MMP I
Exercise Sheet 12

HS 21
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Exercise 1 [Multipole decomposition (6 points)]

a) A density distribution $\rho \in \mathcal{C}_0(\mathbb{R}^3)$ generates the potential

$$\phi(\vec{x}) = \int_{\mathbb{R}^3} \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \mathrm{d}^3 x'.$$

Show that outside of the charge distribution, ϕ can be written as

$$\phi(\vec{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}} q_{lm} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}},$$

where the q_{lm} stand for the so-called *exterior multipole moments* of $\rho(\vec{x}')$. Write down the expression for q_{lm}.

Hint: The denominator is a generating function for Legendre polynomials $P_l(\cos\gamma)$:

$$\frac{1}{\sqrt{r^2 - 2rr'\cos\gamma + r'^2}} = \frac{1}{r_>} \sum_{\ell=0}^{\infty} \left(\frac{r_<}{r_>}\right)^{\ell} P_\ell(\cos\gamma)$$

where $r_< := \min\{|\vec{x}|, |\vec{x}'|\}$ and $r_> := \max\{|\vec{x}|, |\vec{x}'|\}$. Use then the addition theorem for spherical harmonics

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi').$$

What is the meaning of the angle γ?

b) Compute the monopole moment q_{00}. To what quantity is it equivalent in electrostatics or gravitation? Assuming that the particles in the (isolated) system with $\rho(\vec{x})$ are moving around: does the monopole moment vary with time?

– please turn over –

Exercise 2 [Distributions (3 points)]]

a) What is a distribution? What is the definition of the Dirac delta distribution?

b) Show that the spherical coordinate δ distribution may be written as

$$\delta(\mathbf{r} - \mathbf{r}_0) = \delta(r - r_0)\delta(\cos\theta - \cos\theta_0)\delta(\varphi - \varphi_0)/r^2,$$

using the definition of the delta distribution and spherical coordinates.