http://www.itp.uzh.ch/~psaha/

Relativity
Problem Set 1

1. The equation

$$\begin{aligned} A^\alpha B^\beta C_{\beta\gamma} = {} & A^0 B^0 C_{00} + A^0 B^1 C_{10} + A^0 B^2 C_{20} + A^0 B^3 C_{30} \\ & + A^1 B^0 C_{01} + A^1 B^1 C_{11} + A^1 B^2 C_{21} + A^1 B^3 C_{31} \\ & + A^2 B^0 C_{02} + A^2 B^1 C_{12} + A^2 B^2 C_{22} + A^2 B^3 C_{32} \\ & + A^3 B^0 C_{03} + A^3 B^1 C_{13} + A^3 B^2 C_{23} + A^3 B^3 C_{33} \end{aligned}$$

is incorrect under the summation convention. Suggest (i) a modification of the LHS that would be consistent with the given RHS, and (ii) a modification of the RHS that would be consistent with the given LHS. [6]

2. Laptop computers these days have a typical clock rate of 1 GHz. Express one clock 'tick' as a length.

If a new make of laptop computer claims a clock rate of 10 GHz, should one be suspicious? Why? [5]

3. What would be our expression for a boost along x if we were *not* using units with $c = 1$? [5]

4. A boost along by v along x followed by a boost by $-v$ along x takes us back to the original coordinates. Verify this statement by combining two boosts. [6]

5. Consider a rotation about z by $\pi/2$, followed by a boost along x by v, followed by a rotation about z by $-\pi/2$. Can you simplify the resulting transformation? [10]

6. Consider a boost along a direction at θ to the x and y axis. Express the transformation matrix in terms of rotation and boost matrices. [5]

7. Write down examples of pairs of events with (i) spacelike, (ii) timelike, and (iii) null intervals. [3]

8. Show that any two events with a spacelike separation can be made simultaneous, and any two events with a timelike separation can be made spatially coincident, through Lorentz transformations with $|v| < 1$. [6]

9. On a spacetime diagram sketch the locus of events at intervals of (i) $+1$, (ii) -1 from the origin. [4]

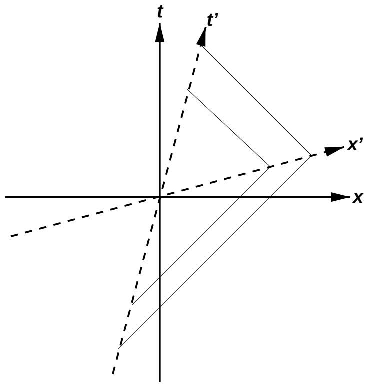

Figure 1: Here the solid axes refer to the observer. The dashed inclined axes refer to a trolley moving forwards along x.

10. Recall the optical experiment on a trolley represented by Figure 1.

(i) Sketch the corresponding diagram in classical (Newtonian) dynamics.

(ii) Sketch the diagram in relativity, but with the trolley moving *backwards* along x. [8]

11. A fish of unit length is swimming forward at speed v_1. Meanwhile a storyteller is chasing after the fish at speed v_2. Compute the length of the fish in the storyteller's coordinate system. [15]

12. A source at rest in a room emits a laser beam at an angle θ with respect to the x axis. A trolley is moving with speed v along the x axis. Show that in the trolley frame the laser beam makes an angle θ' given by

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - v)} \qquad [12]$$

13. An astronomical object is moving with speed v and at an angle θ to us. That is to say, its velocity component towards us is $v\cos\theta$ and its transverse velocity is $v\sin\theta$. The distance is known, and the Earth's motion is negligible. An astronomer observes its motion across the sky and computes the transverse velocity. Unfortunately, this astronomer is good at classical dynamics but clueless about relativity.

Show that they would get a transverse velocity of

$$\frac{v\sin\theta}{1 - v\cos\theta}$$

Can this velocity be faster than light? [15]

Supplementary information

A rotation by φ about z may be represented as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{R}_z(\varphi)\begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{R}_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

and a boost of v along x by

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \mathbf{B}_x(v)\begin{pmatrix} t \\ x \end{pmatrix}, \quad \mathbf{B}_x(v) = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix}, \quad \gamma = 1/\sqrt{1-v^2}.$$

Relativity
Problem Set 2

1. In a spacetime diagram like Figure 1 mark the timelike, spacelike, and null parts. On the same diagram show two timelike vectors whose sum is null. [5]

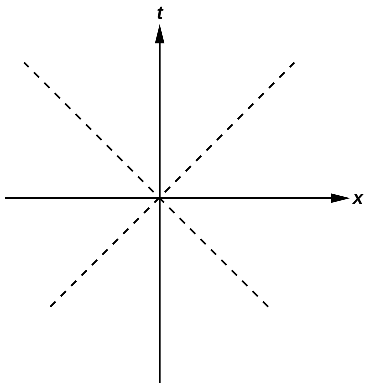

Figure 1: Spacetime diagram.

2. Consider the two equations

$$\begin{aligned} \Lambda^{\alpha}{}_{\gamma}\, \eta_{\alpha\beta}\, \Lambda^{\beta}{}_{\delta} &= \eta_{\gamma\delta} \\ \Lambda_{\mu}{}^{\delta}\, \eta^{\mu\nu}\, \Lambda_{\nu}{}^{\sigma} &= \eta^{\delta\sigma} \end{aligned}$$

Show by index manipulation that the second equation follows from the first. One way is to multiply the left hand sides and right hand sides separately and simplify. [5]

3. If $w_\alpha v^\alpha$ is a scalar for an up vector arbitrary v^α, show that w_α must be a down vector. [5]

4. For 3D velocity $\mathbf{v}$, it is known that $\gamma(1, \mathbf{v})$ a vector under Lorentz transformations. Show that $\gamma(0, \mathbf{v})$, however, cannot be a vector. [5]

5. The following problem is from *Electricity and Magnetism* by E.M. Purcell, and so perfectly phrased that I reproduce it verbatim. The context and notation are different from ours, but the main point is not.

"The divergence of a vector function $\mathbf{F}$ is a scalar, as we know. Suppose we try to define a vector, different from the curl, in this way:

$$\mathbf{G} = \hat{\mathbf{x}}\, \frac{\partial F_x}{\partial x} + \hat{\mathbf{y}}\, \frac{\partial F_y}{\partial y} + \hat{\mathbf{z}}\, \frac{\partial F_z}{\partial z}$$

Can you certify that this creature is a vector, or should we call it instead an *imposter?* (See how it behaves when you rotate the coordinates with respect to which the components are taken. It is enough to bring out the point to consider a 90° rotation about the z axis. The new coordinates are then related to the old in this way: $\hat{\mathbf{x}}' = \hat{\mathbf{y}}$, $\hat{\mathbf{y}}' = -\hat{\mathbf{x}}$, $F'_x = F_y$, etc.)" [5]

6. Let $S^{\alpha\beta}$ be a symmetric and $A^{\alpha\beta}$ be an antisymmetric tensor. That is to say

$$S^{\alpha\beta} = S^{\beta\alpha} \qquad \text{and} \qquad A^{\alpha\beta} = -A^{\beta\alpha}$$

and let $Y^{\alpha\beta}$ be another symmetric tensor. How many independent components are there in each of the following?

(i) $S^{\alpha\beta}$

(ii) $A^{\alpha\beta}$

(iii) $S^{\alpha\beta}A^{\gamma\delta}$

(iv) $S^{\alpha\beta}A_{\beta\delta}$

(v) $S^{\alpha\beta}Y^{\gamma\delta} + S^{\gamma\delta}Y^{\alpha\beta}$ [15]

7. Evaluate $T^{\alpha}{}_{\alpha}$ in special relativity for dust by explicit calculation. Explain also why the answer can in fact be read off directly. [10]

8. Consider two particles of mass m, and velocity components $\left(\frac{3}{13}, \frac{4}{13}, 0\right)$ and $\left(\frac{5}{13}, 0, 0\right)$. What is the total p^α? [5]

9. Consider two particle-accelerator experiments.

In the first experiment, a particle of mass m is accelerated to some γ and then made to collide with a stationary particle, also of mass m; these then form a new particle.

In the second experiment, two particles of mass m are both accelerated to the same γ but moving in opposite directions, and then made to collide and form a new particle.

What is the maximum possible rest mass for the new particle formed in each experiment? [10]

10. A particle of mass m accelerated to some $\mathbf{v}$ amd implied γ collides with a stationary particle, also of mass m. The two particles then fly off with $\mathbf{v}_1, \gamma_1$ and $\mathbf{v}_2, \gamma_2$ respectively.

By comparing four-momentum before and after, show that

$$\gamma = \gamma_1\gamma_2(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)$$

Show that in the low-velocity limit, the particles fly off perpendicular to each other. [15]

11. Consider the collision of a photon with a particle of mass m. Before the collision the photon has wavevector $\mathbf{k}_1$ and the other particle is at rest. After the collision the photon has wavevector $\mathbf{k}_2$ and the other particle has velocity $\mathbf{v}$ and the associated γ. Show that

$$k_1 - k_2 + m = \gamma m \qquad \text{and} \qquad \mathbf{k}_1 - \mathbf{k}_2 = \gamma m \mathbf{v}$$

and hence that

$$\frac{1}{k_2} - \frac{1}{k_1} = \frac{1}{m}(1 - \cos\theta)$$

where θ is the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$. [20]

Supplementary information

Photons have four-momentum of the form $(k, \mathbf{k})$, where $\mathbf{k}$ is the wavevector and k is its magnitude.

Relativity
Problem Set 3

1. Evaluate $F^{\alpha\beta}F_{\alpha\beta}$ where $F^{\alpha\beta}$ is the electromagnetic field tensor given below. [10]

2. Verify that

$$F_{\alpha\beta} = -(A_{\alpha,\beta} - A_{\beta,\alpha})$$

implies

$$F_{\alpha\beta\,,\gamma} + F_{\gamma\alpha\,,\beta} + F_{\beta\gamma\,,\alpha} = 0.$$

Although the second equation consists of 64 equations, only 4 are independent, because of the antisymmetry of $F_{\alpha\beta}$. If any two of α, β, γ are equal, we get a trivial identity. The non-trivial equations are when α, β, γ are all different; among these, permuting α, β, γ still gives the same equation. Write down the four independent equations, by making four choices for α, β, γ. Express these equations in 3D vector notation.

The answer is the homogeneous Maxwell equations, which look a little like the inhomogeneous Maxwell equations (given below). [20]

3. Consider the Lorentz force equation

$$\mathbf{F} = \rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B}$$

in a frame where $\mathbf{E} = \mathbf{0}$, so $\mathbf{J} \times \mathbf{B}$ is the only force. We now change to a frame moving with $\mathbf{J}$, so in the new frame $\mathbf{J}' = 0$. Is there a force in this new frame or not? Explain without equations. [10]

4. A certain scalar in special relativity equals

$$\mathbf{J}\cdot(\mathbf{A}\times\mathbf{B})$$

when there is a magnetic field but no electric field.

What is the scalar equal to when both electric and magnetic fields are present? [30]

5. Electrical engineering books and most physics books use different units (SI units) from what we have been using (relativistic units, sometimes called Heaviside-Lorentz units). The relation is

$$\mathbf{E} = \frac{\sqrt{\epsilon_0}}{c}\mathbf{E}_{\mathrm{SI}}, \quad \mathbf{B} = \frac{1}{c\sqrt{\mu_0}}\mathbf{B}_{\mathrm{SI}},$$

$$\rho_e = \frac{1}{c\sqrt{\epsilon_0}}\rho_{\mathrm{SI}}, \quad \mathbf{J} = \frac{\sqrt{\mu_0}}{c}\mathbf{J}_{\mathrm{SI}}.$$

Here μ_0 and ϵ_0 are conventionally defined constants whose product is $1/c^2$.

Insert these definitions into the inhomogeneous Maxwell equations (given below) and eliminate c, to derive the SI form for these equations. [15]

6. The electromagnetic T^{00} equals $\frac{1}{2}(E^2 + B^2)$. What does it equal in SI units?

If $\mathbf{E}_{\mathrm{SI}} = 1$ it is called an electric field of '1 volt per metre'. If $\mathbf{B}_{\mathrm{SI}} = 1$ it is called a magnetic field of '1 tesla'. We consider an electric field of 1 volt per metre to be weak, but a field of 1 tesla to be very strong. Is this justified?

Hint: In SI units, $c = 3 \times 10^8$. [15]

Supplementary information

The four-vectors of charge-current and electromagnetic potential are, respectively,

$$J^\alpha = (\rho_e, J_x, J_y, J_z), \quad A^\alpha = (\Phi, A_x, A_y, A_z)$$

The field tensor is

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

The inhomogeneous Maxwell equations in 3D notation are

$$\nabla \cdot \mathbf{E} = \rho_e, \quad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \mathbf{t}} = \mathbf{J}$$

Relativity
Problem Set 4

1. Verbatim from Binney (p. 29):

"Consider a star warrior who regains consciousness in a closed cabin some time after being taken prisoner. He reaches for his watch and knocks it to the floor. Fortunately it falls only slowly, so it continues to tick. Is he in a (possibly elastic) accelerating spaceship, or is he on an asteroid? By now fully alert he determines that plumb bobs on either side of the cabin point towards a spot some ten miles away. He instantly concludes that he is either on an asteroid or that opposite sides of his cabin are accelerating away from one another. Moments later he verifies that his bobs have *not* moved apart. Hence he must be on an asteroid. What would he have concluded if he had found that his bobs pointed *away* from a spot thirty yards distant?" [10]

2. Show that a symmetric tensor $S_{\mu\nu}$ remains symmetric under general coordinate transformations. [5]

3. $\Gamma^{\mu}_{\alpha\beta}$ is not a tensor. Can you think of a one-line argument that shows this?

Hint: What is $\Gamma^{\mu}_{\alpha\beta}$ in the freely falling frame? [5]

4. For a down/covariant vector A_i in $3D$, express the components (A_r, A_θ, A_ϕ) in terms of (A_x, A_y, A_z). Here r, θ, ϕ are the usual polar coordinates. [10]

5. Derive the metric tensor in time-plus-polar coordinates (defined below) by starting with $\boldsymbol{\eta}$ in cartesian coordinates and transforming. [20]

6. For some scalar ψ write down the components of $\psi_{,i}$ and $\psi^{,i}$ in 3D spherical polar coordinates. Compare with $\nabla\psi$ in vector calculus. [10]

7. In time-plus-polar coordinates, use the metric (given below) and the Euler-Lagrange equations to write down all four geodesic equations. Comparing with the geodesic equation (also given below) read off all the non-zero Christoffel symbols. [15]

8. For the metric

$$ds^2 = -e^{\lambda} dt^2 + e^{-\lambda} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

where λ is a function of r, compute all the nonzero Christoffel symbols. [25]

Supplementary information

The geodesic equation is

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\,\frac{dx^\mu}{d\tau}\,\frac{dx^\nu}{d\tau} = 0$$

"Time-plus-polar coordinates" refers to (t, r, θ, ϕ) where r, θ, ϕ are the usual 3D polar coordinates

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

In these coordinates the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1/r^2 & \\ & & & 1/(r^2\sin^2\theta) \end{pmatrix}$$

Relativity
Problem Set 5

1. For an arbitrary scalar Φ, simplify $\Phi_{;\alpha\beta} - \Phi_{;\beta\alpha}$ [10]

2. Simplify $g_{\mu\nu;\lambda}$ [10]
Hint: Consider a freely-falling frame.

3. Show that $\left(g^{\alpha\beta} v_\beta\right)_{;\alpha} = g^{\alpha\beta} v_{\beta;\alpha}$ [10]

4. How should ${A_\mu}^{;\lambda}$ be defined? [10]

5. Starting from the definition of $\Gamma^\lambda_{\mu\nu}$ show that

$$\Gamma'^{\gamma}_{\alpha\beta} = {\Lambda_\alpha}^\mu \, {\Lambda_\beta}^\nu \, {\Lambda^\gamma}_\lambda \, \Gamma^\lambda_{\mu\nu} + \frac{\partial^2 x^\nu}{\partial x'^\alpha \partial x'^\beta} \, {\Lambda^\gamma}_\nu$$

and

$$v'_{\alpha,\beta} = {\Lambda_\alpha}^\mu \, {\Lambda_\beta}^\nu \, v_{\mu,\nu} + \frac{\partial^2 x^\mu}{\partial x'^\alpha \partial x'^\beta} \, v_\mu$$

where the ${\Lambda^\alpha}_\mu$ and ${\Lambda_\beta}^\nu$ are as given below. Hence verify that

$$v_{\mu,\nu} - \Gamma^\lambda_{\mu\nu} v_\lambda$$

is a tensor. [35]

6. Show that for a vector A^μ

$$A^\mu{}_{;\mu} = \frac{1}{\sqrt{g}} \left(\sqrt{g} A^\mu\right)_{,\mu}$$

where g stands for $|\det g_{\alpha\beta}|$. [25]

Supplementary information

The Christoffel symbol is

$$\Gamma^\lambda_{\mu\nu} = \left(\frac{\partial x^\lambda}{\partial \xi^\alpha}\right) \left(\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\right) = g^{\lambda\alpha} (g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha})$$

and the transformation matrices are

$${\Lambda^\alpha}_\beta \equiv \frac{\partial x'^\alpha}{\partial x^\beta} \qquad {\Lambda_\alpha}^\beta \equiv \frac{\partial x^\beta}{\partial x'^\alpha}$$

Square matrices satisfy the identity

$$\left(M^{-1}\right)_{ij} = \frac{\partial}{\partial M_{ij}} \ln |\det M|$$

Relativity
Problem Set 6

1. Consider a 2D space with metric

$$ds^2 = e^{\lambda}\, dr^2 + r^2\, d\theta^2$$

where λ is a function of r.
Show that the only nonzero Christoffel symbols are

$$\Gamma^r_{rr} = \tfrac{1}{2}\lambda' \qquad \Gamma^r_{\theta\theta} = -re^{-\lambda} \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \tfrac{1}{r}$$

From this, show that the Ricci tensor is

$$R_{rr} = -\tfrac{1}{2}\frac{\lambda'}{r} \qquad R_{\theta\theta} = -\tfrac{1}{2}r\lambda' e^{-\lambda} \qquad R_{r\theta} = R_{\theta r} = 0$$

and that the Ricci scalar is

$$R = \frac{1}{r}\frac{d}{dr}\,e^{-\lambda}$$

An important special case has $e^{-\lambda} = 1 - r^2$. Can you guess what this surface is? [100]

Supplementary information

The Ricci tensor is

$$R_{\mu\nu} = \Gamma^\alpha_{\alpha\mu,\nu} - \Gamma^\alpha_{\mu\nu,\alpha} + \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu} - \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu}$$

Relativity
Problem Set 7

1. Show that the field equations are equivalent to

$$R_{\mu\nu} = -8\pi G(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu})$$

where T means $g^{\mu\nu}T_{\mu\nu}$. [10]

2. Before arriving at the field equations Einstein tried and rejected other possibilities. One of them was

$$R_{\mu\nu} = -8\pi GT_{\mu\nu}$$

Show that this theory will lead to

$$R^{;\mu} = -16\pi GT^{\mu\nu}{}_{;\nu}$$

Hint: Use the contracted Bianchi identity.

Write down the t component of for dust. What strange physical effect is implied by this equation? [25]

3. Show that the weak field metrics

$$ds^2 = -\left(1 - \frac{r_\mathrm{s}}{r}\right) dt^2 + \left(1 + \frac{r_\mathrm{s}}{r}\right) \left[dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right]$$

$$ds^2 = -\left(1 - \frac{r_\mathrm{s}}{r}\right) dt^2 + \left(1 + \frac{r_\mathrm{s}}{r}\right) dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

(r_s is constant) are physically equivalent, by suggesting a suitable gauge transformation. Assume that terms of order $({r_\mathrm{s}}^2/r^2)$ are negligible. [25]

4. By examining the retarded potential (given below) for the electromagnetic potential, write down the analogous solution of the weak field equations

$$\nabla^2 h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \tfrac{1}{2}T\eta_{\mu\nu}\right)$$

for $h_{\mu\nu}$ in terms of $T_{\mu\nu}$. [10]

5. Consider a stationary (i.e., having $v_i = 0$) perfect fluid, and potentials satisfying

$$\nabla^2\Phi = 4\pi G\rho \qquad \nabla^2\Psi = 4\pi Gp$$

where ρ and p are, as usual, density and pressure. Derive the weak-field metric in terms of Φ and Ψ. [30]

Supplementary information

The wave equation

$$\nabla^2 A^\alpha = -J^\alpha$$

for the electromagnetic potential has the solution

$$A^\alpha(t,\mathbf{x}) = \frac{1}{4\pi}\int \frac{J^\alpha(t-r,\mathbf{x}')}{r}\, d^3\mathbf{x}', \quad r \equiv |\mathbf{x} - \mathbf{x}'|$$

An ideal fluid has energy-momentum tensor

$$T^{\alpha\beta} = \gamma^2\rho\begin{pmatrix} 1 & v_j \\ v_i & v_iv_j\end{pmatrix} + p\begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij}\end{pmatrix} + \gamma^2 p\begin{pmatrix} v^2 & v_j \\ v_i & v_iv_j\end{pmatrix}$$

Relativity
Problem Set 8

1. The Schwarzschild metric has been derived as a solution of the field equations. In fact it also satisfies the rejected field equations

$$R_{\mu\nu} = -8\pi G T_{\mu\nu}$$

Why? (A one-line argument.) [5]

2. Show that any spherically symmetric metric, i.e., a metric of the type

$$ds^2 = g_{tt}\,dt^2 + 2g_{tr}\,dt\,dr + g_{rr}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

conserves angular momentum. [15]

3. In the solar system, one has classical perturbations as well as general relativity. The orbit equation may have the form

$$u'' + u = a^{-1}(1-e^2)^{-2} + \tfrac{3}{2} r_{\rm s} u^2 + k u^n$$

where k is a small constant of the same order as the relativistic term and n is an integer. How will the precession rate be modified by the new term? [20]

4. Consider two circles in the Schwarzschild metric, concentric with the metric and having circumferences $2\pi a_1$ and $2\pi a_2$. What is the radial distance between the circles? [10]

5. A spaceship is in a circular orbit at $r = 30\,\mathrm{km}$ around a black hole with Schwarzschild radius $3\,\mathrm{km}$. The crew travel inwards by $28\,\mathrm{km}$ and yet emerge to tell the tale. Explain why this could happen. (Numerical calculations not required.) [15]

6. Show that a radial path in a Schwarzschild metric, going from r to $r + \Delta r$, covers a distance of

$$\Delta r\left(1 + \frac{r_{\rm s}}{2r}\right)$$

where $r_{\rm s}$ is the Schwarzschild radius, assuming $r_{\rm s}, \Delta r \ll r$. [15]

7. A spaceship is in orbit around a black hole. The orbit is circular and has circumference $2\pi \times 100\,\mathrm{km}$. A pod leaves the ship and settles into another circular orbit, at distance $10\,\mathrm{km}$ inward of the ship's orbit. The pod's orbit is then measured and has circumference $2\pi \times 91\,\mathrm{km}$. Make a rough estimate of the Schwarzschild radius. [15]

8. Explain why an observer inside a Schwarzschild horizon is "carried down to $r = 0$ as surely as you and I are carried into next year." [5]

Supplementary information

Orbit precession (with classical perturbations) is given by

$$\tilde{\phi} = \phi\left(1 - \frac{\frac{3}{2} r_{\rm s}}{a(1-e^2)}\right)$$

Relativity
Problem Set 9

1. Consider a 4D space (x, y, z, w) [w is unrelated to time] and embedded in it the 3D surface

$$x = \sin\psi \sin\theta \cos\phi \quad y = \sin\psi \sin\theta \sin\phi$$
$$z = \sin\psi \cos\theta \qquad w = \cos\psi$$

Show that this surface can be interpreted as a 3-sphere. What is its radius? Show further that the differential *Euclidean* distance between points on this surface is given by

$$ds^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Suggest a new coordinate r that would give

$$ds^2 = \frac{dr^2}{1 - r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Comment on the relation between this ds^2 and the Robertson-Walker metric. [20]

2. In a flat universe containing only dust, show that $a \propto t^{2/3}$. [20]

3. In some early-universe theories, at one epoch the universe was dominated by 'vacuum-energy' with negative pressure, $p = -\rho$.

Show that if $p = -\rho$ and ρ is initially positive, then a will increase exponentially with time. [20]

4. Show that in a Robertson-Walker metric using the conformal time, null geodesics are independent of the scale factor. [40]

Supplementary information

The Friedmann equation is

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2$$

and the scale factor, density, and pressure are related by

$$\frac{d}{da}\left(a^3\rho\right) = -3a^2 p$$