma}\nabla_\beta C'_{\alpha\gamma} - \frac{1}{2}\nabla_\beta C'^{\alpha}{}_{\alpha} - 2C'_\beta{}^\alpha\nabla_\gamma C_\alpha{}^\gamma - \nabla^\alpha B_\beta\nabla_\gamma C_\alpha{}^\gamma + \nabla_\beta B^\alpha\nabla_\gamma C_\alpha{}^\gamma$$
$$- 2C^{\alpha\gamma}\nabla_\gamma C'_{\beta\alpha} + \nabla_\alpha C_{\beta\gamma}\nabla^\gamma B^\alpha - \nabla_\gamma C_{\beta\alpha}\nabla^\gamma B^\alpha - C_{\alpha\gamma}\nabla^\gamma\nabla^\alpha B_\beta + C_{\alpha\gamma}\nabla^\gamma\nabla_\beta B^\alpha) \tag{349}$$

$$R_{\alpha\beta} = 2\bar{h}_{\alpha\beta}\mathcal{H}^2 + \bar{h}_{\alpha\beta}\mathcal{H}' + {}^{(1)}(C''_{\alpha\beta} + 2C'_{\alpha\beta}\mathcal{H} - A'\bar{h}_{\alpha\beta}\mathcal{H} + C'^{\gamma}{}_{\gamma}\bar{h}_{\alpha\beta}\mathcal{H} + 4C_{\alpha\beta}\mathcal{H}^2$$
$$- 4A\bar{h}_{\alpha\beta}\mathcal{H}^2 + 2C_{\alpha\beta}\mathcal{H}' - 2A\bar{h}_{\alpha\beta}\mathcal{H}' + \mathcal{H}\nabla_\alpha B_\beta + \frac{1}{2}\nabla_\alpha B'_\beta + \mathcal{H}\nabla_\beta B_\alpha + \frac{1}{2}\nabla_\beta B'_\alpha$$
$$- \nabla_\beta\nabla_\alpha A - \nabla_\beta\nabla_\alpha C^\gamma{}_\gamma + \bar{h}_{\alpha\beta}\mathcal{H}\nabla_\gamma B^\gamma + \nabla_\gamma\nabla_\alpha C_\beta{}^\gamma + \nabla_\gamma\nabla_\beta C_\alpha{}^\gamma - \nabla_\gamma\nabla^\gamma C_{\alpha\beta})$$
$$+ {}^{(2)}(-A'C'_{\alpha\beta} - 2C'_\alpha{}^\gamma C'_{\beta\gamma} + C'_{\alpha\beta}C'^{\gamma}{}_{\gamma} - 2AC''_{\alpha\beta} + \frac{1}{2}C''_{\alpha\beta} - 2A'C_{\alpha\beta}\mathcal{H}$$

$$
\begin{aligned}
&- 4AC'_{\alpha\beta}\mathcal{H} + 2C_{\alpha\beta}C'^{\gamma}{}_{\gamma}\mathcal{H} + C'_{\alpha\beta}\mathcal{H} + 4AA'\bar{h}_{\alpha\beta}\mathcal{H} - \frac{1}{2}A'\bar{h}_{\alpha\beta}\mathcal{H} - B^{\gamma}B'_{\gamma}\bar{h}_{\alpha\beta}\mathcal{H} \\
&- 2C^{\gamma\delta}C'_{\gamma\delta}\bar{h}_{\alpha\beta}\mathcal{H} - 2AC'^{\gamma}{}_{\gamma}\bar{h}_{\alpha\beta}\mathcal{H} + \frac{1}{2}C'^{\gamma}{}_{\gamma}\bar{h}_{\alpha\beta}\mathcal{H} - 8AC_{\alpha\beta}\mathcal{H}^2 + 2C_{\alpha\beta}\mathcal{H}^2 \\
&+ 8A^2\bar{h}_{\alpha\beta}\mathcal{H}^2 - 2A\bar{h}_{\alpha\beta}\mathcal{H}^2 - 2B_{\gamma}B^{\gamma}\bar{h}_{\alpha\beta}\mathcal{H}^2 - 4AC_{\alpha\beta}\mathcal{H}' + C_{\alpha\beta}\mathcal{H}' + 4A^2\bar{h}_{\alpha\beta}\mathcal{H}' \\
&- A\bar{h}_{\alpha\beta}\mathcal{H}' - B_{\gamma}B^{\gamma}\bar{h}_{\alpha\beta}\mathcal{H}' - \frac{1}{2}A'\nabla_{\alpha}B_{\beta} + \frac{1}{2}C'^{\gamma}{}_{\gamma}\nabla_{\alpha}B_{\beta} - 2A\mathcal{H}\nabla_{\alpha}B_{\beta} - A\nabla_{\alpha}B'_{\beta} \\
&+ \frac{1}{2}\mathcal{H}\nabla_{\alpha}B_{\beta} + \frac{1}{4}\nabla_{\alpha}B'_{\beta} - B'^{\gamma}\nabla_{\alpha}C_{\beta\gamma} - 2B^{\gamma}\mathcal{H}\nabla_{\alpha}C_{\beta\gamma} - B^{\gamma}\nabla_{\alpha}C'_{\beta\gamma} + \nabla_{\alpha}A\nabla_{\beta}A \\
&- \frac{1}{2}A'\nabla_{\beta}B_{\alpha} + \frac{1}{2}C'^{\gamma}{}_{\gamma}\nabla_{\beta}B_{\alpha} - 2A\mathcal{H}\nabla_{\beta}B_{\alpha} - \frac{1}{2}\nabla_{\alpha}B^{\gamma}\nabla_{\beta}B_{\gamma} - A\nabla_{\beta}B'_{\alpha} + \frac{1}{2}\mathcal{H}\nabla_{\beta}B_{\alpha} \\
&+ \frac{1}{4}\nabla_{\beta}B'_{\alpha} - B'^{\gamma}\nabla_{\beta}C_{\alpha\gamma} - 2B^{\gamma}\mathcal{H}\nabla_{\beta}C_{\alpha\gamma} + \nabla_{\alpha}C^{\gamma\delta}\nabla_{\beta}C_{\gamma\delta} - B^{\gamma}\nabla_{\beta}C'_{\alpha\gamma} + 2A\nabla_{\beta}\nabla_{\alpha}A \\
&- \frac{1}{2}\nabla_{\beta}\nabla_{\alpha}A - B^{\gamma}\nabla_{\beta}\nabla_{\alpha}B_{\gamma} + 2C^{\gamma\delta}\nabla_{\beta}\nabla_{\alpha}C_{\gamma\delta} - \frac{1}{2}\nabla_{\beta}\nabla_{\alpha}C^{\gamma}{}_{\gamma} - B^{\gamma}\bar{h}_{\alpha\beta}\mathcal{H}\nabla_{\gamma}A \\
&+ C'_{\alpha\beta}\nabla_{\gamma}B^{\gamma} + 2C_{\alpha\beta}\mathcal{H}\nabla_{\gamma}B^{\gamma} - 2A\bar{h}_{\alpha\beta}\mathcal{H}\nabla_{\gamma}B^{\gamma} + \frac{1}{2}\nabla_{\alpha}B_{\beta}\nabla_{\gamma}B^{\gamma} + \frac{1}{2}\nabla_{\beta}B_{\alpha}\nabla_{\gamma}B^{\gamma} \\
&+ \frac{1}{2}\bar{h}_{\alpha\beta}\mathcal{H}\nabla_{\gamma}B^{\gamma} + B'^{\gamma}\nabla_{\gamma}C_{\alpha\beta} + 2B^{\gamma}\mathcal{H}\nabla_{\gamma}C_{\alpha\beta} + B^{\gamma}\bar{h}_{\alpha\beta}\mathcal{H}\nabla_{\gamma}C^{l}{}_{l} + \nabla_{\alpha}C_{\beta}{}^{\gamma}\nabla_{\gamma}C^{l}{}_{l} \\
&+ \nabla_{\beta}C_{\alpha}{}^{\gamma}\nabla_{\gamma}C^{l}{}_{l} + 2B^{\gamma}\nabla_{\gamma}C'_{\alpha\beta} + \frac{1}{2}B^{\gamma}\nabla_{\gamma}\nabla_{\alpha}B_{\beta} + \frac{1}{2}\nabla_{\gamma}\nabla_{\alpha}C_{\beta}{}^{\gamma} + \frac{1}{2}B^{\gamma}\nabla_{\gamma}\nabla_{\beta}B_{\alpha} \\
&+ \frac{1}{2}\nabla_{\gamma}\nabla_{\beta}C_{\alpha}{}^{\gamma} - \frac{1}{2}\nabla_{\gamma}\nabla^{\gamma}C_{\alpha\beta} + \nabla_{\alpha}C_{\beta\gamma}\nabla^{\gamma}A + \nabla_{\beta}C_{\alpha\gamma}\nabla^{\gamma}A - \nabla_{\gamma}C_{\alpha\beta}\nabla^{\gamma}A \\
&- C'_{\beta\gamma}\nabla^{\gamma}B_{\alpha} - \frac{1}{2}\nabla_{\gamma}B_{\beta}\nabla^{\gamma}B_{\alpha} - C'_{\alpha\gamma}\nabla^{\gamma}B_{\beta} - \nabla_{\gamma}C^{l}{}_{l}\nabla^{\gamma}C_{\alpha\beta} - 2B^{\gamma}\bar{h}_{\alpha\beta}\mathcal{H}\nabla_{l}C_{\gamma}{}^{l} \\
&- 2\nabla_{\alpha}C_{\beta}{}^{\gamma}\nabla_{l}C_{\gamma}{}^{l} - 2\nabla_{\beta}C_{\alpha}{}^{\gamma}\nabla_{l}C_{\gamma}{}^{l} + 2\nabla^{\gamma}C_{\alpha\beta}\nabla_{l}C_{\gamma}{}^{l} - 2C^{\gamma\delta}\nabla_{l}\nabla_{\alpha}C_{\beta\gamma} - 2C^{\gamma\delta}\nabla_{l}\nabla_{\beta}C_{\alpha\gamma} \\
&+ 2C^{\gamma\delta}\nabla_{l}\nabla_{\gamma}C_{\alpha\beta} - 2C_{\gamma\delta}\bar{h}_{\alpha\beta}\mathcal{H}\nabla^{l}B^{\gamma} - 2\nabla_{\gamma}C_{\beta\delta}\nabla^{l}C_{\alpha}{}^{\gamma} + 2\nabla_{l}C_{\beta\gamma}\nabla^{l}C_{\alpha}{}^{\gamma})
\end{aligned}
\tag{350}
$$

$$
\begin{aligned}
R =&\frac{1}{a^2}\Big[6\mathcal{H}^2 + 6\mathcal{H}' + {}^{(1)}(2C''^{\alpha}{}_{\alpha} - 6A'\mathcal{H} + 6C'^{\alpha}{}_{\alpha}\mathcal{H} - 12A\mathcal{H}^2 - 12A\mathcal{H}' + 6\mathcal{H}\nabla_{\alpha}B^{\alpha} + 2\nabla_{\alpha}B'^{\alpha} \\
&- 2\nabla_{\alpha}\nabla^{\alpha}A + 2\nabla_{\beta}\nabla_{\alpha}C^{\alpha\beta} - 2\nabla_{\beta}\nabla^{\beta}C^{\alpha}{}_{\alpha}) + {}^{(2)}(-2A'C'^{\alpha}{}_{\alpha} - 3C'{}_{\alpha\beta}C'^{\alpha\beta} + C'^{\alpha}{}_{\alpha}C'^{\beta}{}_{\beta} \\
&- 4C^{\alpha\beta}C''{}_{\alpha\beta} - 4AC''^{\alpha}{}_{\alpha} + C''^{\alpha}{}_{\alpha} + 24AA'\mathcal{H} - 3A'\mathcal{H} - 6B^{\alpha}B'{}_{\alpha}\mathcal{H} - 12C^{\alpha\beta}C'{}_{\alpha\beta}\mathcal{H} \\
&- 12AC'^{\alpha}{}_{\alpha}\mathcal{H} + 3C'^{\alpha}{}_{\alpha}\mathcal{H} + 24A^2\mathcal{H}^2 - 6A\mathcal{H}^2 - 6B_{\alpha}B^{\alpha}\mathcal{H}^2 + 24A^2\mathcal{H}' - 6A\mathcal{H}' \\
&- 6B_{\alpha}B^{\alpha}\mathcal{H}' - 6B^{\alpha}\mathcal{H}\nabla_{\alpha}A - 2A'\nabla_{\alpha}B^{\alpha} + 2C'^{\beta}{}_{\beta}\nabla_{\alpha}B^{\alpha} - 12A\mathcal{H}\nabla_{\alpha}B^{\alpha} - 4A\nabla_{\alpha}B'^{\alpha} \\
&+ 3\mathcal{H}\nabla_{\alpha}B^{\alpha} + \nabla_{\alpha}B'^{\alpha} + 2B'^{\alpha}\nabla_{\alpha}C^{\beta}{}_{\beta} + 6B^{\alpha}\mathcal{H}\nabla_{\alpha}C^{\beta}{}_{\beta} + 4B^{\alpha}\nabla_{\alpha}C'^{\beta}{}_{\beta} + 4A\nabla_{\alpha}\nabla^{\alpha}A \\
&- \nabla_{\alpha}\nabla^{\alpha}A + 2B^{\alpha}\nabla_{\alpha}\nabla_{\beta}B^{\beta} + 2\nabla_{\alpha}A\nabla^{\alpha}A - 2\nabla_{\alpha}C^{\beta}{}_{\beta}\nabla^{\alpha}A + \nabla_{\alpha}B^{\alpha}\nabla_{\beta}B^{\beta} - 4B'^{\alpha}\nabla_{\beta}C_{\alpha}{}^{\beta} \\
&- 12B^{\alpha}\mathcal{H}\nabla_{\beta}C_{\alpha}{}^{\beta} + 4\nabla^{\alpha}A\nabla_{\beta}C_{\alpha}{}^{\beta} - 4B^{\alpha}\nabla_{\beta}C'{}_{\alpha}{}^{\beta} + 4C^{\alpha\beta}\nabla_{\beta}\nabla_{\alpha}C^{\gamma}{}_{\gamma} + \nabla_{\beta}\nabla_{\alpha}C^{\alpha\beta} \\
&- 2B^{\alpha}\nabla_{\beta}\nabla^{\beta}B_{\alpha} - \nabla_{\beta}\nabla^{\beta}C^{\alpha}{}_{\alpha} - 8C^{\alpha\beta}\nabla_{\beta}\nabla_{\gamma}C_{\alpha}{}^{\gamma} - 2C'{}_{\alpha\beta}\nabla^{\beta}B^{\alpha} - 12C_{\alpha\beta}\mathcal{H}\nabla^{\beta}B^{\alpha} \\
&+ \frac{1}{2}\nabla_{\alpha}B_{\beta}\nabla^{\beta}B^{\alpha} - \frac{1}{2}3\nabla_{\beta}B_{\alpha}\nabla^{\beta}B^{\alpha} - 4C_{\alpha\beta}\nabla^{\beta}B'^{\alpha} - \nabla_{\beta}C^{\gamma}{}_{\gamma}\nabla^{\beta}C^{\alpha}{}_{\alpha} + 4C_{\alpha\beta}\nabla^{\beta}\nabla^{\alpha}A \\
&- 4\nabla_{\alpha}C^{\alpha\beta}\nabla_{\gamma}C_{\beta}{}^{\gamma} + 4\nabla^{\beta}C^{\alpha}{}_{\alpha}\nabla_{\gamma}C_{\beta}{}^{\gamma} + 4C^{\alpha\beta}\nabla_{\gamma}\nabla^{\gamma}C_{\alpha\beta} - 2\nabla_{\beta}C_{\alpha\gamma}\nabla^{\gamma}C^{\alpha\beta} + 3\nabla_{\gamma}C_{\alpha\beta}\nabla^{\gamma}C^{\alpha\beta})\Big]
\end{aligned}
\tag{351}
$$

Part VI
References

References

[1] T. L. Chow, *Gravity, black holes, and the very early universe : an introduction to general relativity and cosmology.* Springer, 2008.

[2] G. F. R. Ellis, "Issues in the Philosophy of Cosmology," 2 2006.

[3] E. Hubble, "A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae," *Proceedings of the National Academy of Science*, vol. 15, pp. 168–173, 3 1929.

[4] Fazio and Giovanni G, "The Encyclopedia of Cosmology," tech. rep.

[5] M. S. Longair, "A Brief History of Cosmology," *Carnegie Observatories Astrophysics Series*, vol. 2, 2004.

[6] A. Einstein, "Näherungsweise Integration der Feldgleichungen der Gravitation," *Sitzungsberichte Preussischen Akademie der Wissenschaften*, pp. 688–696, 1916.

[7] A. Einstein, "Über Gravitationswellen," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, vol. 1918, pp. 154–167, 1918.

[8] S. Vitale, "The first 5 years of gravitational-wave astrophysics," *Science*, vol. 372, p. eabc7397, 6 2021.

[9] V. Gladyshev and I. Fomin, "The Early Universe as a Source of Gravitational Waves," in *Progress in Relativity* (C. G. Buzea, M. Agop, and L. Butler, eds.), p. Ch. 13, Rijeka: IntechOpen, 2019.

[10] R.-G. Cai, Z. Cao, Z.-K. Guo, S.-J. Wang, and T. Yang, "The gravitational-wave physics," *National Science Review*, vol. 4, pp. 687–706, 9 2017.

[11] B. P. Abbott and et al., "ASTROPHYSICAL IMPLICATIONS OF THE BINARY BLACK HOLE MERGER GW150914," *The Astrophysical Journal Letters*, vol. 818, p. L22, 2 2016.

[12] The LISA Collaboration, P. Amaro-Seoane, and et al., "Laser Interferometer Space Antenna," 2 2017.

[13] N. Christensen, "Stochastic Gravitational Wave Backgrounds," *Rept. Prog. Phys.*, vol. 82, no. 1, p. 16903, 2019.

[14] N. van Remortel, K. Janssens, and K. Turbang, "Stochastic gravitational wave background: Methods and implications," *Prog. Part. Nucl. Phys.*, vol. 128, p. 104003, 2023.

[15] N. Bartolo, D. Bertacca, S. Matarrese, M. Peloso, A. Ricciardone, A. Riotto, and G. Tasinato, "Characterizing the cosmological gravitational wave background: Anisotropies and non-Gaussianity," *Physical Review D*, vol. 102, p. 023527, 7 2020.

[16] A. Malhotra, E. Dimastrogiovanni, G. Domènech, M. Fasiello, and G. Tasinato, "New universal property of cosmological gravitational wave anisotropies," *Physical Review D*, vol. 107, p. 103502, 5 2023.

[17] S. Babak, C. Caprini, D. G. Figueroa, N. Karnesis, P. Marcoccia, G. Nardini, M. Pieroni, A. Ricciardone, A. Sesana, and J. Torrado, "Stochastic gravitational wave background from stellar origin binary black holes in LISA," 5 2023.

[18] H. Noh and J. c. Hwang, "Second-order perturbations of the Friedmann world model," *Physical Review D - Particles, Fields, Gravitation and Cosmology*, vol. 69, no. 10, 2004.

[19] J.-O. Gong, J.-c. Hwang, H. Noh, D. C. L. Wu, and J. Yoo, "Exact non-linear equations for cosmological perturbations," *Journal of Cosmology and Astroparticle Physics*, vol. 2017, pp. 027–027, 10 2017.

[20] D. Baumann, P. Steinhardt, K. Takahashi, and K. Ichiki, "Gravitational wave spectrum induced by primordial scalar perturbations," *Physical Review D - Particles, Fields, Gravitation and Cosmology*, vol. 76, 10 2007.

[21] J.-c. Hwang, D. Jeong, and H. Noh, "Gauge Dependence of Gravitational Waves Generated from Scalar Perturbations," *The Astrophysical Journal*, vol. 842, p. 46, 6 2017.

[22] J.-O. Gong, "Analytic Integral Solutions for Induced Gravitational Waves," *Astrophys. J.*, vol. 925, no. 1, p. 102, 2022.

[23] M. Sipp and B. M. Schäfer, "Scalar-induced gravitational waves in a Λ CDM cosmology," *Physical Review D*, vol. 107, p. 063538, 3 2023.

[24] A. Einstein, "On The influence of gravitation on the propagation of light," *Annalen Phys.*, vol. 35, pp. 898–908, 1911.

[25] A. Einstein, "The foundation of the general theory of relativity.," *Annalen Phys.*, vol. 49, no. 7, pp. 769–822, 1916.

[26] J. A. Wheeler, "Geometrodynamics and the issue of final state," in *Les Houches Summer Shcool of Theoretical Physics: Relativity, Groups and Topology*, pp. 317–522, 1964.

[27] S. Carroll, *Spacetime and geometry: An introduction to general relativity.* Pearson Education Limited, 2014.

[28] D. Hilbert, "Die Grundlagen der Physik . (Erste Mitteilung.)," *Nachrichten von der Koeniglichen Gesellschaft der Wissenschaften zu Goettingen, Math-physik. Klasse*, pp. 395–407, 1915.

[29] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* W. H. Freeman, 1973.

[30] P. Jetzer and A. Borde, "General Relativity Script: Autumn Semester 2019," 2019.

[31] A. Palatini, "Deduzione invariantiva delle equazioni gravitazionali dal principio di Hamilton," *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, vol. 43, pp. 203–212, 12 1919.

[32] c. Eckart, “The thermodynamics of Irreversible Processes. III Relativistic Theory of the Simple Fluid,” *Physical Review*, vol. 58, pp. 919–924, 11 1940.

[33] N. Andersson and G. L. C. Comer, “Relativistic fluid dynamics: physics for many different scales,” 8 2020.

[34] J. Yoo, “Lecture notes: Physical Cosmology, Spring Semester 2022,” 2022.

[35] A. Riddle, *An introduction to modern cosmology.* third ed., 2015.

[36] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* 1972.

[37] E. A. Milne, *Relativity, Gravitation and World Structure.* Oxford: Clarendon Press, 1935.

[38] H. Mo, F. van de Bosch, and H. White, *Galaxy formation and evolution.* 2010.

[39] G. F. R. Ellis, R. Maartens, and M. A. H. MacCallum, *Relativistic Cosmology.* Cambridge University Press, 3 2012.

[40] E. W. Kolb and M. S. Turner, *The early universe*, vol. 69. 1990.

[41] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time.* Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1973.

[42] L. Bergström and A. Goobar, *Cosmology and Particle Astrophysics.* 3 2004.

[43] S. Dodelson and F. Schmidt, *Modern Cosmology.* Elsevier, 2020.

[44] K. A. Malik and D. R. Matravers, “A concise introduction to perturbation theory in cosmology,” *Classical and Quantum Gravity*, vol. 25, p. 193001, 10 2008.

[45] K. Nakamura, “Second-Order Gauge-Invariant Cosmological Perturbation Theory: Current Status,” *Advances in Astronomy*, vol. 2010, 2010.

[46] M. Bruni, S. Matarrese, S. Mollerach, and S. Sonego, “Classical and Quantum Gravity Perturbations of spacetime: gauge transformations and gauge invariance at second order and beyond,” *Class. Quantum Grav*, vol. 14, pp. 2585–2606, 1997.

[47] M. Bruni and S. Sonego, “Observables and gauge invariance in the theory of nonlinear spacetime perturbations,” *Classical and Quantum Gravity*, vol. 16, p. L29, 7 1999.

[48] M. Magi and J. Yoo, “Second-order gauge-invariant formalism for the cosmological observables: Complete verification of their gauge-invariance,” *Journal of Cosmology and Astroparticle Physics*, vol. 2022, 9 2022.

[49] T. Clifton, C. S. Gallagher, S. Goldberg, and K. A. Malik, “Viable gauge choices in cosmologies with nonlinear structures,” *Physical Review D*, vol. 101, p. 063530, 3 2020.

[50] W. Cui, L. Liu, X. Yang, Y. Wang, L. Feng, and V. Springel, “An Ideal Mass Assignment Scheme for Measuring the Power Spectrum with Fast Fourier Transforms,” *The Astrophysical Journal*, vol. 687, pp. 738–744, 11 2008.

[51] R. Durrer, *The Cosmic Microwave Background.* Cambridge University Press, 12 2020.

[52] E. Massara, F. Villaescusa-Navarro, C. Hahn, M. M. Abidi, M. Eickenberg, S. Ho, P. Lemos, A. M. Dizgah, and B. R.-S. Blancard, "Cosmological Information in the Marked Power Spectrum of the Galaxy Field," 6 2022.

[53] P. Paykari, F. Lanusse, J.-L. Starck, F. Sureau, and J. Bobin, "PRISM: Sparse recovery of the primordial power spectrum," *Astronomy & Astrophysics*, vol. 566, p. A77, 6 2014.

[54] S. L. Bridle, A. M. Lewis, J. Weller, and G. Efstathiou, "Reconstructing the primordial power spectrum," 2 2003.

[55] M. Maggiore, *Gravitational Waves Volume 2: Astrophysics and Cosmology.* Oxford University Press, 2018.

[56] Planck Collaboration, N. Aghanim, and et al., "Planck 2018 results VI. Cosmological parameters," *Astronomy & Astrophysics*, vol. 641, p. A6, 9 2020.

[57] A. H. Guth, "Infiationary universe: A possible solution to the horizon and flatness problems," *Physical Review D*, vol. 23, no. 2, pp. 347–356, 1980.

[58] D. Bailin and A. Love, *Cosmology in Gauge Field Theory and String Theory.* Taylor & Francis, 2004.

[59] A. H. Guth and E. J. Weinberg, "Could the universe have recovered from a slow first-order phase transition?," *Nuclear Physics B*, vol. 212, pp. 321–364, 2 1983.

[60] A. D. Linde, "A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems," *Physics Letters B*, vol. 108, pp. 389–393, 2 1982.

[61] A. Albrecht and P. J. Steinhardt, "Cosmology for Grand Unified Theories with Radiatively Induced Symmetriy Breaking," *Physical Review Letters*, vol. 48, no. 17, pp. 1220–1223, 1982.

[62] G. Dvali and S. Kachru, "New Old Inflation," 9 2003.

[63] S. Weinberg, *The quantum theory of fields. Vol. 2: Modern applications.* Cambridge University Press, 4 2013.

[64] A. H. Guth, "Inflation and the New Era of High-Precision Cosmology," in *2002 Annual physics@mit* (C. Breen, ed.), vol. 15, pp. 28–39, 2002.

[65] A. Albrecht, "Cosmic Inflation and the Arrow of Time," 10 2002.

[66] K. El Bourakadi, "Preheating and Reheating after Standard Inflation," 4 2021.

[67] M. A. Amin, M. P. Hertzberg, D. I. Kaiser, and J. Karouby, "Nonperturbative Dynamics Of Reheating After Inflation: A Review," *Int. J. Mod. Phys. D*, vol. 24, p. 1530003, 2014.

[68] D. H. Lyth and A. R. Liddle, *The Primordial Density Perturbation.* Cambridge University Press, 6 2009.

[69] R. Allahverdi, R. Brandenberger, F.-Y. Cyr-Racine, and A. Mazumdar, “Reheating in Inflationary Cosmology: Theory and Applications,” *Annual Review of Nuclear and Particle Science*, vol. 60, pp. 27–51, 10 2010.

[70] V. Mukhanov, *Physical foundation of cosmology.* 2005.

[71] B. Ryden, *Introduction to Cosmology.* Cambridge University Press, 11 2016.

[72] A. R. Liddle and D. H. Lyth, *Cosmological Inflation and Large-Scale Structure.* Cambridge University Press, 4 2000.

[73] A. D. Dolgov and A. D. Linde, “Baryon asymmetry in the inflationary universe,” *Physics Letters B*, vol. 116, pp. 329–334, 10 1982.

[74] A. D. Dolgov and S. H. Hansen, “Equation of motion of a classical scalar field with back reaction of produced particles,” *Nuclear Physics B*, vol. 548, pp. 408–426, 5 1999.

[75] L. F. Abbott, E. Farhi, and M. B. Wise, “Particle production in the new inflationary cosmology,” *Physics Letters B*, vol. 117, pp. 29–33, 11 1982.

[76] G. Lazarides, “Basics of inflationary cosmology,” *J. Phys.: Conf. Ser*, vol. 53, p. 528, 2006.

[77] P. J. Steinhardt and M. S. Turner, “Prescription for successful new inflation,” *Physical Review2162*, vol. 29, no. 10, pp. 2162–2171, 1984.

[78] A. Riotto, “Inflation and the Theory of Cosmological Perturbations,” 10 2002.

[79] J. Yoo, “AST802 Advanced Topics of Theoretical Cosmology,” tech. rep., 2022.

[80] D. Baumann, “TASI Lectures on Inflation,” 7 2009.

[81] R. H. Brandenberger, “Inflationary Cosmology: Progress and Problems,” 10 1999.

[82] J. Maldacena, “Non-Gaussian features of primordial fluctuations in single field inflationary models,” 10 2002.

[83] V. F. Mukhanov, “Quantum Theory of Gauge Invariant Cosmological Perturbations,” *Sov. Phys. JETP*, vol. 67, pp. 1297–1302, 1988.

[84] M. Sasaki, “Large Scale Quantum Fluctuations in the Inflationary Universe,” *Progress of theoretical physics.*, vol. 76, no. 5, pp. 1036–1046, 1986.

[85] S. Kundu, “Prepared for submission to JCAP Inflation with general initial conditions for scalar perturbations,” 2012.

[86] E. D. Stewart and D. H. Lyth, “A more accurate analytic calculation of the spectrum of cosmological perturbations produced during inflation,” 2 1993.

[87] S. A. Hughes, “Gravitational wave astronomy and cosmology,” 5 2014.

[88] T. L. S. The LIGO Scientific Collaboration and t. V. the Virgo Collaboration, “GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence,” 6 2016.

[89] The LIGO Scientific Collaboration, The Virgo Collaboration, B. P. Abbott, and et al., "Binary Black Hole Mergers in the first Advanced LIGO Observing Run," 6 2016.

[90] T. L. S. The LIGO Scientific Collaboration and t. V. the Virgo Collaboration, "Observation of Gravitational Waves from a Binary Black Hole Merger," 2 2016.

[91] The LIGO Scientific Collaboration, The Virgo Collaboration, B. P. Abbott, and et al., "GW170608: Observation of a 19-solar-mass Binary Black Hole Coalescence," 11 2017.

[92] C. The LIGO Scientific Collaboration and The Virgo Collaboration, "GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral," 10 2017.

[93] The LIGO Scientific Collaboration, The Virgo Collaboration, B. P. Abbott, and et al., "GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence," 9 2017.

[94] The LIGO Scientific Collaboration, The Virgo Collaboration, B. P. Abbott, and et al., "GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2," 6 2017.

[95] The LIGO Scientific Collaboration, the Virgo Collaboration, R. Abbott, and et al., "GW190412: Observation of a Binary-Black-Hole Coalescence with Asymmetric Masses," 4 2020.

[96] The LIGO Scientific Collaboration, the Virgo Collaboration, R. Abbott, and et al., "GW190814: Gravitational Waves from the Coalescence of a 23 M$_\odot$ Black Hole with a 2.6 M$_\odot$ Compact Object," 6 2020.

[97] The LIGO Scientific Collaboration, The Virgo Collaboration, B. P. Abbott, and et al., "GW190425: Observation of a Compact Binary Coalescence with Total Mass $\sim 3.4 M_{\odot}$," 1 2020.

[98] R. Abbott, T. D. Abbott, S. Abraham, and et al., "GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run," 10 2020.

[99] T. L. S. The LIGO Scientific Collaboration, t. V. the Virgo Collaboration, R. Abbott, and et al., "GW190521: A Binary Black Hole Merger with a Total Mass of $150 ~ M_{\odot}$," 9 2020.

[100] T. L. S. The LIGO Scientific Collaboration, t. V. the Virgo Collaboration, t. K. the KAGRA Collaboration, and et al., "Observation of gravitational waves from two neutron star-black hole coalescences," 6 2021.

[101] C. Caprini and D. G. Figueroa, "Cosmological Backgrounds of Gravitational Waves," 1 2018.

[102] S. Kawamura and et al., "The Japanese space gravitational wave antenna - DECIGO," *Journal of Physics: Conference Series*, vol. 122, p. 012006, 7 2008.

[103] M. Maggiore, "Gravitational wave experiments and early universe cosmology," *Physics Reports*, vol. 331, pp. 283–367, 7 2000.

[104] G. Domènech, "Induced gravitational waves in a general cosmological background," *International Journal of Modern Physics D*, vol. 29, p. 2050028, 1 2020.

[105] S. Rowan and J. Hough, "The detection of gravitational waves," in *1998 European School of High-Energy Physics*, pp. 301–311, 1998.

[106] G. Domenech, "Scalar Induced Gravitational Waves Review," *Universe*, vol. 7, p. 398, 10 2021.

[107] A. Dirkes, "Gravitational waves — A review on the theoretical foundations of gravitational radiation," *International Journal of Modern Physics A*, vol. 33, p. 1830013, 5 2018.

[108] P. Jetzer, *Applications of General Relativity.* Springer cham, 2022.

[109] M. Maggiore, *Gravitational Waves Volume 1: Theory and Experiments-Oxford University Press, USA (2007).* 2007.

[110] J. D. Romano and N. J. Cornish, "Detection methods for stochastic gravitational-wave backgrounds: a unified treatment," *Living Rev. Rel.*, vol. 20, no. 1, p. 2, 2017.

[111] A. Buonanno, "Gravitational waves," in *Les Houches Summer School - Session 86: Particle Physics and Cosmology: The Fabric of Spacetime*, 4 2007.

[112] C. Grojean and G. Servant, "Gravitational waves from phase transitions at the electroweak scale and beyond," *Physical Review D*, vol. 75, p. 043507, 2 2007.

[113] A. I. Renzini, B. Goncharov, A. C. Jenkins, and P. M. Meyers, "Stochastic Gravitational-Wave Backgrounds: Current Detection Efforts and Future Prospects," *Galaxies*, vol. 10, no. 1, 2022.

[114] D. G. Figueroa and F. Torrenti, "Gravitational wave production from preheating: parameter dependence," *JCAP*, vol. 10, p. 57, 2017.

[115] B. A. Bassett, D. I. Kaiser, and R. Maartens, "General relativistic effects in preheating," *Physics Letters B*, vol. 455, pp. 84–89, 1998.

[116] F. Finelli and R. H. Brandenberger, "Parametric amplification of metric fluctuations during reheating in two field models," *Phys. Rev. D*, vol. 62, p. 83502, 2000.

[117] S. Y. Khlebnikov and I. I. Tkachev, "Relic gravitational waves produced after preheating," *Phys. Rev. D*, vol. 56, pp. 653–660, 1997.

[118] R. Easther, J. T. Giblin Jr., and E. A. Lim, "Gravitational Wave Production At The End Of Inflation," *Phys. Rev. Lett.*, vol. 99, p. 221301, 2007.

[119] J. Garcia-Bellido, D. G. Figueroa, and A. Sastre, "A Gravitational Wave Background from Reheating after Hybrid Inflation," *Phys. Rev. D*, vol. 77, p. 43517, 2008.

[120] D. G. Figueroa, J. Garcia-Bellido, and F. Torrenti, "Gravitational wave production from the decay of the standard model Higgs field after inflation," *Phys. Rev. D*, vol. 93, no. 10, p. 103521, 2016.

[121] M. Lemoine, J. Martin, and P. Peter, *Inflationary cosmology.* 2008.

[122] C. Caprini and D. G. Figueroa, “Stochastic Gravitational Wave Backgrounds of Cosmological Origin,” in *Handbook of Gravitational Wave Astronomy* (C. Bambi, S. Katsanevas, and K. D. Kokkotas, eds.), pp. 1–54, Singapore: Springer Singapore, 2020.

[123] R. Durrer, “Gravitational waves from cosmological phase transitions,” *Journal of Physics: Conference Series*, vol. 222, p. 012021, 4 2010.

[124] C. J. Hogan, “Gravitational radiation from cosmological phase transitions,” vol. 218, pp. 629–636, 2 1986.

[125] A. Kosowsky, M. S. Turner, and R. Watkins, “Gravitational radiation from colliding vacuum bubbles,” vol. 45, pp. 4514–4535, 6 1992.

[126] M. Kamionkowski, A. Kosowsky, and M. S. Turner, “Gravitational radiation from first-order phase transitions,” vol. 49, pp. 2837–2851, 3 1994.

[127] M. B. Hindmarsh, M. Lüben, J. Lumma, and M. Pauly, “Phase transitions in the early universe,” *arXiv e-prints*, p. arXiv:2008.09136, 8 2020.

[128] T. Nakamura, M. Sasaki, T. Tanaka, and K. S. Thorne, “Gravitational Waves from Coalescing Black Hole MACHO Binaries,” *The Astrophysical Journal*, vol. 487, pp. L139–L142, 10 1997.

[129] M. Raidal, C. Spethmann, V. Vaskonen, and H. Veermäe, “Formation and evolution of primordial black hole binaries in the early universe,” *Journal of Cosmology and Astroparticle Physics*, vol. 2019, pp. 018–018, 2 2019.

[130] R. Dong, W. H. Kinney, and D. Stojkovic, “Gravitational wave production by Hawking radiation from rotating primordial black holes,” *JCAP*, vol. 10, p. 34, 2016.

[131] M. C. Guzzetti, N. Bartolo, M. Liguori, and S. Matarrese, “Gravitational waves from inflation,” *Riv. Nuovo Cim.*, vol. 39, no. 9, pp. 399–495, 2016.

[132] J.-c. Hwang, D. Jeong, and H. Noh, “Gauge Dependence of Gravitational Waves Generated from Scalar Perturbations,” *The Astrophysical Journal*, vol. 842, p. 46, 6 2017.

[133] K. N. Ananda, C. Clarkson, and D. Wands, “Cosmological gravitational wave background from primordial density perturbations,” *Physical Review D*, vol. 75, p. 123518, 6 2007.

[134] K. Kohri and T. Terada, “Semianalytic calculation of gravitational wave spectrum nonlinearly induced from primordial curvature perturbations,” *Physical Review D*, vol. 97, p. 123532, 6 2018.

[135] Y. Watanabe and E. Komatsu, “Improved Calculation of the Primordial Gravitational Wave Spectrum in the Standard Model,” tech. rep., 2006.

[136] K. Inomata and T. Terada, “Gauge independence of induced gravitational waves,” *Physical Review D*, vol. 101, p. 023523, 1 2020.

[137] A. Ali, Y. Gong, and Y. Lu, “Gauge transformation of scalar induced tensor perturbation during matter domination,” vol. 103, p. 43516, 2 2021.

[138] D. Blas, J. Lesgourgues, and T. Tram, "The Cosmic Linear Anisotropy Solving System (CLASS). Part II: Approximation schemes," *Journal of Cosmology and Astroparticle Physics*, vol. 2011, pp. 034–034, 7 2011.