1. Hamiltonian Dynamics

Solar system dynamics is very unusual, as dynamical systems go. The reasons are (i) deviations from the exactly soluble two-body force law are small ($M_{\rm Jup} \simeq 10^{-3} M_\odot$ and tidal forces also mostly small), but (ii) the time scales are very long, $\sim 10^{10}$ orbits even. Thus it is crucial to be able to study small perturbations of the two-body problem over very long times. The Hamiltonian formulation is an elegant way of doing this and was developed in the 19th century mainly to study solar system dynamics. It got cleaned up a lot in the 20th century, and developed in some unexpected ways, and continues to be quite an active research area. Not all problems in solar system dynamics are Hamiltonian (in particular, dissipative forces are excluded) but many are. And this is our reason for studying Hamiltonians at this point.

Hamiltonian dynamics may looks very abstract and apparently unconnected with things going round in the solar system, but it's really a glorified coordinate transformation technique: the general idea is to change from positions and velocities to some abstract variables in which the dynamics is simple or trivial.

Hamilton's principle and Hamilton's equations

In Hamiltonian dynamics the variables are the 'canonical coordinates' usually $\mathbf{q}$, and the 'canonical momenta' usually $\mathbf{p}$. These may be ordinary positions and velocities or they may be different. Say there are N each of coordinates and momenta. Then we say there are N 'degrees of freedom'. However there are really $2N$ variables, bceause the $\mathbf{p}$ and $\mathbf{q}$ really are mutually independent variables; the $2N$-dimensional space $(\mathbf{p}, \mathbf{q})$ is called phase space.

Now the whole dynamics is run by one scalar function $H(\mathbf{p}, \mathbf{q}, t)$ called the Hamiltonian. That is to say, the particle paths $\mathbf{q}(t)$ and their momenta $\mathbf{p}(t)$ are governed by H. The paths do whatever it takes to make a certain functional stationary, specifically

$$\begin{aligned} &\delta \int \mathbf{p} \cdot d\mathbf{q} - H\, dt = 0 \\ &\text{or} \\ &\delta \int (\mathbf{p} \cdot \dot{\mathbf{q}} - H)\, dt = 0 \end{aligned} \tag{1.1}$$

with $\mathbf{q}, t$ fixed at the ends.

'Hamilton's principle' (1.1) is in fact equivalent to 'Hamilton's equations':[1]

$$\boxed{\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}\,, \qquad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}} \tag{1.2}$$

Proof Doing the usual calculus-of-variations thing in (1.1) we have

$$\int \left(\dot{\mathbf{q}} \cdot \delta\mathbf{p} + \mathbf{p} \cdot \delta\dot{\mathbf{q}} - \frac{\partial H}{\partial \mathbf{q}}\, \delta\mathbf{q} - \frac{\partial H}{\partial \mathbf{p}}\, \delta\mathbf{p} \right) dt = 0, \tag{1.3}$$

[1] You will be expected to know by heart any equation in boxes. The sections marked 'proof' or 'digression' are optional. Everything else you should understand but need not memorize.

or

$$\mathbf{p}\cdot\delta\mathbf{q}\Big|+\int\left(\dot{\mathbf{q}}-\frac{\partial H}{\partial\mathbf{p}}\right)\delta\mathbf{p}\,dt-\int\left(\dot{\mathbf{p}}+\frac{\partial H}{\partial\mathbf{q}}\right)\delta\mathbf{q}\,dt=0 \tag{1.4}$$

and since $\delta\mathbf{q}$ is fixed at the ends but otherwise the variations are arbitrary we have (1.2) □

EXAMPLE [Particle in a potential] In this simple (but important) case we have

$$H=\frac{\mathbf{p}^2}{2m}+V(\mathbf{q}) \tag{1.5}$$

Hamilton's equations are then $\dot{\mathbf{p}}=-\partial V/\partial\mathbf{q}$ (i.e., Newton's second law) and $\dot{\mathbf{q}}=\mathbf{p}/m$ relating momentum and velocity. Combining these we get the familiar

$$m\ddot{\mathbf{q}}=-\frac{\partial V}{\partial\mathbf{q}} \tag{1.6}$$

with $\mathbf{p}$ being the usual momentum. □

PROBLEM 1.1: But $\mathbf{p}$ isn't always $m\dot{\mathbf{q}}$, and we need to say 'canonical momentum' for $\mathbf{p}$ to avoid confusion with the ordinary momentum $m\dot{\mathbf{q}}$. For example, a charged particle in a magnetic field has

$$H=\frac{1}{2m}(\mathbf{p}-e\mathring{A})^2$$

where e is the charge and A is the vector potential of the magnetic field.

Write down Hamilton's equations for this case, assuming that $\mathbf{A}$ depends on $\mathbf{q}$ but not on t. Then play around a bit (you will get a term $\dot{A}$, and will have to think about what it means and what to do with it) to show that

$$m\ddot{\mathbf{q}}=e\,\dot{\mathbf{q}}\times\nabla\times\mathbf{A}$$

which is the Lorentz force.

Each problem comes with a rating **1**: [easiest] to **4**: [hardest]. [**4**]

Autonomous systems

The time-derivative of H along trajectories equals the time-derivative at fixed $(\mathbf{p}, \mathbf{q})$, that is

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{1.7}$$

which we can see by expanding out the total derivative and using Hamilton's equations. Thus if H has no explicit t dependence (we call such systems autonomous) then H is a constant of motion, in fact the conserved energy.

An autonomous Hamiltonian system with N degrees of freedom can always be written as a (non-autonomous) Hamiltonian system with $N-1$ degrees of freedom (by treating one of the q variables as a formal time).

PROOF Suppose we are given $H(\mathbf{p}, \mathbf{q})$. Define

$$\tau \equiv -q_N \qquad \text{and} \qquad F(p_1 \dots p_{N-1}, q_1 \dots q_{N-1}, \tau) \equiv p_N. \tag{1.8}$$

Then

$$\mathbf{p} \cdot d\mathbf{q} - H dt = p_1 dq_1 + \ldots + p_{N-1} dq_{N-1} - F d\tau - d(tH) + t dH. \tag{1.9}$$

Since $dH = 0$ for an autonomous system, we can replace (1.1)) by

$$\delta \int p_1 dq_1 + \ldots + p_{N-1} dq_{N-1} - F d\tau = 0 \tag{1.10}$$

which amounts to a (non-autonomous) Hamiltonian system with $N-1$ degrees of freedom. □

Conversely, a non-autonomous Hamiltonian system with N degrees of freedom can always be replaced by an autonomous one with $N+1$ degrees of freedom (by making t into q_{N+1}. Thus, no generality is lost by considering only autonomous Hamiltonians.

PROOF Suppose we are given $H(\mathbf{p}, \mathbf{q}, t)$. We proceed to treat t as q_{N+1} and introduce the new variables p_{N+1} and τ. Defining

$$F(p_1 \dots p_N, p_{N+1}, q_1 \dots q_N, t) \equiv H(p_1 \dots p_N, q_1 \dots q_N, t) + p_{N+1} \tag{1.11}$$

the equations

$$\begin{aligned} &\frac{dq_i}{d\tau} = \frac{\partial F}{\partial p_i}, \quad \frac{dp_i}{d\tau} = -\frac{\partial F}{\partial q_i}, \qquad i = 1 \dots N, \\ &\frac{dt}{d\tau} = \frac{\partial F}{\partial p_{N+1}}, \quad \frac{dp_{N+1}}{d\tau} = -\frac{\partial F}{\partial t} = -\frac{\partial H}{\partial t}, \end{aligned} \tag{1.12}$$

which together define an autonomous Hamiltonian system with $N+1$ degrees of freedom, are equivalent to the original (non-autonomous) system. □

Canonical transformations

Going back to the variational principle (1.1) it is clear that if we add a path-independent term to $\mathbf{p}\cdot d\mathbf{q} - H\,dt$, then the stationary paths will remain the same, i.e., the dynamics will not be affected.

Suppose someone else has different variables $(\mathbf{P}, \mathbf{Q})$ and some other Hamiltonian $F(\mathbf{P}, \mathbf{Q}, t)$. That means they have

$$\delta \int \mathbf{P}\cdot d\mathbf{Q} - F\,dt = 0 \tag{1.13}$$

with $\mathbf{Q}, t$ fixed at ends. They'll have the same dynamics as us if their integrand differs from ours by a path independent term, say dS, i.e.,

$$\oint \mathbf{P}\cdot d\mathbf{Q} - \mathbf{p}\cdot d\mathbf{q} = 0, \quad F = H - \frac{\partial S}{\partial t}\,. \tag{1.14}$$

In other words, variable changes that preserve $\oint \mathbf{p}\cdot d\mathbf{q}$ preserve dynamics (with appropriately modified H). Such variable changes are known as 'canonical transformations'. They are sort of area-preserving transformations.

This freedom to add path-independent terms is very important. Functions of the type S above can be used to specify canonical transformations. One useful route is to take a function $S(\mathbf{P}, \mathbf{q}, t)$ and write

$$\mathbf{p}\cdot d\mathbf{q} - H\,dt - \mathbf{P}\cdot d\mathbf{Q} + F\,dt = dS - d(\mathbf{Q}.\mathbf{P}). \tag{1.15}$$

Expanding this out and comparing coefficients we have

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}}\,, \quad \mathbf{Q} = \frac{\partial S}{\partial \mathbf{P}}\,, \quad F = H + \frac{\partial S}{\partial t}\,, \tag{1.16}$$

which gives $(\mathbf{p}, \mathbf{q}) \to (\mathbf{P}, \mathbf{Q})$ implicitly. Here S is called the generating function.

Example [Polar canonical variables] The generating function

$$S(P, q) = \int \sqrt{2P - q^2}\,dq \tag{1.17}$$

leads to the canonical transformation

$$\boxed{p = \sqrt{2P}\cos Q, \quad q = \sqrt{2P}\sin Q} \tag{1.18}$$

which we'll often use. In this case the S has no t-dependence, so $H = P$. P, Q are sort of polar variables.

Notice that if $H = \frac{1}{2}(p^2 + q^2)$ (a harmonic oscillator) then $F = P$. □

PROBLEM 1.2: Consider a system with two degrees of freedom and coordinates (q_x, q_y) and momenta (p_x, p_y), and Hamiltonian

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + V(q_x, q_y, t).$$

Now the potential V is rotating, which is to say that actually $V = U(q'_x, q'_y)$ where

$$\begin{pmatrix} q'_x \\ q'_y \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} q_x \\ q_y \end{pmatrix}$$

Consider the generating function

$$S = \begin{pmatrix} P_x & P_y \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} q_x \\ q_y \end{pmatrix}$$

and use (1.16) to

(i) work out (Q_x, Q_y) in terms of (q_x, q_y),

(ii) work out (p_x, p_y) in terms of (P_x, P_y),

(iii) show that the transformed Hamiltonian is

$$F = \tfrac{1}{2}(P_x^2 + P_y^2) + U(Q_x, Q_y) + Q_x P_y - Q_y P_x.$$

Notice that this F has no explicit time dependence. It is a constant of motion and you have seen it before—where? [**3**]

EXAMPLE [Another generating function] A rather strange type of canonical transformation is specified in terms of the mean variables

$$\bar{\mathbf{p}} = \tfrac{1}{2}(\mathbf{p} + \mathbf{P}), \qquad \bar{\mathbf{q}} = \tfrac{1}{2}(\mathbf{q} + \mathbf{Q}). \tag{1.19}$$

Consider

$$\mathbf{p} \cdot d\mathbf{q} - \mathbf{P} \cdot d\mathbf{Q} = dS(\bar{\mathbf{p}}, \bar{\mathbf{q}}) + \tfrac{1}{2} d(\mathbf{p} \cdot \mathbf{q} - \mathbf{P} \cdot \mathbf{Q}) + d(\bar{\mathbf{p}} \cdot \mathbf{q} - \bar{\mathbf{q}} \cdot \mathbf{p}). \tag{1.20}$$

Expanding and eliminating $d\mathbf{Q}$ and $d\mathbf{P}$ gives

$$(\mathbf{p} - \mathbf{P}) \cdot d\bar{\mathbf{q}} - (\mathbf{q} - \mathbf{Q}) \cdot d\bar{\mathbf{p}} = dS \tag{1.21}$$

and expanding and comparing coefficients again gives

$$\mathbf{P} = \mathbf{p} - \frac{\partial S}{\partial \bar{\mathbf{q}}}, \qquad \mathbf{Q} = \mathbf{q} + \frac{\partial S}{\partial \bar{\mathbf{p}}}. \tag{1.22}$$

Notice that this is also implicit. □

The Hamilton-Jacobi Equation

Imagine canonical transformations taken to extremes; let's say we want the transformed Hamiltonian to be identically zero. That would make the dynamics simple indeed: $\mathbf{P}, \mathbf{Q}$ would just be constants. Using (1.16) we can see that this will happen if $S(\mathbf{P}, \mathbf{q})$ is such that

$$H\left(\frac{\partial S}{\partial \mathbf{q}}, \mathbf{q}, t\right) + \frac{\partial S}{\partial t} = 0. \tag{1.23}$$

This is known as the Hamilton-Jacobi equation, and it's a partial differential equation for $S(\mathbf{q})$. S will depend on $\mathbf{P}$ as well, but the $\mathbf{P}$, being constant, are parameters, not variables.

If H is autonomous, we can write

$$S(\mathbf{q}, t) = W(\mathbf{q}) - Et, \tag{1.24}$$

and hence

$$H(\frac{\partial W}{\partial \mathbf{q}}, \mathbf{q}) = E. \tag{1.25}$$

> Digression [The Schrödinger Equation] For a particle of mass m in a potential $V(\mathbf{q})$ the Schrödinger equation for the wavefunction $\psi(\mathbf{q})$ is
>
> $$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \mathbf{q}^2} + V(\mathbf{q})\right)\psi = i\hbar\frac{\partial \psi}{\partial t}\,. \tag{1.26}$$
>
> If we write
>
> $$\psi = \exp\left(\frac{i}{\hbar}S\right) \tag{1.27}$$
>
> and assume that S is slowly varying, we get
>
> $$\frac{1}{2m}\left(\frac{\partial S}{\partial \mathbf{q}}\right)^2 + V(\mathbf{q}) + \frac{\partial S}{\partial t} = 0, \tag{1.28}$$
>
> i.e., the Hamilton-Jacobi equation for a particle in a potential.
>
> Part of Schrödinger thinking when he dreamed up his equation was to have something that reduced to the Hamilton-Jacobi equation in some limit. And that is how Hamiltonians got into quantum mechanics. □

Upon solving The Hamilton-Jacobi equation for $S(\mathbf{q})$, the $\mathbf{P}$ are just constants of integration and their numerical values are arbitrary. But it's useful to have some convention for assigning the values of $\mathbf{P}$. The convention is to choose

$$\mathbf{P} = \frac{1}{2\pi}\oint \mathbf{p}\cdot d\mathbf{q}, \qquad \mathbf{Q} = \frac{\partial S}{\partial \mathbf{P}} \tag{1.29}$$

where the $\oint$ are taken over suitable closed paths, assuming these exist. (Note that the second equation in (1.29) involves differentiating with respect to parameters.) An advantage of this convention is that someone else who happened to start with a different canonical set $(\mathbf{p}', \mathbf{q}')$ will end up with the same $(\mathbf{P}, \mathbf{Q})$ if they choose the same closed curves.

Considering autonomous systems, we wil find that for all exactly soluble cases the transformed Hamiltonian will we $F(\mathbf{P})$, i.e., independent of $\mathbf{Q}$. In which case, examining (1.24), (1.25) and (1.29) we see that the solution will be

$$\mathbf{P} = \text{const}, \qquad \mathbf{Q} = \left(\frac{\partial F}{\partial \mathbf{P}}\right) t. \tag{1.30}$$

Such $(\mathbf{P}, \mathbf{Q})$ are known as action-angle variables.

In fact the existence of action-angles, in which variables the solution is of the form (1.30) is generic for integrable systems, and thus can be used as a *definition* of integrability. This is nowadays called the Liouville-Arnol'd theorem, but we won't go into it.

PROBLEM 1.3: What are the action-angle variables for a harmonic oscillator

$$H = \tfrac{1}{2}(p^2 + q^2)?$$

Trying applying (1.18) first and then considering the Hamilton-Jacobi equation in the new variables. [**2**]

Integrable systems are rare, so it's only for very few systems that we can solve the Hamilton-Jacobi equation and find action-angles. The two-body problem will be one. But once we've found them, we can use the $(\mathbf{P}, \mathbf{Q})$ to try and simplify the dynamics of nearby Hamiltonians. We write

$$F(\mathbf{P}) + \Delta F(\mathbf{P}, \mathbf{Q}) = F(\mathbf{P}) + \sum_n \Delta F_{\mathbf{n}}(\mathbf{P}) \exp(i\mathbf{n} \cdot \mathbf{Q}). \tag{1.31}$$

Here $(\mathbf{P}, \mathbf{Q})$ are action-angles for F but not for $F + \Delta F$, but they are perfectly legal canonical variables for *any* Hamiltonian system. What now happens is that we make further canonical transformations to transform away $\Delta F_{\mathbf{n}}$ term by term. This is perturbation theory, which we'll come to a bit later.

Poisson brackets

If we are offered a transformation $(\mathbf{p}, \mathbf{q}) \to (\mathbf{P}, \mathbf{Q})$ it's useful to be able to easily test that it's canonical. Such a test is

$$[\mathbf{Q}, \mathbf{P}] = \mathbf{1}, \qquad [\mathbf{P}, \mathbf{P}] = 0, \qquad [\mathbf{Q}, \mathbf{Q}] = 0 \tag{1.32}$$

where the square brackets denote Poisson brackets defined by

$$\boxed{[A, B] \equiv \left(\frac{\partial A}{\partial \mathbf{q}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{q}} \right)} \tag{1.33}$$

Poisson brackets are invariant under canonical transformations.

Proof It's useful to introduce the notation

$$\mathbf{z} = (\mathbf{p}, \mathbf{q}) \text{ or } (p_1 . . p_N, q_1 . . q_N) \tag{1.34}$$

and

$$\mathbf{J} = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \tag{1.35}$$

where the $\mathbf{1}$'s denote $N \times N$ identity matrices. In this notation Hamilton's equations become[2]

$$\dot{z}_i = J_{ij} \frac{\partial H}{\partial z_j} , \tag{1.36}$$

and Poisson brackets become

$$[A, B] = J_{ij} \frac{\partial A}{\partial z_i} \frac{\partial B}{\partial z_j} . \tag{1.37}$$

Now back to what we need to prove. For a canonical transformation we want $\mathbf{P} \cdot d\mathbf{Q} - \mathbf{p} \cdot d\mathbf{q}$ to be path-independent, i.e., an exact differential. The condition for being an exact differential won't change if we add an extra exact differential to it; let us add $\frac{1}{2} d/dt(\mathbf{p} \cdot \mathbf{q} - \mathbf{P} \cdot \mathbf{Q})$. Simplifying, our condition becomes

$$\mathbf{P} \cdot d\mathbf{Q} - \mathbf{Q} \cdot d\mathbf{P} - \mathbf{p} \cdot d\mathbf{q} + \mathbf{q} \cdot d\mathbf{p} = J_{ij}(z_i dz_j - Z_i dZ_j) \quad \text{exact diff.} \tag{1.38}$$

or

$$J_{ij} \left(z_i \frac{\partial z_j}{\partial Z_k} - Z_i \delta_{jk} \right) dZ_k \quad \text{exact diff.} \tag{1.39}$$

Now, an expression of the type $F_k dZ_k$ is an exact differential if and only if

$$\frac{\partial F_k}{\partial Z_l} = \frac{\partial F_l}{\partial Z_k} . \tag{1.40}$$

Applying this to (1.40) and simplifying yields

$$J_{kl} = J_{ij} \frac{\partial z_i}{\partial Z_k} \frac{\partial z_j}{\partial Z_l} \equiv \{Z_k, Z_l\} \quad \text{(say)}. \tag{1.41}$$

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[2] We'll often switch between vector and index notation in these notes. Index notation will use the summation convention.

If this footnote does not make sense *please ask.*

The expression $\{Z_k, Z_l\}$ is called a Lagrange bracket.

Remembering that $J_{ij}J_{jk} = -\delta_{jk}$ (which follows from the definition) it is readily seen that

$$\{Z_i, Z_j\}[Z_j, Z_k] = -\delta_{ik} \tag{1.42}$$

and using this identity, (1.41) is equivalent to

$$[z_i, z_j] = J_{ij}, \tag{1.43}$$

which is nothing but the Poisson bracket condition (1.32).

Applying the chain rule and (1.43) gives immediately

$$J_{ij}\,\frac{\partial A}{\partial Z_i}\,\frac{\partial B}{\partial Z_j} = J_{kl}\,\frac{\partial A}{\partial z_k}\,\frac{\partial B}{\partial z_l}\,, \tag{1.44}$$

which is to say that an arbitrary Poisson bracket $[A, B]$ is invariant under canonical transformations. □

Time evolution under Hamilton's equations is itself a canonical transformation, that is, the Poisson bracket condition (1.32) is true if $(\mathbf{P}, \mathbf{Q})$ happen to be just $(\mathbf{p}, \mathbf{q})$ at later times.

PROOF We need to show that the Poisson bracket $[z_i(t), z_j(t)] = J_{ij}$ remains true under time evolution, which follows from

$$\begin{aligned}\frac{d}{dt}\,[z_i(t), z_j(t)] &= J_{kl}\left(\frac{\partial}{\partial z_k}\,\dot{z}_i(t)\,\frac{\partial}{\partial z_l}\,z_j(t) + \frac{\partial}{\partial z_k}\,z_i(t)\,\frac{\partial}{\partial z_l}\,\dot{z}_j(t)\right)\\ &= J_{kj}\,\frac{\partial}{\partial z_k}\left(J_{il}\,\frac{\partial H}{\partial z_l}\right) + J_{il}\,\frac{\partial}{\partial z_l}\left(J_{jk}\,\frac{\partial H}{\partial z_k}\right) = 0,\end{aligned} \tag{1.45}$$

where the last step follows from the antisymmetry of J_{jk}. □

Lie transforms and perturbation theory

The total time derivative operator is

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \dot{\mathbf{q}} \frac{\partial}{\partial \mathbf{q}} + \dot{\mathbf{p}} \frac{\partial}{\partial \mathbf{p}} \tag{1.46}$$

Inserting Hamilton's equations here we get

$$\frac{d}{dt} = \frac{\partial}{\partial t} + [\ , H] \tag{1.47}$$

where $[\ , H]f$ means $[f, H]$. We can then write a formal time-development operator $\mathcal{T}(t)$ by exponentiating (1.47):

$$\mathcal{T}(t)f = \exp\left[t\left(\frac{\partial}{\partial t} + [\ , H]\right)\right] f \Big|_{t=0}. \tag{1.48}$$

We can (formally at least) use the time development operator (1.48) on any function. For example we might use it on some function $W(\mathbf{p}, \mathbf{q}, \epsilon)$ where ϵ is some parameter. We could even—and this is the cunning bit—swap the roles of t and ϵ, and the roles of H and W, and follow the ϵ-evolution of H under W. It gives us a potential method for getting rid of undesirable terms in H.

An ϵ-evolution operator of this type is called a Lie Transform. The operator $\mathcal{L}$ can be built out of its own generating function W as a power series in ϵ

$$\begin{aligned} W &= W_1 + \epsilon W_2 + \epsilon^2 W_3 + \ldots, \\ \mathcal{L} &= 1 + \epsilon \mathcal{L}_1 + \epsilon^2 \mathcal{L}_2 + \ldots, \\ \mathcal{L}_n &= \frac{1}{n} \sum_{m=1}^{n} [W_m, \mathcal{L}_{n-m}], \qquad \mathcal{L}_0 = 1, \end{aligned} \tag{1.49}$$

and applied thus

$$F(\mathbf{P}, \mathbf{Q}) = \mathcal{L}(\mathbf{p}, \mathbf{q}), \qquad \mathbf{p} = \mathcal{L}\mathbf{P}, \quad \mathbf{q} = \mathcal{L}\mathbf{Q}. \tag{1.50}$$

Note from (1.50) that the transform on functions goes from old-to-new, but the transform on the variables themselves goes from new-to-old.

Proof Differentiating (1.48) we obtain

$$\frac{\partial}{\partial t} \mathcal{T} = \mathcal{T}[\ , H], \tag{1.51}$$

which by the way is a differential equation that, with initial condition $\mathcal{T}|_{t=0} = 1$ specifies $\mathcal{T}(t)$. Using (1.51) and $\mathcal{T}^{-1}\mathcal{T} = \mathcal{T}\mathcal{T}^{-1} = 1$ we have

$$\frac{\partial}{\partial t} \mathcal{T}^{-1} = -[\ , H]\mathcal{T}^{-1}. \tag{1.52}$$

Now replacing t with ϵ and $\mathcal{T}^{-1}$ with $\mathcal{L}$ we get

$$\frac{\partial}{\partial \epsilon} \mathcal{L} = [W, \ ]\mathcal{L}. \tag{1.53}$$

Inserting the series expansions from the first two lines of (1.49) into (1.53) and sorting through the indices we can derive the third line of (1.49). □

To first order in ϵ we have

$$\boxed{\begin{aligned} &F(\mathbf{P},\mathbf{Q}) = \Big(1+\epsilon[W,\ ]\Big)H(\mathbf{p},\mathbf{q}),\\ &\mathbf{p} = \Big(1+\epsilon[W,\ ]\Big)\mathbf{P}, \quad \mathbf{q} = \Big(1+\epsilon[W,\ ]\Big)\mathbf{Q} \end{aligned}} \tag{1.54}$$

and the trick is to choose W so as to annihilate the leading-order $\mathbf{Q}$ dependence in F.

EXAMPLE [The pendulum] The pendulum Hamiltonian

$$H = \tfrac{1}{2}p^2 - \cos q \tag{1.55}$$

has two kinds of trajectories: if p is large then q circulates, if p is small then q oscillates (or librates) about 0. We'll find first order perturbative solutions for both cases by sticking an ϵ in and setting $\epsilon = 1$ at the end of the calculation.

Consider the large-p (circulating) case first, and write

$$H = \tfrac{1}{2}p^2 - \epsilon\cos q. \tag{1.56}$$

Using (1.54) we can annihilate the $\epsilon\cos q$ term to leading order by setting

$$W(p,q) = \frac{\sin q}{p} \tag{1.57}$$

which leads to the solution

$$\begin{aligned} &p = P + \epsilon\frac{\cos Q}{P}, \quad q = Q,\\ &P = \text{const}, \quad Q = Pt. \end{aligned} \tag{1.58}$$

In the librating case both p, q are small, and we keep the two leading terms of the cosine and write

$$H = \tfrac{1}{2}(p^2+q^2) - \tfrac{1}{24}\epsilon q^4. \tag{1.59}$$

Whenever $\frac{1}{2}(p^2+q^2)$ occurs, it usually pays to transform using (1.18) first; so we do this and then rename P, Q as p, q because we will need the capitals again soon. We get

$$H = p - \tfrac{1}{6}\epsilon p^2\cos^4 q = p - \tfrac{1}{16}\epsilon p^2 - \epsilon p^2\left(\tfrac{1}{12}\cos 2q + \tfrac{1}{48}\cos 4q\right). \tag{1.60}$$

This time we can annihilate the q-dependent terms to leading order via

$$W(p,q) = p^2\left(\tfrac{1}{24}\sin 2q + \tfrac{1}{192}\sin 4q\right) \tag{1.61}$$

which leads to the solution

$$\begin{aligned} &p = P + \epsilon P^2\left(\tfrac{1}{12}\cos 2Q + \tfrac{1}{48}\cos 4Q\right), \quad q = Q - \epsilon P\left(\tfrac{1}{12}\sin 2Q + \tfrac{1}{96}\sin 4Q\right),\\ &P = \text{const}, \quad Q = (1 - \tfrac{1}{8}\epsilon P)t. \end{aligned} \tag{1.62}$$

Variants of the pendulum come up again and again in the solar system. □

2. More about orbital elements

In the last chapter we developed Hamiltonian perturbation theory but the examples were all toy problems. In real solar system problems the same method applies, only the algebra becomes a truly awful mess.

What I mean by the last statement is that most solar system dynamics is about a system of the type

$$H_{\text{Kep}}(\mathbf{P}) + \epsilon H_{\text{pert}}(\mathbf{P}, \mathbf{Q}) \tag{2.1}$$

where H_{Kep} is a Keplerian or two-body Hamiltonian (or the sum of several of them), $(\mathbf{P}, \mathbf{Q})$ are the action-angles for it, and ϵH_{pert} is the perturbation from other bodies. Perturbation theory in the $(\mathbf{P}, \mathbf{Q})$ works exactly as in the previous chapter. What makes life difficult is that $(\mathbf{P}, \mathbf{Q})$ are not simple functions of cartesian variables. So while H_{pert} is usually simple in terms of cartesian $(\mathbf{p}, \mathbf{q})$ it is a horrible function of $(\mathbf{P}, \mathbf{Q})$.

We'll now spend some time discussing the action-angle variables for the two-body problem

$$H = \tfrac{1}{2}\mathbf{p}^2 - \frac{1}{r} \tag{2.2}$$

We jump the gun a little bit by writing down some definitions for the quantities u, f, M, which involve the as-yet-undefined quantities a, e.

$$\begin{aligned} r &= a(1 - e\cos u), \\ \tan\tfrac{1}{2}f &= \sqrt{\frac{1+e}{1-e}}\tan\tfrac{1}{2}u, \\ M &= u - e\sin u. \end{aligned} \tag{2.3}$$

We know what these are going to turn out to be of course, but right now a, e are just some constants and u, f, M are functions of r.

The actions

One can associate an action with each of the spherical polar coordinates, let's call them (P_r, P_θ, P_ϕ). (This is a general property of Hamiltonian that are spherically symmetric.) In terms of the familiar orbital elements they are

$$\begin{aligned} P_\phi &= \sqrt{a(1-e^2)}\cos I \\ P_\theta + P_\phi &= \sqrt{a(1-e^2)} \\ P_r + P_\theta + P_\phi &= \sqrt{a} \end{aligned} \tag{2.4}$$

and the Hamiltonian in terms of these is

$$H = -\frac{1}{2(P_r + P_\theta + P_\phi)^2}. \tag{2.5}$$

As you might guess from the form of (2.5) these actions translate into quantum numbers for the corresponding quantum problem, the Hydrogen atom.

PROOF The Hamilton-Jacobi equation is

$$\left(\frac{\partial W}{\partial r}\right)^2 + \left(\frac{1}{r}\,\frac{\partial W}{\partial \theta}\right)^2 \left(\frac{1}{r\sin\theta}\,\frac{\partial W}{\partial \phi}\right)^2 - \frac{1}{r} = 2E \tag{2.6}$$

which easily separates into

$$\begin{aligned} p_\phi &= \frac{\partial W}{\partial \phi} = L_z \\ p_\theta &= \frac{\partial W}{\partial \theta} = \sqrt{L^2 - \frac{L_z}{\sin^2\theta}} \\ p_r &= \frac{\partial W}{\partial r} = \sqrt{2E + \frac{2}{r} - \frac{L^2}{r^2}} \end{aligned} \tag{2.7}$$

with L_z, L, E being constants of integration.

At this point we introduce some auxiliary quantities, I, θ_0, a, and e, via

$$L_z = L\cos I = L\sin\theta_0, \qquad L^2 = a(1-e^2), \qquad E = -\frac{1}{2a}. \tag{2.8}$$

The new constants are just surrogates for the constants of integration E, L, L_z. With these substitutions we have

$$\sqrt{2E + \frac{2}{r} - \frac{L^2}{r^2}} = e\sqrt{a}\frac{\sin u}{r}. \tag{2.9}$$

We also note the identities

$$\sqrt{1 - \frac{\sin^2\theta_0}{\sin^2\theta}} = \frac{\cos\theta_0}{\sin\theta}\sqrt{1 - \frac{\cos^2\theta}{\cos^2\theta_0}} = \cos\theta_0\sqrt{1 - \frac{\cot^2\theta}{\cot^2\theta_0}} \tag{2.10}$$

and

$$\frac{1}{\pi}\int_0^{2\pi} \frac{\sin^2 u}{1 + e\cos u} = \frac{1}{e^2}\left(1 - \sqrt{1-e^2}\right). \tag{2.11}$$

Now working out $\oint p_r\,dr = \frac{1}{\pi}\int_{\mathrm{rmin}}^{\mathrm{rmax}} p_r\,dr$ and so on gives (2.4). □

PROBLEM 2.1: The Hamiltonian (2.2) should really be

$$H = \frac{\mathbf{p}^2}{2m} - \frac{GMm}{r}.$$

what we have done is chosen units so that $m = 1$ and $GM = 1$. What would $P_r + P_\theta + P_\phi$ be with the constants G, M, m all included? Work this out by dimensional arguments.

On this subject

$$\begin{aligned} GM_\odot &= 1.327124400 \times 10^{20}\,\mathrm{m}^3\,\mathrm{sec}^{-2} \\ G &= 6.672 * 10^{-11}\mathrm{kg}^{-1}\,\mathrm{m}^3\,\mathrm{sec}^{-2} \\ M_\odot &= 1.98911 \times 10^{30}. \end{aligned}$$

How come we have many more digits for $GM_\odot$ than for G or $M_\odot$? What does this say about SI units? [**3**]

The actions P_r, P_θ, P_ϕ are not unique. In fact a set of action-angles variables is transformed into another set of action-angles by

$$Q_i' = N_{ij} Q_j, \qquad P_i' = P_j N_{ji}^{-1} \tag{2.12}$$

for N_{ij} being any integer matrix with unit determinant. (This is a canonical transformation for any N_{ij} that is constant or only time-dependent, as can be verified using the generating function $S(\mathbf{P}', \mathbf{Q}) = P_i' N_{ij} Q_j$. The extra property of being an integer matrix with unit determinant ensures that the Q_i are still angles.) So let us change to a new set of actions

$$\begin{aligned} P_1 &= P_\phi \\ P_2 &= P_\theta + P_\phi \\ P_1 &= P_r + P_\theta + P_\phi. \end{aligned} \tag{2.13}$$

PROBLEM 2.2: We will calculate the angles Q_1, Q_2, Q_3 below, but will not bother with the angles Q_r, Q_θ, Q_ϕ associated with the original actions P_r, P_θ, P_ϕ. But if we wanted them, how could we recover Q_r, Q_θ, Q_ϕ from Q_1, Q_2, Q_3? [**1**]

THE ANGLES

The angles turn out to be

$$\begin{aligned} Q_1 &= M \\ Q_2 &= \omega \quad \text{(argument of perihelion)}, \\ Q_3 &= \Omega \quad \text{(longitude of ascending node)}. \end{aligned} \tag{2.14}$$

Since $H = -\frac{1}{2} P_1^{-2}$, we have $\dot{Q}_1 = (\partial H / \partial P_1) = P_1^{-3} = a^{-\frac{3}{2}}$ as expected, while Q_2 and Q_3 are constant.

PROOF Substituting the actions (2.13) via (2.8) into (2.7) we get

$$W = \int \sqrt{-\frac{1}{P_1^2} + \frac{2}{r} - \frac{P_2^2}{r^2}}\, dr + \int \sqrt{P_2^2 - \frac{P_1^2}{\sin^2\theta}}\, d\theta + sqrt P_3^2 \phi. \tag{2.15}$$

Taking partial derivatives with respect to P_1, P_2, P_3 we have

$$\begin{aligned} Q_1 &= \frac{1}{P_2^3} \int \left(-\frac{1}{P_1^2} + \frac{2}{r} - \frac{P_2^2}{r^2} \right)^{-\frac{1}{2}} dr \\ Q_2 &= -P_2 \int \frac{1}{r^2} \left(-\frac{1}{P_1^2} + \frac{2}{r} - \frac{P_2^2}{r^2} \right)^{-\frac{1}{2}} dr + P_2 \left(P_2^2 - \frac{P_3^2}{\sin^2\theta} \right)^{-\frac{1}{2}} d\theta \\ Q_3 &= -P_3 \int \frac{1}{\sin^2\theta} \left(P_2^2 - \frac{P_3^2}{\sin^2\theta} \right)^{-\frac{1}{2}} d\theta \pm \phi \end{aligned} \tag{2.16}$$

and working out the integrals gives

$$\begin{aligned} Q_1 &= u - e \sin u, \\ Q_2 &= -f(e,u) + \operatorname{sgn}(\dot{\theta}) \arcsin\left(\frac{\cos\theta}{\cos\theta_0} \right), \\ Q_3 &= \operatorname{sgn}(L_z)\phi - \operatorname{sgn}(\dot{\theta}) \arcsin\left(\frac{\tan\theta_0}{\cos\theta} \right). \end{aligned} \tag{2.17}$$

Now the last equation in (2.17) is $\sin(\phi - Q_3)\tan\theta = \operatorname{sgn}(\dot\theta)\tan\theta_0$, which shows that the motion is in a plane with $\theta = \theta_0$ (equivalently, at inclination I). The second equation in (2.17) is $\sin(f + Q_2)\cos\theta_0 = \operatorname{sgn}(\dot\theta)\cos\theta$, which shows that $f + Q_2$ is the angle with the ascending node; but since $f = 0$ at $u = 0$ or perihelion, Q_2 must be the argument of perihelion. □

The set $(P_1, P_2, P_3, Q_1, Q_2, Q_3)$ are the Delaunay elements, traditionally denoted by (L, G, H, l, g, h). They were introduced by Delaunay c. 1860 for (what else?) perturbation theory.

By applying the transpose-inverse pair of matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.18}$$

to (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) respectively we get yet another set of action-angles, the Poincaré elements

$$\begin{aligned} P_\lambda &= \sqrt{a}, & \lambda &= M + \omega + \Omega, \\ P_\varpi &= \sqrt{a}\left(1 - \sqrt{1-e^2}\right), & -\varpi &= -\omega - \Omega, \\ P_\Omega &= \sqrt{a(1-e^2)}(1 - \cos I), & -\Omega. & \end{aligned} \tag{2.19}$$

The Poincaré elements have the advantage of remaining well defined in the limit $I \to 0$. This limit makes it convenient to rederive the usual geometrical interpretation of the orbit and the anomalies. Incidentally, changing e to $-e$ swaps the main focus and the empty focus.

Proof From (2.17) and (2.18) we get

$$\begin{aligned} \lambda &= M - f + \operatorname{sgn}(L_z)\phi + \theta_1 - \theta_2, \\ -\varpi &= f - \operatorname{sgn}(L_z)\phi - \theta_1 + \theta_2, \\ -\Omega &= -\operatorname{sgn}(L_z)\phi + \theta_2, \end{aligned} \tag{2.20}$$

where

$$\theta_1 = \operatorname{sgn}(\dot\theta)\arcsin\left(\frac{\cos\theta}{\cos\theta_0}\right), \qquad \theta_2 = \operatorname{sgn}(\dot\theta)\arcsin\left(\frac{\tan\theta_0}{\cos\theta}\right). \tag{2.21}$$

We now apply appropriate rotations to make $I = 0$, so that $\theta = \theta_0 = \pi/2$, and also to make $\varpi = 0$. We then have

$$\tan\tfrac{1}{2}\phi = \sqrt{\frac{1+e}{1-e}}\tan\tfrac{1}{2}u, \qquad \text{or} \qquad \cos u = \frac{e + \cos\phi}{1 + e\cos\phi}, \tag{2.22}$$

so that the orbit is the ellipse

$$r = \frac{a(1-e^2)}{1 + e\cos\phi} \qquad \text{or} \qquad \frac{(x+ea)^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1. \tag{2.23}$$

The last equation shows that changing the sign of e swaps main focus and empty focus. □

PROBLEM 2.3: Show that

$$\cos f = \cos u - e \sin^2 u + O(e^2),$$

while

$$\cos M = \cos u + e \sin^2 u + O(e^2).$$

Explain why this shows that to $O(e)$ the Moon always keeps the same face to the empty focus. [**4**]

WHEREFOR THIS MACHINERY?

The eventual use of the angles is to Fourier-expand $H_{\rm pert}$ in (2.1). Then integration over the $\mathbf{Q}$, needed for perturbation theory, becomes trivial. Integration over the $\mathbf{P}$ are not needed, only derivatives are, so it's not essential to write the $\mathbf{P}$ dependence explicitly—it's okay to leave $H_{\rm pert}$ in terms of $(a, e, I, \lambda, \varpi, \Omega)$ or similar.

Now $H_{\rm pert}$ is usually simply in cartesian variables. Cartesian variables are expressible in terms of elementary functions of $(a, e, I, u, \varpi, \Omega)$. The complication is that u is not an elementary function of the angle variables: one has to solve the last equation in (2.3), which is Kepler's equation, to obtain u in terms of M. And this fact is the source of the complexity of celestial mechanics.

Kepler's equation can be solved as a series involving Bessel functions:

$$\begin{aligned} u &= M + 2\sum_{n=1}^{\infty} \tfrac{1}{n} J_n(ne) \sin nM \\ &= M + e\sin M + e^2\left(\tfrac{1}{2}\sin 2M\right) + e^3\left(\tfrac{3}{8}\sin 3M - \tfrac{1}{8}\sin M\right) + \ldots \end{aligned} \tag{2.24}$$

from which may be derived

$$\begin{aligned} \frac{r}{a} &= 1 - e\cos M + \tfrac{1}{2}e^2(1-\cos 2M) + \tfrac{3}{8}e^3(\cos M - \cos 3M) + \ldots \\ \cos f &= \cos M + e(\cos 2M - 1) + \tfrac{9}{8}e^2(\cos 3M - \cos M) + \tfrac{4}{3}e^3(\cos 4M - \cos 2M) + \ldots \\ \sin f &= \sin M + e\sin 2M + e^2(\tfrac{9}{8}\sin 3M - \tfrac{7}{8}\cos M) + e^3(\tfrac{4}{3}\sin 4M - \tfrac{7}{6}\sin 2M) + \ldots \end{aligned} \tag{2.25}$$

3. Perturbation Theory for the Restricted Problem

The restricted three-body problem (two point masses in circular orbits with an infinitesimal third mass in the same plane) is sort of a prototype celestial mechanics problems. The variety of motions possible for the third mass is so great that people still do research on it.

We'll confine ourselves to the case where one the two main masses is much less than the other, and the third body's eccentricity is small.

The Hamiltonian

We choose a mass unit to make the first mass 1 and the second μ, and rotating coordinate (x, y) and a length unit such that the bodies are at $(-\mu, 0)$ and $(1, 0)$ respectively. With a time unit chosen to make $G = 1$ the third body's Hamiltonian is (cf. Problem 2.1)

$$H = \tfrac{1}{2}\mathbf{p}^2 + (xp_y - yp_x) - \left[(x+\mu)^2 + y^2\right]^{-\frac{1}{2}} - \mu\left[(x-1)^2 + y^2\right]^{-\frac{1}{2}}. \tag{3.1}$$

We rewrite this as

$$\begin{aligned} H &= H_{\text{sol}} + H_{\text{pert}} \\ H_{\text{sol}} &= \tfrac{1}{2}\mathbf{p}^2 - \frac{1}{r} + (xp_y - yp_x) \\ H_{\text{pert}} &= \frac{1}{r} - \left[(x+\mu)^2 + y^2\right]^{-\frac{1}{2}} - \mu\left[(x-1)^2 + y^2\right]^{-\frac{1}{2}}. \end{aligned} \tag{3.2}$$

H_{sol} (for soluble!) is a two-body problem plus a $(xp_y - yp_x)$ term, while H_{pert} of $O(\mu)$. Let us change to Keplerian action-angles $(P_\lambda, P_\varpi, \lambda, -\varpi)$. (Since we are already in a plane, we can forget about P_ϖ and $-\Omega$.) In these variables

$$H_{\text{sol}} = -\tfrac{1}{2}P_\lambda^{-2} + P_\varpi - P_\lambda. \tag{3.3}$$

Proof To see that

$$xp_y - y_p x = P_\varpi - P_\lambda \tag{3.4}$$

recall the Hamilton-Jacobi function $W(r, \phi)$, with $\theta = \pi/2$ now.

$$xp_y - y_p x = x\,\frac{\partial W}{\partial y} - y\,\frac{\partial W}{\partial x} = \frac{\partial y}{\partial \phi}\,\frac{\partial W}{\partial y} + \frac{\partial x}{\partial \phi}\,\frac{\partial W}{\partial x} = \frac{\partial W}{\partial \phi}.$$

But $\partial W/\partial\phi = L_z = P_\lambda - P_\varpi$ with a possible sign ambiguity. □

In fact the origin of $P_\varpi - P_\lambda$ is easy to see. By changing to rotating coordinates we are in effect subtracting 1 from $\dot\lambda$ and $\dot\varpi$; adding $P_\varpi - P_\lambda$ has precisely this effect because λ and $-\varpi$ are the conjugate coordinates to P_λ and P_ϖ.

We now need to express $H_{\text{pert}}(x, y)$ in terms of $(P_\lambda, P_\varpi, \lambda, -\varpi)$; the variables (a, e, λ, ϖ) will do.

The disturbing function

By disturbing function we mean $H_{\rm pert}$, the potential energy of the perturbation from the exactly-soluble problem.[1] We will work it out to first order in μ and e, but even so it will be a mess.

Changing to polar coordinates we have

$$H_{\rm pert} = \frac{1}{r} - \left(r^2 + \mu^2 + 2\mu r\cos\phi\right)^{-\frac{1}{2}} - \mu\left(1 + r^2 - 2r\cos\phi\right)^{-\frac{1}{2}}. \tag{3.5}$$

The first two terms are easy, a binomial expansion gives

$$\frac{1}{r} - \left(r^2 + \mu^2 + 2\mu r\cos\phi\right)^{-\frac{1}{2}} = \frac{\mu}{r^2}\cos\phi + O(\mu^2), \tag{3.6}$$

which we call the indirect term.[2] The last term in (3.5) is called the direct term, and it is the source of all the trouble. We want to get that r and ϕ into the numerator first, and the way we do it is by Taylor-expanding r around a and then Fourier-expanding for ϕ. Taylor-expanding the direct term around $r = a$ and suppressing a factor of $-\mu$ for now, we have

$$\left(1 + a^2 - 2a\cos\phi\right)^{-\frac{1}{2}} + (a - \cos\phi)(ae\cos u)\left(1 + a^2 - 2a\cos\phi\right)^{-\frac{3}{2}} + O(e^2). \tag{3.7}$$

We will assume that r is either well within or well outside of 1. The case of r near 1, involving the possibility of collisions, we will leave alone.

Now we Fourier-expand the terms with $\cos\phi$. The necessary Fourier coefficients are known as Laplace coefficients $b_s^{(j)}$, and these are defined by

$$\begin{aligned}
&\left(1 + a^2 - 2a\cos\phi\right)^{-s} = \sum_{j=0}^{\infty} b_s^{(j)}(a)\cos j\phi, \\
&b_s^{(j>0)}(a) = \frac{2}{\pi}\int_0^\pi \frac{\cos j\phi\, d\phi}{(1 + a^2 - 2a\cos\phi)^s}, \quad b_s^{(0)}(a) = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{(1 + a^2 - 2a\cos\phi)^s}.
\end{aligned} \tag{3.8}$$

The $b_s^{(j)}(a)$ have series expansions that converge for $a = 1$, but since from (3.8)

$$b_s^{(j)}(a) = a^{-s} b_s^{(j)}(a^{-1}), \tag{3.9}$$

they are well-defined for $a > 1$ as well. Anyway they are well studied and tabulated,[3] so we can treat them as standard functions.

Substituting Laplace coefficients into (3.7) we have

$$\sum_{j=0}^{\infty} b_{\frac{1}{2}}^{(j)}(a)\cos j\phi + ae\cos u(a - \cos\phi)\sum_{j=0}^{\infty} b_{\frac{3}{2}}^{(j)}(a)\cos j\phi + O(e^2) \tag{3.10}$$

---

[1] Traditionally, disturbing function meant $-H_{\rm pert}$, so the force was the gradient of the disturbing function rather than minus gradient of the potential as is the familiar convention now.

[2] Again, in traditional usage the indirect term is similar but not quite the same.

[3] But standard usage has $b_s^{(0)}$ defined as twice the value here.

and combining with (3.6) and (3.3) gives us

$$\begin{aligned} H = &-\tfrac{1}{2}P_\lambda^{-2} + P_\varpi - P_\lambda \\ &+ \mu\Bigg[a^{-2}(1 + 2e\cos M)\cos\phi - \sum_{j=0}^{\infty}\Big(b_{\frac{1}{2}}^{(j)}(a)\cos j\phi\Big) \\ &\quad - ae\cos M(a - \cos\phi)\sum_{j=0}^{\infty}\Big(b_{\frac{3}{2}}^{(j)}(a)\cos j\phi\Big)\Bigg] + O(\mu^2 + \mu e^2) \end{aligned} \tag{3.11}$$

Now ϕ is $f + \varpi$, and from the expansions for $\cos f$ and $\sin f$ in terms of M, we have

$$\cos j\phi = \cos j\lambda - 2je\sin M\sin j\lambda + O(e^2). \tag{3.12}$$

Substituting into (3.11) we get

$$\begin{aligned} H = &-\tfrac{1}{2}P_\lambda^{-2} + P_\varpi - P_\lambda \\ &+ \mu\Bigg[a^{-2}(\cos\lambda + 2e\cos\varpi) - \sum_{j=0}^{\infty} b_{\frac{1}{2}}^{(j)}(a)\Big(\cos j\lambda - 2je\ \sin M\sin j\lambda\Big) \\ &\quad - ae\cos M\sum_{j=0}^{\infty} b_{\frac{3}{2}}^{(j)}(a)\Big(a\cos j\lambda - \tfrac{1}{2}\cos(j+1)\lambda - \tfrac{1}{2}\cos(j-1)\lambda\Big)\Bigg] \\ &+ O(\mu^2 + \mu e^2). \end{aligned} \tag{3.13}$$

We can expand the trig products in (3.13) using

$$\begin{aligned} -2\sin M\sin j\lambda &= \cos[(j+1)\lambda - \varpi] - \cos[(j-1)\lambda + \varpi], \\ 2\cos M\cos j\lambda &= \cos[(j+1)\lambda - \varpi] + \cos[(j-1)\lambda + \varpi], \end{aligned} \tag{3.14}$$

which puts the Hamiltonian into the form

$$H = -\tfrac{1}{2}P_\lambda^{-2} + P_\varpi - P_\lambda + \mu\sum_{j,k} c_{kl}(a,e)\cos(j\lambda + k\varpi). \tag{3.15}$$

PERTURBATION THEORY

We now apply the Lie transform perturbation method. The (λ, ϖ) dependence of H can be annihilated to first order by the generating function

$$W = -\sum_{jk} \frac{\sin(j\lambda + k\varpi)}{j(P_\lambda^{-3} - 1) - k} \tag{3.16}$$

whence the perturbed solution is

$$\begin{aligned} P_\lambda &\leftarrow P_\lambda + \mu \frac{\partial W}{\partial \lambda}, \qquad & P_\varpi &\leftarrow P_\varpi - \mu \frac{\partial W}{\partial \varpi}, \\ \lambda &\leftarrow \lambda - \mu \frac{\partial W}{\partial P_\lambda}, \qquad & \varpi &\leftarrow \varpi + \mu \frac{\partial W}{\partial P_\varpi}, \end{aligned} \tag{3.17}$$

the right hand sides in (3.17) being evaluated at

$$\begin{aligned} P_\lambda &= \text{const}, & P_\varpi &= \text{const}, \\ \lambda &= \left(P_\lambda^{-3} - 1 + \mu \frac{\partial c_{00}(a,e)}{\partial P_\lambda} \right) t, \qquad & \varpi &= -\left(1 + \mu \frac{\partial c_{00}(a,e)}{\partial P_\varpi} \right) t. \end{aligned} \tag{3.18}$$

That strange-looking terms with derivatives of $c_{00}(a,e)$ come because any term in (3.13) with $b_s^{(0)}$ in fact contains no (λ, ϖ) dependence, so it is in effect part of the unperturbed Hamiltonian. We can work out from (3.13) that

$$\frac{\partial}{\partial P_\lambda} c_{00}(a,e) = \tfrac{1}{2} P_\lambda^{-1} b_{\frac{1}{2}}'^{(0)}(P_\lambda^2), \qquad \frac{\partial}{\partial P_\varpi} c_{00}(a,e) = 0, \tag{3.19}$$

but that's not so important.

What the perturbative solution is showing is basically a bunch of periodic variations in (P_λ, P_ϖ), and a bunch of modulations in (λ, ϖ). In particular, notice that variations in P_λ (which is $\sqrt{a}$) come from terms in the disturbing function involving λ while variations in P_ϖ (which is $\simeq \frac{1}{2}\sqrt{a}e^2$) come from terms in the disturbing function involving ϖ. (At higher orders in perturbation theory, things get increasingly mixed and complicated, but at first order this is the case.) But all of this periodic variation is not necessarily interesting; for example if a has a small periodic variation over a time scale faster even than one orbit, but no change in its mean, then that's not very exciting.

Resonances

What are interesting are slow changes. These will happen when $\cos(j\lambda + k\varpi)$ in (3.15) vary slowly, i.e., when the denominator in (3.16) is small. In fact if

$$P_\lambda = \left(\frac{j}{j+k}\right)^{\frac{1}{3}} \tag{3.20}$$

the denominator becomes zero. For any nearby value of P_λ the denominator becomes small; this is the infamous 'problem of small denominators' and it's a problem because it makes the sort of perturbation theory here diverge catastrophically.

Does that mean that the theory we spent so much effort on is useless in any interesting situation? No not quite. As an example, let's take the case of $j = 3, k = -1$, of which several terms occur in the disturbing function in (3.13). We change to new momenta

$$\Delta P_\lambda = P_\lambda - \left(\tfrac{3}{2}\right)^{\frac{1}{3}}, \quad \Delta P_\varpi = P_\varpi - \text{const} \tag{3.21}$$

(a canonical transformation with no change in λ, ϖ) and assume $\Delta P_\lambda, \Delta P_\varpi$ are small. We then focus on the part of the Hamiltonian (call it H_{res}) which would give small denominators. The rest just gives small fast variations which we now ignore. We have

$$\begin{aligned} H_{\text{res}} &= -\tfrac{1}{2}\left(\left(\tfrac{3}{2}\right)^{\frac{1}{3}} + \Delta P_\lambda\right)^{-2} + \Delta P_\varpi - \Delta P_\lambda + \mu e B\cos(3\lambda - \varpi), \\ B &= -2b^{(2)}_{\frac{1}{2}} - \tfrac{1}{2}a^2 b^{(2)}_{\frac{3}{2}} + a\tfrac{1}{4}b^{(1)}_{\frac{3}{2}} + a\tfrac{1}{4}b^{(3)}_{\frac{3}{2}}, \end{aligned} \tag{3.22}$$

where the $b^{(j)}_s$ are all evaluated at $a = \left(\frac{3}{2}\right)^{\frac{2}{3}}$ and hence are constant. Now we change to yet another set of action-angles

$$\begin{pmatrix} \theta_{\text{res}} \\ \theta_{\text{fast}} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \lambda \\ -\varpi \end{pmatrix}, \qquad \begin{pmatrix} \Delta P_\lambda \\ \Delta P_\varpi \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} P_{\text{res}} \\ P_{\text{fast}} \end{pmatrix}. \tag{3.23}$$

This gives

$$H_{\text{res}} = -\tfrac{1}{2}\left(\left(\tfrac{3}{2}\right)^{\frac{1}{3}} + 3P_{\text{res}} + 2P_{\text{fast}}\right)^{-2} - 2P_{\text{res}} - P_{\text{fast}} + \mu e B\cos\theta_{\text{res}}. \tag{3.24}$$

If we now set P_{fast} to 0 (its average value) Taylor-expand around $P_{\text{res}} = 0$ we get

$$H_{\text{res}} = -9\left(\tfrac{2}{3}\right)^{\frac{1}{3}} P^2_{\text{res}} + \mu e B\cos\theta_{\text{res}} \tag{3.25}$$

which is a pendulum equation, and θ_{res} can librate about 0 or π.

The above is a crude theory for Pluto's orbit. Pluto's orbit is in a resonance with Neptune's, which prevents them colliding. (This is a crude theory because it is ignoring inclination and going only to first order in e, but it's still interesting.) In terms of longitudes in a non-rotating frame:

$$\begin{aligned} \lambda &= \lambda_{\text{Pl}} - \lambda_{\text{Ne}}, \\ \varpi &= \varpi_{\text{Pl}} - \lambda_{\text{Ne}}, \end{aligned} \qquad 3\lambda - \varpi = 3\lambda_{\text{Pl}} - 2\lambda_{\text{Ne}} - \varpi_{\text{Pl}}. \tag{3.26}$$

So what looks like a 3 : 1 resonance in the rotating frame is in fact a 3 : 2 resonance in the non-rotating frame.[4] The angle $3\lambda_{\text{Pl}} - 2\lambda_{\text{Ne}} - \varpi_{\text{Pl}}$ librates about π.

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[4] Also note that in the non-rotating frame the sum of coefficients of the longitudes is always 0. This is one of the so-called d'Alembert properties.

PROBLEM 3.1: Another example of a $3:2$ resonance occurs in the asteroid belt. This time the third body's orbit is inside the second body's. The Hilda group of asteroids has a mean motion $\frac{3}{2}$ that of Jupiter (not $\frac{2}{3}$).

Work out the resonant Hamiltonian, analogous to (3.25) for this case. [**4**]

4. Some Qualitative Dynamics

Having done lots of algebraic things, we'll now step back a bit and discuss some qualitative aspects of the dynamics. We saw while doing perturbation theory that the pendulum tends to reappear a lot, and we'll continue to use it to for examples, but the things we'll discuss are generic.

Phase space plots and surfaces of section

One often get a lot of information about dynamics just by plotting up the Hamiltonian in phase space, or in sections of phase space. Figure 4.1 does that for the pendulum, in two sets of canonical variables. The librating and circulating solutions show up as two different kinds of contours. Separating them is the transitional case where the pendulum just barely gets to the top; it's an infinite-period solution. This is called the separatrix, generically as separatrices.

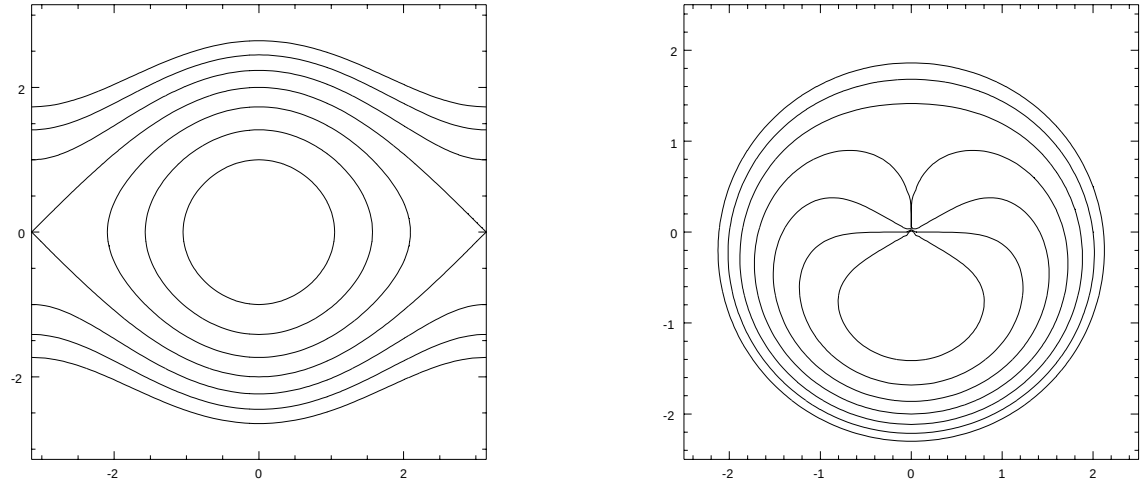

Figure 4.1: Contour map of the pendulum Hamiltonian $\frac{1}{2}p^2 - \cos q$. The left panel uses the variables (q, p). The right panel uses polar coordinates, with $\sqrt{2p}$ as the radial coordinate and q as the angular coordinate. The curves correspond to $H = -1, \ldots, 2.5$ in steps of 0.5, with $H = 1$ being the transitional case (separatrix).

The pendulum by itself isn't very exciting. Things can quickly get complicated though, when we add one or more extra degrees of freedom. Consider adding a periodic time dependence, such as putting the pendulum on a turntable. We now have to worry about the Hamiltonian in three variables, and Figure 4.1 with just (p, q) isn't enough. Still, two-dimensional plots are useful things, so what people often do is plot a two-dimensional section of phase space. For example, we could plot $H(p, q, t = nT)$ where T is the period of the turntable. For a very weak time-dependence, the "surface of section" would look much like Figure 4.1, but the interpretation would be different. It would a stroboscopic map: trajectories wouldn't just move along one of the contours on the figure; they'd intersect it at one point, then leave and intersect it at another point on the same curve.

In general a surface of section is the intersection of trajectories through some chosen surface in phase space. A 'Poincaré map' takes the trajectories as they pass through

the surface and maps them to where they pass next time round. (It is therefore a canonical transformation). If we take bunches of trajectories with particular values of H, the surface of section is a contour map of H. But not every surface of section is a contour map of H, it depends on what we choose. People usually take some canonical variables (p, q) as the axes of a surface of section, in which case the Poincaré map is area-preserving.

Discrete maps and Leapfrog

The idea of discrete maps is very useful, not least for numerical work, because after all computers are discrete things.

Consider a Hamiltonian of the type

$$H = H_{\mathrm{A}} + H_{\mathrm{B}}, \tag{4.1}$$

where H_{A} and H_{B} are easy to solve for by themselves, but not together. The obvious example is

$$H_{\mathrm{A}} = \frac{\mathbf{p}^2}{2m}, \quad H_{\mathrm{B}} = V(\mathbf{q}). \tag{4.2}$$

If we now take a stroboscopic time-step τ and do the sequence of steps

$$\boxed{\begin{array}{l}\langle\text{Evolve under } H_{\mathrm{A}} \text{ only for } \tfrac{1}{2}\tau\rangle \\ \langle\text{Evolve under } H_{\mathrm{B}} \text{ only for } \tau\rangle \\ \langle\text{Evolve under } H_{\mathrm{A}} \text{ only for } \tfrac{1}{2}\tau\rangle\end{array}} \tag{4.3}$$

it amounts to evolving for τ under

$$H_{\mathrm{A}} + H_{\mathrm{B}} + \frac{\tau^2}{12}\{\{H_{\mathrm{A}}, H_{\mathrm{B}}\}, H_{\mathrm{B}} + \tfrac{1}{2}H_{\mathrm{A}}\} + O(\tau^4). \tag{4.4}$$

Proof It follows from the operator identity (sometimes called the Baker-Campbell-Hausdorff identity)

$$e^{\frac{1}{2}\tau\{\ ,H_{\mathrm{A}}\}}\, e^{\tau\{\ ,H_{\mathrm{B}}\}}\, e^{\frac{1}{2}\tau\{\ ,H_{\mathrm{A}}\}} \equiv e^{\tau\{\ ,H_{\mathrm{A}}+H_{\mathrm{B}}+H_{\mathrm{err}}\}} \tag{4.5}$$

where H_{err} is a formal power series in τ starting at $O(\tau^2)$ and consisting of nested Poisson brackets of H_{A} and H_{B}:

$$H_{\mathrm{err}} = \frac{\tau^2}{12}\{\{H_{\mathrm{A}}, H_{\mathrm{B}}\}, H_{\mathrm{B}} + \tfrac{1}{2}H_{\mathrm{A}}\} + O(\tau^4). \tag{4.6}$$

□

The sequence of steps (4.3) is known as 'Leapfrog', and is the basis for a range of 'symplectic integrators' (numerical integrators that preserve the Hamiltonian property of the equations, and eliminate errors like spurious dissipation).

EXAMPLE [Standard Map] If we apply leapfrog to the pendulum Hamiltonian in the form

$$H = \tfrac{1}{2}p^2 - k\cos q, \tag{4.7}$$

taking $\tau = 1$, we get

$$q \to q + p, \qquad p \to p + k\sin q. \tag{4.8}$$

This is called the 'standard map', and much used by researchers on chaos. □

PROBLEM 4.1: Consider the Hamiltonian

$$H = -\tfrac{1}{2}P_\lambda^{-2} + P_\varpi - P_\lambda + \dot\lambda_{\rm sec}P_\lambda + +\dot\varpi_{\rm sec}P_\varpi + \mu c_{kl}\cos(j\lambda + k\varpi). \tag{4.9}$$

Here $\dot\lambda_{\rm sec}$ and $\dot\varpi_{\rm sec}$ are constants, which are basically the c_{00} terms taken out. We assume we are near a resonance where λ and ϖ vary quickly but $j\lambda + k\varpi$ varies slowly, and have discarded all the fast terms.

Apply leapfrog with timestep $\tau = 1$ to this Hamiltonian, to derive a map, analogous to the standard map. [**3**]

CHAOS

Well, all this discussion of the perturbed pendulum, separatrices, and discrete maps can end only in one place...

Resonances, whether in a perturbed pendulum or elsewhere are generically associated with chaos. Sometimes the chaos is incipient, and confined to the borders of the resonance; sometimes it's pervasive. But it's always there, except in systems that have only one degree of freedom, or can be decoupled into separate systems with one degree of freedom each, like the Kepler problem.

The first person to figure out about chaos seems to have been Poincaré c. 1902, and he gave a beautiful proof of its existence. Nobody else seems to have realized how important it was till the 1960s. But the ideas about chaos and the butterfly effect that have become well known since the 1960s are perfectly described in Poincaré's writings from early this century.

Here's a sketch of Poincaré's argument.

Consider a pendulum at its unstable equilibrium, $p = 0, q = \pi$, so $H = \frac{1}{2}p^2 - \cos q = 1$. Now consider curves of $H = 1$ leading away from the unstable equilibrium: they involve moving q away from π a bit and introducing a small p. There are four such curves leading away from the unstable equilibrium point, corresponding to two signs each for the possible changes in p and q. Two of these will lead back to the equilibrium position and two away from it, depending on whether the velocity opposes the displacement or reinforces it. These curves are known as the stable and unstable manifolds, (we'll call them $\mathcal{H}^+$ and $\mathcal{H}^-$) respectively. Figure 4.2 illustrates. Because of the symmetry of the pendulum, $\mathcal{H}^+$ is the same as $\mathcal{H}^-$ with q reflected about 0; an 'away' perturbation will lead back into the equilibrium after a full circle.

Now let's complicate the system by attaching a baby pendulum at the bob of our pendulum. The baby pendulum (with its own canonical variables say p', q') acts as a perturbation on the main pendulum. The system now has two degrees of freedom and four dimensions in phase space, so we can't visualize the full phase space any more, we

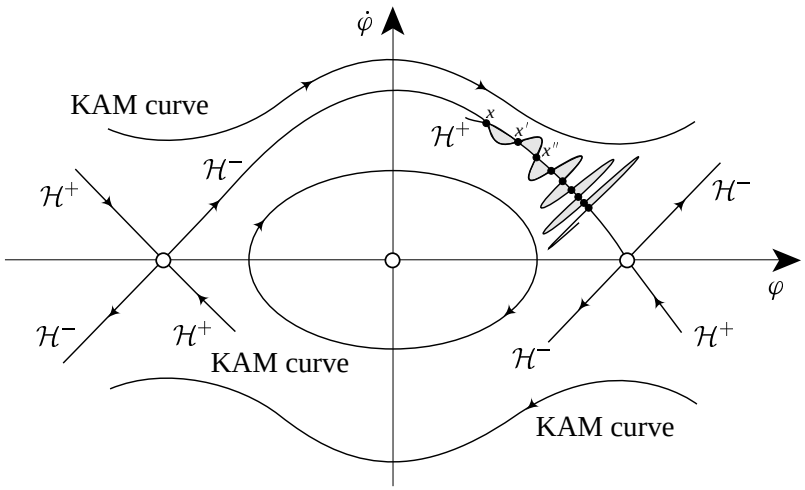

Figure 4.2: How the left panel Figure 4.1 is modified under perturbation. Libration solutions (inner ellipse) and the circulating solutions (labelled 'KAM curves') still exist, but near the separatrix there is chaos.

have to take a section of it. Say we take the section having $p' = q' = 0$ instantaneously, and consider Figure 4.2 again, which is now a surface of section. The existence of the unstable equilibrium and the two $\mathcal{H}^+$ and two $\mathcal{H}^-$ curves is generic and still there, but there is an important change: $\mathcal{H}^+$ is no longer a reflection of $\mathcal{H}^-$, because a small 'away' perturbation will give some energy to the baby pendulum as the main pendulum swings around the circle, so there may not be enough energy to take it back to the unstable equilibrium position after one swing.

Now there are two important facts about $\mathcal{H}^+$ and $\mathcal{H}^-$, which we'll state with some heuristic justification but without proof.

(1) Neither $\mathcal{H}^+$ nor $\mathcal{H}^-$ can intersect itself. This follows from a continuity argument.

(2) $\mathcal{H}^+$ and $\mathcal{H}^-$ may intersect each other, but if they intersect once they must intersect infinitely many times. This also follows from a continuity argument. Thus $\mathcal{H}^+$ and $\mathcal{H}^-$ must oscillate wildly, to make infinite mutual intersections while avoiding self-intersections, as Figure 4.2 starts to show.

Applying the Poincaré map to any of the self-intersections will naturally give us another self-intersection, though not necessarily a neighbouring one. But if we take any closed loop involving two self-intersections and Poincaré-map it, we'll get another closed loop involving two self-intersections; and the two loops will have the same area. This means that the wiggles in $\mathcal{H}^+$ and $\mathcal{H}^-$ must have the same area in them. This makes the curves themselves dense with mutual intersection points.

We've been talking about pendulums to help our intuition, but the picture is completely generic.

Lyapunov exponents

The defining feature of chaos is extreme sensitivity to initial conditions, exponential sensitivity. That is, two trajectories that start out with some small separation (say $d(0)$) have their separation in phase space $d(t)$ growing with time like $\exp(\lambda t)$. This λ is called the Lyapunov exponent. It is usually defined as

$$\lim_{t\to\infty} \frac{1}{t} \ln \frac{d(t)}{d(0)}. \tag{4.10}$$

Basically it quantifies the complex stretching and folding of separatrices. The reciprocal of the Lyapunov exponent is sometimes called the e-folding time.

Problem 4.2: The simplest 'interesting' map of all is called the logistic map:

$$x_{n+1} = kx_n(1 - x_n),$$

with k a parameter between 0 and 4. This becomes chaotic for $k > 3.57$. The logistic map is not Hamiltonian, it's just a particularly simple illustration of chaos.

Using the variable $x_n = \sin\theta_n$, show that the logistic map is chaotic for $k = 4$. What you are looking for is amplification of arbitrarily fine details in the initial conditions. What can you say about the Lyapunov exponent? [**4**]