**QFT methods for BSM theories**
**Solution 1.**

HS 2019
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## Exercise 1. *Power-counting of soft fields*

*Analogously to the example of scalar fields in the lecture, determine the power-counting of a soft fermion and a soft gauge boson.*

**Solution.** For a soft fermion and soft gauge boson we have respectively,

$$\begin{aligned}\langle 0|T\{\Psi(0)\Psi(x)\}|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{\not{k}+m}{k^2-m^2} \sim \quad \mathcal{O}(E^4)\cdot\mathcal{O}(E^{-1}) = \mathcal{O}(\lambda^3 M^3), \\ \Rightarrow \Psi^2 \sim \lambda^3 M^3 \quad \Rightarrow \quad \Psi \sim \lambda^{3/2} M^{3/2}.\end{aligned} \tag{S.1}$$

and

$$\begin{aligned}\langle 0|T\{A^\mu(0)A^\nu(x)\}|0\rangle &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2-m^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}\right), \\ &\sim \mathcal{O}(E^4)\cdot\big(\mathcal{O}(E^{-2}) - \mathcal{O}(1)\big) = \mathcal{O}(\lambda^2 M^2) \\ &\Rightarrow A^\mu A^\nu \sim \lambda^2 M^2 \quad \Rightarrow \quad A^\mu \sim \lambda M.\end{aligned} \tag{S.2}$$

where we have retained the leading contribution provided by the longitudinal part of the vector field.

## Exercise 2. *An Effective Field Theory example*

*Consider the following Lagrangian*

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{M^2}{2}\Phi^2 + \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{m^2}{2}\varphi^2 - \frac{\kappa}{4!}\varphi^4 - \frac{\lambda}{3!}\varphi^3\Phi, \tag{1}$$

*where $M \gg m$. We now want to consider processes at energies $E \ll \Lambda \sim M$.*

(a) *Compute the effective Lagrangian $\mathcal{L}_{\text{eff}}$ at tree-level both diagrammatically and with the aid of the equations of motion, up to $\mathcal{O}(1/\Lambda^2)$ in the power-counting of the effective theory.*

(b) *Perform the one-loop EFT matching diagrammatically using dimensional regularization with the $\overline{\text{MS}}$ subtraction scheme, without the method of regions.*

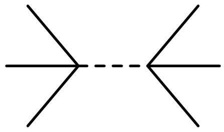
(c) *In the calculation of the one-loop diagrams of the full theory in part (ii) separate the integrals in hard $k \sim M$ and soft regions $k \sim m \ll M$. What do you observe with respect to the corresponding terms in the diagrams of the EFT?*

(d) *Compute the anomalous dimension matrix of the matching coefficients. Take $\alpha = 0$ at the matching scale $\mu = M$, where $\alpha$ is the coefficient of the $\varphi^4$ operator in the EFT and examine the RG evolution of the couplings. What do you observe at $\mu = m$? Why is a term of the type $\varphi^3$ not generated by the running?*

(e) ***Bonus:*** *Compute the effective Lagrangian at one-loop order using the background field method.*

**Solution.**

(a) At tree-level the operator $\varphi^6$ is generated by the diagram,



$$= \frac{6\cdot 2\cdot 3\cdot 2}{2!(3!)^2}(-i\lambda)^2\sum_{\pi(p)}\frac{i}{p^2-M^2}\approx i\frac{10\lambda^2}{M^2}, \tag{S.3}$$

where we have written the symmetry factors for a single topology explicitly and summed over the possible permutations of external momenta[1]. After normalizing the effective operator with the factor 6! the tree-level effective Lagrangian is,

$$\mathcal{L}_{\text{eff}}^{\text{tree}} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{m^2}{2}\varphi^2 - \frac{\kappa}{4!}\varphi^4 + \frac{\lambda^2}{72M^2}\varphi^6, \tag{S.4}$$

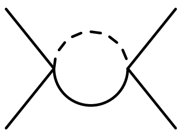
Alternatively, by solving the equations of motion for the $\Phi$ field we obtain,

$$\Phi = -\frac{\lambda}{6M^2}\varphi^3 + \mathcal{O}(\Lambda^{-4}), \tag{S.5}$$

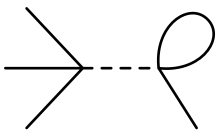
that, upon substituting in 1 gives S.4.

(b) We consider in the matching procedure one-loop diagrams with only light fields, since they are present in both the full theory and the effective theory amplitudes and, accordingly, cancel out in the matching.

For the model under discussion there is no contribution to the two- and three-point Green functions involving heavy particles in the loop. The diagrams contributing to the matching of the four-point Green function are given by



$$\begin{aligned} &= \frac{6\cdot 2\cdot 3\cdot 2}{2!(3!)^2}(-i\lambda)^2\sum_{\pi(p)}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-M^2}\frac{i}{(k+p)^2-m^2} \\ &= \frac{i}{16\pi^2}3\lambda^2\left[1+\frac{1}{\epsilon}+\ln\left(\frac{\mu^2}{M^2}\right)+\frac{m^2}{M^2}\ln\left(\frac{m^2}{M^2}\right)+\frac{s+t+u}{6M^2}\right] \end{aligned}$$

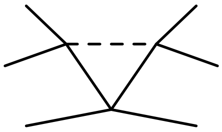


$$\begin{aligned} &= \frac{4!\cdot\binom{3}{2}\cdot 2}{2!(3!)^2}(-i\lambda)^2\int\frac{d^dk}{(2\pi)^d}\frac{i}{p^2-M^2}\frac{i}{(k+p')^2-m^2} \\ &= \frac{i}{16\pi^2}2\lambda^2\left[-\frac{m^2}{M^2}\left(1+\frac{1}{\epsilon}-\ln\left(\frac{m^2}{\mu^2}\right)\right)\right]. \end{aligned} \tag{S.6}$$

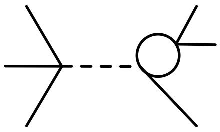
Notice that the divergence in the various diagrams of either the full or the effective theory are cancelled by the appropriate counterterms once the theories are renormalized. Therefore they shall not affect the matching procedure.

[1] For your convenience, we will be providing those details for the rest of the solutions as well. However, unless there is a momentum dependence, we shall include in the nominator together with the symmetry factor the number of all equivalent topologies.

The next contribution to the one-loop effective theory comes from the six-point Green function. The full theory provides two diagrams for the matching:



$$
\begin{aligned}
&= \frac{6!\cdot\binom{3}{1}^2\cdot\binom{4}{2}\cdot 2\cdot\binom{3}{2}}{(3!)^3 4!}(-i\kappa)(-i\lambda)^2 \\
&\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-M^2}\frac{i}{(k+p)^2-m^2}\frac{i}{(k+p')^2-m^2} \\
&= \frac{i}{16\pi^2}45\kappa\lambda^2\left[\frac{1}{M^2}+\frac{1}{M^2}\ln\left(\frac{m^2}{M^2}\right)\right],
\end{aligned}
\tag{S.7}
$$

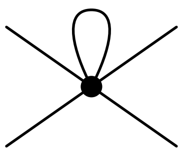


$$
\begin{aligned}
&= \frac{6!\cdot\binom{3}{2}\cdot 2\cdot\binom{4}{2}\cdot 2\cdot\binom{3}{2}}{(3!)^3 4!}(-i\kappa)(-i\lambda)^2 \\
&\int\frac{d^dk}{(2\pi)^d}\frac{i}{p^2-M^2}\frac{i}{(k+p)^2-m^2}\frac{i}{(k+p')^2-m^2} \\
&= \frac{i}{16\pi^2}\frac{30\kappa\lambda^2}{M^2}\left[-\frac{1}{\epsilon}+\ln\left(\frac{m^2}{\mu^2}\right)\right].
\end{aligned}
$$

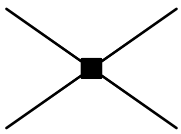
The results of the full theory allow for an educated guess regarding the set operators that shall arise in the EFT. In particular, we would need to include: a dimension-6 operator with four light fields and derivatives (that would generate the momentum dependent part), the 4- and 6-light field operators already present in $\mathcal{L}_{\text{eff}}^{\text{tree}}$ that would receive one-loop corrections. The effective Lagrangian matched at one-loop becomes then:

$$
\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{\text{tree}} + \frac{\alpha}{4!}\varphi^4 + \frac{\beta}{4!M^2}\varphi^2\partial^2\varphi^2 + \frac{\gamma}{6!M^2}\varphi^6, \tag{S.8}
$$

The EFT contributions to the four-point Green function read

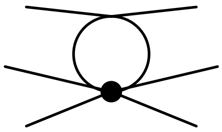


$$
\begin{aligned}
&= \frac{6\cdot 5\cdot 4\cdot 3}{72}\frac{(i\lambda^2)}{M^2}\sum_{s,u,t}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} \\
&= -\frac{i}{16\pi^2}\frac{5\lambda^2}{M^2}m^2\left[1+\frac{1}{\epsilon}-\ln\left(\frac{m^2}{\mu^2}\right)\right],
\end{aligned}
\tag{S.9}
$$

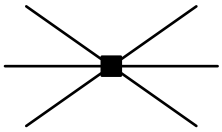


$$
= i\alpha - i\frac{\beta}{3M^2}(s+t+u).
$$

The six-point effective theory amplitude gives,



$$= \frac{6!\cdot\binom{6}{2}\cdot 2\cdot\binom{4}{2}\cdot 2}{2!\cdot 72\cdot 4!}\frac{(-i\kappa)(i\lambda^2)}{M^2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{(k+p')^2-m^2}\frac{i}{(k+p)^2-m^2}$$
$$= -\frac{i}{16\pi^2}\frac{75\kappa\lambda^2}{M^2}\left[\frac{1}{\epsilon}-\ln\left(\frac{m^2}{\mu^2}\right)\right],$$



$$= i\frac{\gamma}{M^2}. \tag{S.10}$$

Finally, we may perform the matching.

$$\alpha = \frac{3\lambda^2}{16\pi^2}\left[1+\frac{m^2}{M^2}+\left(1+\frac{m^2}{M^2}\right)\ln\left(\frac{\mu^2}{M^2}\right)\right],\quad \beta = -\frac{3\lambda^2}{32\pi^2}\quad\text{and}\quad \gamma = \frac{45\kappa\lambda^2}{16\pi^2}\left[1+\ln\left(\frac{\mu^2}{M^2}\right)\right]. \tag{S.11}$$

(c) We repeat the calculations of the full theory this time by seperating the contributions from the hard and soft regions.

$$\begin{aligned}
(\text{S.6a}) &= \lambda^2\sum_{\pi(p)}\left(\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-M^2}\frac{1}{(k+p)^2}+\int\frac{d^dk}{(2\pi)^d}\frac{1}{-M^2}\frac{1}{(k+p)^2-m^2}\right)\\
&= \frac{i}{16\pi^2}3\lambda^2\left[1+\frac{1}{\epsilon}+\frac{m^2}{M^2}\left(1+\frac{1}{\epsilon}+\ln\left(\frac{\mu^2}{M^2}\right)\right)+\frac{s+t+u}{6M^2}+\ln\left(\frac{\mu^2}{M^2}\right)\right]\Bigg|_{\text{hard}}\\
&+\frac{i}{16\pi^2}3\lambda^2\left[-\frac{m^2}{M^2}\left(1+\frac{1}{\epsilon}-\ln\left(\frac{m^2}{\mu^2}\right)\right)\right]\Bigg|_{\text{soft}}\\
(\text{S.6b}) &= 2\lambda^2\left(\int\frac{d^dk}{(2\pi)^d}\frac{1}{-M^2}\frac{1}{k^2}+\int\frac{d^dk}{(2\pi)^d}\frac{1}{-M^2}\frac{1}{(k+p')^2-m^2}\right)\\
&= \frac{i}{16\pi^2}2\lambda^2\left[-\frac{m^2}{M^2}\left(1+\frac{1}{\epsilon}-\ln\left(\frac{m^2}{\mu^2}\right)\right)\right]\Bigg|_{\text{soft}}
\end{aligned} \tag{S.12}$$

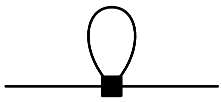
$$\begin{aligned}
(\text{S.7a}) &= 45\kappa\lambda^2\left(\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-M^2}\frac{1}{(k+p)^2}\frac{1}{(k+p')^2}+\int\frac{d^dk}{(2\pi)^d}\frac{1}{-M^2}\frac{1}{(k+p)^2-m^2}\frac{1}{(k+p')^2-m^2}\right)\\
&= \frac{i}{16\pi^2}45\kappa\lambda^2\left[\frac{1}{M^2}\left(1+\frac{1}{\epsilon}+\ln\left(\frac{\mu^2}{M^2}\right)\right)\right]\Bigg|_{\text{hard}}+\frac{i}{16\pi^2}45\kappa\lambda^2\left[-\frac{1}{M^2}\left(\frac{1}{\epsilon}-\ln\left(\frac{m^2}{\mu^2}\right)\right)\right]\Bigg|_{\text{soft}}\\
(\text{S.7b}) &= 30\kappa\lambda^2\left(\int\frac{d^dk}{(2\pi)^d}\frac{1}{-M^2}\frac{1}{(k+p)^2-m^2}\frac{1}{(k+p')^2-m^2}+\int\frac{d^dk}{(2\pi)^d}\frac{1}{-M^2}\frac{1}{(k+p)^2}\frac{1}{(k+p')^2}\right)\\
&= \frac{i}{16\pi^2}30\kappa\lambda^2\left[-\frac{1}{M^2}\left(\frac{1}{\epsilon}-\ln\left(\frac{m^2}{\mu^2}\right)\right)\right]\Bigg|_{\text{soft}}
\end{aligned} \tag{S.13}$$

One may thus explicitly verify one of the main messages from the lecture, namely that the matching coefficients are determined solely by the hard part of the one-loop full-theory amplitudes and the soft components, which contain divergences match exactly the one-loop diagrams in the EFT.

(d) Next, we examine the running of the matching coefficients from the scale of the matching $\mu = M$ down to the scales relevant to the process $\mu \approx M$. To this end, we renormalize the effective Lagrangian S.8, by assigning,

$$\alpha(1+\delta_\alpha) = Z_\varphi^2 \alpha^{(0)}, \quad \beta(1+\delta_\beta) = Z_\varphi^2 \beta^{(0)} \quad \text{and} \quad \gamma(1+\delta_\gamma) = Z_\varphi^4 \gamma^{(0)}, \tag{S.14}$$

where the subscript (0) denotes the matching coefficients in Eq. (S.11) after reabsorbing the respective coefficients of the tree-level generated $\phi^4$ and $\phi^6$ operators and $Z_\varphi$ is the wave function renormalization computed from the following tadpole diagram[2],



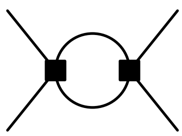
$$\begin{aligned} &= \frac{2\cdot\binom{4}{2}}{4!}\left[(i\alpha)\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} + \frac{i\beta}{M^2}\int\frac{d^dk}{(2\pi)^d}\frac{iV[k,-k,p,-p]}{k^2-m^2}\right] \\ &= -\frac{i}{16\pi^2}\frac{\alpha m^2}{2}\left[1+\frac{1}{\epsilon}-\ln\left(\frac{m^2}{\mu^2}\right)\right] + \frac{i}{16\pi^2}\frac{16\beta m^2}{M^2}(m^2+p^2)\left[1+\frac{1}{\epsilon}-\ln\left(\frac{m^2}{\mu^2}\right)\right] \\ &\Rightarrow Z_\varphi = 1 - \frac{1}{16\pi^2}\frac{16m^2\beta}{M^2}\frac{1}{\epsilon}. \end{aligned} \tag{S.15}$$

where

$$V[p_1,p_2,p_3,p_4] = -4\left[(p_1+p_2)^2+(p_1+p_3)^2+(p_1+p_4)^2+(p_2+p_3)^2+(p_2+p_4)^2+(p_3+p_4)^2\right] \tag{S.16}$$

is the vertex function derived by the Feynman rule for the $\varphi^2\partial^2\varphi^2$ operator.

In order to determine the counterterms we need to compute the one-loop diagrams that are present in the EFT. In addition to the ones already evaluated in part (a), we have also the following diagram (which would arise at two-loop level in the full theory)



$$\begin{aligned} &= i\frac{8\cdot3\cdot4\cdot3\cdot2}{2!(4!)^2}\sum_{\pi(p)}\left[(i\alpha)^2\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2}\frac{i}{(k+p)^2-m^2}\right. \\ &\left.+\frac{(i\alpha)(i\beta)}{M^2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2}\frac{iV[p,p',-k-p-p',k]}{(k+p)^2-m^2}\right] \\ &= -\frac{i}{16\pi^2}\frac{3}{2}\alpha^2\left(\frac{1}{\epsilon}-\ln\left(\frac{m^2}{\mu^2}\right)+\frac{s+t+u}{18m^2}\right) \\ &+\frac{i}{16\pi^2}\frac{12\alpha\beta}{M^2}\left[8m^2\left(\frac{3}{2}+\frac{1}{\epsilon}-\ln\left(\frac{m^2}{\mu^2}\right)\right)+(s+t+u)\left(2+\frac{1}{\epsilon}-\ln\left(\frac{m^2}{\mu^2}\right)\right)\right]. \end{aligned} \tag{S.17}$$

The loop diagrams (S.9a) and (S.10a) are rescaled with respect to the EFT operators and we extract the divergences $-\frac{i}{16\pi^2}\frac{\gamma m^2}{2M^2}\frac{1}{\epsilon}$ and $-\frac{i}{16\pi^2}\frac{15\alpha\gamma}{2M^2}\frac{1}{\epsilon}$, respectively.

[2] Remember that the counterterm for the propagator is defined as $i(p^2\delta_{Z_\phi}-\delta_m)$.

Now we are ready to read off the counterterms of the effective operators for the 4- and 6-field vertices:

$$\delta_\alpha = \frac{1}{16\pi^2}\left(\frac{\gamma m^2}{2\alpha M^2} + \frac{3\alpha}{2} - \frac{96\beta m^2}{M^2}\right)\frac{1}{\epsilon}, \quad \delta_\beta = -\frac{1}{16\pi^2}12\alpha\frac{1}{\epsilon} \quad \text{and} \quad \delta_\gamma = \frac{1}{16\pi^2}\frac{15\alpha}{2}\frac{1}{\epsilon}. \tag{S.18}$$

In general the RG equations read,

$$\mu\frac{d\mathcal{C}_i^{(0)}}{d\mu} = 0 \Rightarrow \mu\frac{d}{d\mu}\left(\frac{\mathcal{C}_i(1+\delta_{\mathcal{C}_i})}{Z_\phi^n}\right) = 0 \Rightarrow \mu\frac{d\mathcal{C}_i}{d\mu} = -\mathcal{C}_i\mu\frac{d}{d\mu}\left[(\delta_{\mathcal{C}_i} - n\delta_\varphi)\right]. \tag{S.19}$$

and in the RHS of the Equation we replace the derivatives of the coefficients with the leading-order term,

$$\mu\frac{d\mathcal{C}_i}{d\mu} = -[\mathcal{C}_i]\mathcal{C}_i, \tag{S.20}$$

where $[\mathcal{C}_i]$ is the dimension of the coefficient $\mathcal{C}_i$ in the $D = 4-2\epsilon$ dimensional regularization scheme. In our case, for $\alpha$ and $\beta$ we have $n = 2$ and $[\alpha] = [\beta] = 2\epsilon$ and for $\gamma$ we have $n = 3$ and $[\gamma] = 4\epsilon$.

After some algebra, we get the equations,

$$\mu\frac{d\alpha}{d\mu} = \frac{3\alpha^2}{16\pi^2} + \frac{m^2}{M^2}\left(-\frac{8\alpha\beta}{\pi^2} + \frac{\gamma}{16\pi^2}\right) \tag{S.21}$$

$$\mu\frac{d\beta}{d\mu} = -\frac{3\alpha\beta}{2\pi^2} + \frac{m^2}{M^2}\frac{4\beta^2}{\pi^2} \tag{S.22}$$

$$\mu\frac{d\gamma}{d\mu} = \frac{15\alpha\gamma}{8\pi^2} + \frac{m^2}{M^2}\frac{6\beta\gamma}{\pi^2}, \tag{S.23}$$

We solve the system of differential equations order by order in $m^2/M^2$. At $\mathcal{O}(1)$ we have,

$$\begin{aligned}\alpha(\mu) &= \frac{\alpha(M)}{1 - \frac{3}{32\pi^2}\alpha(M)\ln\left(\frac{\mu^2}{M^2}\right)}\\ \beta(\mu) &= \frac{\beta(M)}{\left[1 - \frac{3}{32\pi^2}\alpha(M)\ln\left(\frac{\mu^2}{M^2}\right)\right]^{-8}}\\ \gamma(\mu) &= \frac{\gamma(M)}{\left[1 - \frac{3}{32\pi^2}\alpha(M)\ln\left(\frac{\mu^2}{M^2}\right)\right]^{10}}.\end{aligned} \tag{S.24}$$

We plug these back in Eq. (S.19) and solve the equations again at $\mathcal{O}(m^2/M^2)$ this time choosing $\alpha(M) = 0$,

$$\begin{aligned}\alpha(\mu) &= \frac{m^2}{M^2}\frac{\gamma(M)}{16\pi^2}\ln\left(\frac{\mu^2}{M^2}\right)\\ \beta(\mu) &= \beta(M) - \frac{m^2}{M^2}\frac{2\beta(M)}{\pi^2}\ln\left(\frac{\mu^2}{M^2}\right)\\ \gamma(\mu) &= \gamma(M) - \frac{m^2}{M^2}\frac{3\beta(M)\gamma(M)}{\pi^2}\ln\left(\frac{\mu^2}{M^2}\right).\end{aligned} \tag{S.25}$$

We confirm that the $\varphi^4$ operator is generated by quantum effects even if it is absent at the UV scale.

Finally, a term of the type $\varphi^3$ is not generated, because the original Lagrangian respectes a $Z_2$ symmetry, which can not broken by the RGE effects.

(e) We proceed by using the background field method as explained in the lecture: $\Phi \to \Phi_{\rm cl} + \Phi$ and $\varphi \to \varphi_{\rm cl} + \varphi$. Then we define the vector $\eta = (\Phi, \varphi)^\intercal$. and the fluctuation operator:

$$\begin{aligned} \Delta_H(x, \partial) &= -\partial^2 - M^2, \\ \Delta_S(x, \partial) &= -\partial^2 - m^2 - \frac{\kappa}{2}\varphi_{\rm cl}^2 - \lambda \varphi_{\rm cl}\Phi_{\rm cl}, \\ X_{SH}(x, \partial) &= -\frac{\lambda}{2}\varphi_{\rm cl}^2, \end{aligned} \tag{S.26}$$

In order to construct $\widetilde{\Delta}_H(x, \partial + ik)$ in Eq. (1.69) we need to determine $\Delta_S^{-1}(x, \partial + ik)$:

$$\begin{aligned} \Delta_S(x, \partial + ik) &= k^2 - m^2 - 2ik_\mu \partial^\mu - \partial^2 - \frac{\kappa}{2}\varphi_{\rm cl}^2 - \lambda\varphi_{\rm cl}\Phi_{\rm cl}, \\ \Delta_S^{-1}(x, \partial + ik) &= \frac{1}{k^2}\left(1 + \frac{m^2}{k^2}\right) + \frac{1}{k^4}\left(2ik_\mu\partial^\mu + \partial^2 + \frac{\kappa}{2}\varphi_{\rm cl}^2\right) - 4\frac{k_\mu k_\nu}{k^6}\partial^\mu\partial^\nu + \mathcal{O}(\Lambda^{-5}). \end{aligned} \tag{S.27}$$

Now, we use this result and focus on the part $U = X_{SH}^\dagger \Delta_S^{-1} X_{SH}$, that generates interactions between the fields $\varphi_{\rm cl}$,

$$U(x, \partial_x + ip) = \frac{\lambda^2}{4}\varphi_{\rm cl}^2\left[\frac{1}{k^2}\left(1 + \frac{m^2}{k^2}\right) + \frac{1}{k^4}\left(2ik_\mu\partial^\mu + \partial^2 + \frac{\kappa}{2}\varphi_{\rm cl}^2\right) - 4\frac{k_\mu k_\nu}{k^6}\partial^\mu\partial^\nu\right]\varphi_{\rm cl}^2. \tag{S.28}$$

Inserting this operator in Eq. (1.83), we notice that at order $\mathcal{O}(\Lambda^{-2})$ only the $n = 1$ term contributes, with

$$\mathcal{L}_{\rm EFT}^{\rm 1loop} = -\frac{i}{2}\int \frac{d^dk}{(2\pi)^d}\frac{U(x, \partial_x + ik)}{k^2 - M^2}. \tag{S.29}$$

The momentum integration can be readily performed: In the $\overline{\rm MS}$ regularization scheme with $\mu = M$ we finally obtain

$$\mathcal{L}_{\rm EFT}^{\rm 1loop} = \frac{\lambda^2}{16(16\pi^2)}\left[2\left(1 + \frac{m^2}{M^2}\right)\varphi_{\rm cl}^4 - \frac{1}{M^2}\varphi_{\rm cl}^2\partial^2\varphi_{\rm cl}^2 + \frac{\kappa}{M^2}\varphi_{\rm cl}^6\right]. \tag{S.30}$$

**Exercise 3.** ***Redundant operators***

*Consider the following effective Lagrangian,*

$$\mathcal{L}_{\rm eff} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 + \frac{\mathcal{C}_1}{6!\Lambda^2}\varphi^6 + \frac{\mathcal{C}_2}{3!\Lambda^2}\varphi^3\partial^2\varphi + \frac{\mathcal{C}_3}{3!\Lambda^2}(\partial^\mu\varphi^3)\partial_\mu\varphi + \mathcal{O}\left(\frac{1}{\Lambda^3}\right). \tag{2}$$

*(a) Use the equations of motion and integration-by-parts identities to reduce the operator basis to a non-redundant set.*

*(b) Show that the use of the equations of motion is equivalent to a field redefinition of the form $\varphi \to \varphi + \alpha\varphi^3$, with an appropriate choice of $\alpha$.*

**Solution.**

(a) Let us start with the final term in 2. Using integration by parts under the Lagrangian density spacetime integral we get:

$$\frac{\mathcal{C}_3}{3!\Lambda^2}(\partial^\mu\varphi^3)\partial_\mu\varphi = -\frac{\mathcal{C}_3}{3!\Lambda^2}\varphi^3\partial^2\varphi. \tag{S.31}$$

So, we may define the Wilson Coefficient of the $\varphi^3\partial^2\varphi$ operator as:

$$\frac{\mathcal{C}_2'}{3!\Lambda^2} = \frac{\mathcal{C}_2 - \mathcal{C}_3}{3!\Lambda^2}. \tag{S.32}$$

Now, the equation of motion read,

$$-\partial^2\varphi - m^2\varphi - \frac{\lambda}{3!}\varphi^3 + \frac{\mathcal{C}_1}{5!\Lambda^2}\varphi^5 + \frac{\mathcal{C}_2'}{2!\Lambda^2}\varphi^2\partial^2\varphi = 0 \tag{S.33}$$

Solving the above equation for $\partial^2\varphi$ and replacing back in the Lagrangian keeping only the terms up to order $\mathcal{O}\left(\frac{1}{\Lambda^3}\right)$ yields:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{m^2}{2}\varphi^2 - \frac{\lambda'}{4!}\varphi^4 + \frac{\mathcal{C}_1'}{6!\Lambda^2}\varphi^6 + \mathcal{O}\left(\frac{1}{\Lambda^3}\right). \tag{S.34}$$

where $\lambda' = \lambda + \frac{4\mathcal{C}_2' m^2}{\Lambda^2}$ and $\mathcal{C}_1' = \mathcal{C}_1 - \frac{20\mathcal{C}_2'\lambda}{\Lambda^2}$.

(b) The field redefinition $\varphi \to \varphi + \alpha\varphi^3$ gives,

$$\frac{1}{2}(\partial_\mu\varphi)^2 \to \frac{1}{2}(\partial_\mu\varphi)^2 - \alpha\frac{1}{\Lambda^2}\varphi^3\partial^2\varphi. \tag{S.35}$$

If $\alpha = \frac{\mathcal{C}_2'}{3!}$ then the term $\varphi^3\partial^2\varphi$ is cancelled. The rest of the shifts give,

$$\begin{aligned}
&\frac{m^2}{2}\varphi^2 \to \frac{m^2}{2}\varphi^2 + \frac{m^2\mathcal{C}_2'}{3!\Lambda^2}\varphi^4 \\
&\frac{\lambda}{4!}\varphi^4 \to \frac{\lambda}{4!}\varphi^4 + \frac{20\lambda\mathcal{C}_2'}{6!\Lambda^2}\varphi^6 \\
&\frac{\mathcal{C}_1}{6!\Lambda^2}\varphi^6 \to \frac{\mathcal{C}_1}{6!\Lambda^2}\varphi^6 \\
&\frac{\mathcal{C}_2'}{3!\Lambda^2}\varphi^3\partial^2\varphi \to \frac{\mathcal{C}_2'}{3!\Lambda^2}\varphi^3\partial^2\varphi
\end{aligned}$$

which results in S.34.