Problem Set 1

1. Simplify the following

(a) $\cos 120^\circ$

(b) $\sin 75^\circ \sin 15^\circ$ [10]

2. Find $\int \sin^4 x\, dx$ and $\int_0^{2\pi} \sin^4 x\, dx$. [10]

3. Show that for positive integers m, n

$$\int_0^{2\pi} \sin mx \sin nx\, dx = \pi \delta_{mn}$$
$$\int_0^{2\pi} \cos mx \cos nx\, dx = \pi \delta_{mn}$$

where

$$\delta_{mn} \equiv \begin{cases} 1 \text{ if } m = n \\ 0 \text{ otherwise.} \end{cases}$$ [10]

4. Show that if

$$I_n \equiv \int_0^{2\pi} x^n \exp(ix)\, dx$$

then

$$I_n = i\left(nI_{n-1} - (2\pi)^n\right).$$

Hence evaluate $\int_0^{2\pi} x^2 \sin(x)\, dx$. [15]

5. Find dy/dx where

$$x = u - \tfrac{1}{2}\sin 2u, \qquad y = \sin^2 u$$

and simplify it. [5]

6. Use rescaling to evaluate

$$\int \frac{dx}{a^2 + x^2} \qquad \text{and} \qquad \int_0^\infty x^n e^{-ax}\, dx$$

where a is a constant and n is a non-negative integer. [You may assume the answers for $a = 1$.] [10]

7. Let $x = \cos\phi$, $y = \sin\phi$, $f = x^2 + y^2$. Compute $df/d\phi$.

Try and interpret your answer geometrically. [10]

8. Given $y = u^v$, where u and v are functions of x, use the chain rule to derive

$$y' = u^v \left(v' \ln u + \frac{v}{u} u' \right). \qquad [10]$$

9. Show that $\int_0^\infty \sin x \, e^{-x} \, dx$ is finite but $\int_0^\infty \tan x \, e^{-x} \, dx$ diverges. [10]

10. Compute $\int_0^\pi \sin x \, dx$ using the trapezoidal rule with 6 intervals (i.e., evaluate the sine at 30°, 60°, etc.)

How accurate is the answer? [10]

Prize question. *This question is optional. The first correct solution (any time up to the end of term) wins £10. Solutions to Dr. Saha please.*

Show that

$$\int_0^{\pi/2} \ln(\sin x) \, dx = -\tfrac{1}{2}\pi \ln 2.$$

Hint: Try and relate the LHS to $\int_0^{\pi/2} \ln(\cos x) \, dx$. Then add both integrals and use some cunning. . .

Problem Set 2

1. Suppose we are given

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and

$$\frac{df}{dx} = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)\,.$$

Derive A_0, A_n, B_n in terms of a_0, a_n, b_n with brief explanations. [10]

2. Show that for $-\pi < x < \pi$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nx}{n^2}$$ [20]

3. Show that for $-\pi < x < \pi$

$$x = 2 \sum_{n=1} \frac{(-)^{n+1}}{n} \sin nx\ . \tag{1}$$

Now, with the help of (1)

(a) Derive

$$1 - \tfrac{1}{3} + \tfrac{1}{5} + \ldots = \frac{\pi}{4}\ .$$

(b) Use Parseval's relation [which recall is $\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ in the usual notation] to derive an infinite series for π^2. [25]

4. Write down the Fourier series for

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ x & \text{if } 0 < x < \pi. \end{cases}$$

Simplify the series at $x = \pi$ to derive a series for π^2. [20]

5. We are given an odd function $f(x)$ and we want to approximate it in the domain $[-\pi, \pi]$ by a finite series

$$\sum_{n=1}^{N} s_n \sin nx$$

with suitable coefficients s_n. To find the optimal s_n we first define a function which measures the 'badness' of the approximation

$$R = \int_0^{\pi} \left(f(x) - \textstyle\sum_{n=1}^{N} s_n \sin nx\right)^2 dx$$

and then minimize R by requiring

$$\frac{\partial R}{\partial s_m} = 0.$$

Solve this last equation for s_m and compare with the usual Fourier sine coefficients b_m. [25]

Problem Set 3

1. We want to find a path $y(x)$ between two given endpoints such that

$$\int (y'^2 - y^2)\, dx$$

is extremized. Determine the Euler-Lagrange equation for the problem and its general solution.

What is the particular solution if the endpoints are $(0, 0)$ and $(\frac{\pi}{2}, \frac{\pi}{2})$? [20]

2. Give the common name for the curve

$$x = \phi \cos \phi \qquad y = \phi \sin \phi$$

with a brief explanation or sketch. [The common name will be obvious from a sketch.]

Show that arc length along this curve is

$$\int \sqrt{1 + \phi^2}\, d\phi$$

and then solve the integral. [Try the substitution $\phi = \sinh x$; the identity $\cosh 2x = 2\cosh^2 x - 1$ is also useful.] [20]

3. Let $X(x, y)$ and $Y(x, y)$ be well-behaved functions. Show that the Euler-Lagrange equation for extremizing the integral

$$\int X\, dx + Y\, dy = \int (X + Y y')\, dx$$

between given endpoints is

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}\,.$$

What does this condition say about the dependence of the integral on the path? [20]

4. Consider a drag strip in the shape of the curve $y(x)$. If we measure y downwards, the speed of a body released at $(0, 0)$ and sliding down without friction is $\sqrt{y}$. What is the time taken for such a body to get to some point $y(x_1)$ along the drag strip? Derive an Euler-Lagrange equation for a shape that extremizes the time. Verify that a cycloid satisfies that equation.

This is known as the brachistochrone problem. [20]

5. The brachistochrone problem is the most famous problem in the calculus of variations. Look it up on the web and write a mini essay (100 words or less) on why you think it became famous. [20]

Note: Half the marks are for looking up the relevant historical facts, the rest are for making some intelligent comment on them.

Problem Set 4

1. Work out the angle between the vectors $2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$ and $4\,\mathbf{i} + \mathbf{j} - \mathbf{k}$. [8]

2. Describe each of the following surfaces in three dimensional space. [For example, $x^2 + y^2 = 1$ is a cylinder of radius 1 whose axis is the z axis.]

(a) $z = 0$

(b) $x^2 + y^2 + z^2 - 2y - 4z + 1 = 0$

(c) $x^2 + y^2 = z$ [16]

3. Consider a sphere of radius 3 centred at the origin.

(a) Express the surface of the sphere in the form $f(x, y, z) = 0$.

(b) Find a plane tangent to the sphere at the point $\mathbf{r}_1 = (1, 2, 2)$.

(c) Find the position vector for a line through $\mathbf{r}_1$ and $\mathbf{r}_2 = (2, 1, 2)$.

(d) Find the position vector of a line through $\mathbf{r}_1$ and normal to the sphere. [20]

4. For any vector $\mathbf{v}$ (having nonzero magnitude v) relate dv/dt to $d\mathbf{v}/dt$ and hence show that

$$\frac{dv}{dt} = 0 \Rightarrow \frac{d\mathbf{v}}{dt} \perp \mathbf{v} \, . \qquad [12]$$

5. Given an arbitrary constant vector $\mathbf{a}$, simplify $\nabla|\mathbf{r} - \mathbf{a}|$. [8]

Hint: Can you think of a substitution that would simplify the derivatives?

6. Calculate the following

$$\nabla \cdot \mathbf{r}, \quad \nabla\times\mathbf{r}, \quad \mathbf{r}.\nabla\mathbf{r}, \quad \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) \, . \qquad [20]$$

7. If $\nabla^2\Phi = 0$, show that $\nabla\Phi$ has zero divergence and curl. [8]

8. Consider a rotation by 90° about the z axis, taking $x \to x'$ etc. Evaluate $x'^2\,\mathbf{i}' + y'^2\,\mathbf{j}' + z'^2\,\mathbf{k}'$ under this transformation. [8]

Prize question. *The first correct solution received by Dr. Saha before the end of term wins £10.*

For the differential equations

$$\ddot{\mathbf{r}} = -\nabla(r^{-1})$$

show that

$$\dot{\mathbf{r}}\times(\mathbf{r}\times\dot{\mathbf{r}}) - \hat{\mathbf{r}} = \text{const.}$$

Problem Set 5

Derive the following vector identities using index notation.

You may assume the identity

$$\epsilon_{ijk}\ \epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{km}\delta_{jn}.$$

1. $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$ [20]

2. $\nabla\times(\psi\mathbf{u}) = (\nabla\psi)\times\mathbf{u} + \psi\nabla\times\mathbf{u}$ [20]

3. $\nabla\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\nabla\times\mathbf{u} - \mathbf{u}\cdot\nabla\times\mathbf{v}$ [20]

4. $\nabla\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\nabla\cdot\mathbf{v} - \mathbf{v}\nabla\cdot\mathbf{u} + \mathbf{v}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{v}$ [20]

5. $\nabla\times\nabla\times\mathbf{u} = \nabla\,\nabla\cdot\mathbf{u} - \nabla\cdot\nabla\mathbf{u}$ [20]

Problem Set 6

1. Find the Earth polar coordinates of the following cities:

(a) Antananarivo (Tananarive)

(b) Chihuahua,

(c) Makkah (Mecca),

(d) Singapore.

Answers may be rounded to the nearest degree. [16]

2. How would you explain the shape of the segments of an orange to someone who knows about spherical polar coordinates but has never seen an orange?

[Must be done without pictures.] [12]

3. In the accompanying figure the vector $\hat{\mathbf{e}}_1$ is recognizable as $\hat{\mathbf{e}}_z$ since it points towards increasing z.

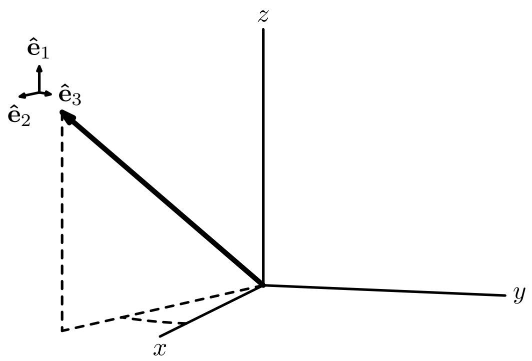


Figure for problem 3.

Identify the other basis vectors $\hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_6$, including minus signs if needed. Justify your identification in a few sentences. [16]

4. Invert the relation

$$\begin{pmatrix} \hat{\mathbf{e}}_\rho \\ \hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}$$

for cylindrical coordinates, to $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in terms of $\hat{\mathbf{e}}_\rho$, $\hat{\mathbf{e}}_\phi$, $\hat{\mathbf{e}}_z$.

Hence show that the position vector $\mathbf{r}$ can be expressed as $\rho\,\hat{\mathbf{e}}_\rho + z\,\hat{\mathbf{e}}_z$. [12]

5. Show that in cylindrical coordinates

$$\dot{\hat{\mathbf{e}}}_\rho = \dot{\phi}\,\hat{\mathbf{e}}_\phi \qquad \dot{\hat{\mathbf{e}}}_\phi = -\dot{\phi}\,\hat{\mathbf{e}}_\rho$$

and hence show that

$$\ddot{\mathbf{r}} = (\ddot{\rho} - \rho\dot{\phi}^2)\,\hat{\mathbf{e}}_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\,\hat{\mathbf{e}}_\phi + \ddot{z}\,\hat{\mathbf{e}}_z. \qquad [12]$$

6. Consider the two-dimensional coordinate system (α, β) defined through

$$x = \sinh\alpha\,\sin\beta, \quad y = \cosh\alpha\,\cos\beta.$$

Curves of constant α are ellipses, while curves of constant β are hyperbolae.

Show that this coordinate system is orthogonal.

Hint: Can you show that $\hat{\mathbf{e}}_\alpha \cdot \hat{\mathbf{e}}_\beta = 0$? [16]

7. Show that about 4% of the Earth's surface lies north of the Arctic circle.

Hint: No calculator is needed, but remembering Taylor series will help. [16]

Prize question. *The first correct solution received by Dr. Saha before the end of term wins £10.*

Show

$$\int_0^\infty x^{n-\frac{1}{2}}\,e^{-x}\,dx = (n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots\tfrac{1}{2}\,\sqrt{\pi}$$

where n is an integer ≥ 0.

Problem Set 7

1. Calculate

$$\oint \mathbf{F} \cdot d\mathbf{r}, \quad \mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$$

around the circle $x^2 + y^2 = 1$ in the x, y plane. [16]

2. Compute

$$\int (\nabla \times \mathbf{F}) \cdot d\mathbf{S}, \quad \mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$$

over (i) the disc $z = 0,\ \rho \leq 1$, and (ii) the unit hemisphere $r = 1,\ z > 0$. [20]

3. Evaluate

$$\int \frac{\mathbf{r} \times d\mathbf{r}}{r^2}$$

along the line $(0, a, t)$, where a is a positive constant and $-\infty < t < \infty$. [20]

4. Consider a line integral $\int \mathbf{F} \cdot d\mathbf{r}$ of each of the following vector fields.

(a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

(b) $\mathbf{F} = x\,\mathbf{i} + z\,\mathbf{j} + y\,\mathbf{k}$

(c) $\mathbf{F} = x\,\mathbf{i} + z\,\mathbf{j} - y\,\mathbf{k}$

(d) $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$

In each case, state whether the line integral will depend on the path. Give brief reasons in your answer. [20]

5. Show that

$$\int \frac{r^{n-3}\,dV}{(1+r)^{n+1}}$$

over a sphere of radius R centred at the origin equals $(4\pi/n)(1 + 1/R)^{-n}$. [24]

Problem Set 8

1. Show that over any closed surface

$$\oint \mathbf{r} \cdot d\mathbf{S} = 3 \langle \text{enclosed volume} \rangle \ . \qquad [8]$$

2. An important theorem in complex variables (called Cauchy's theorem) states that if there are two well-behaved functions $u(x, y)$ and $v(x, y)$ satisfying

$$\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial v}{\partial y}\right) \quad \left(\frac{\partial u}{\partial y}\right) = -\left(\frac{\partial v}{\partial x}\right)$$

and an arbitrary closed curve C in the x, y plane then

$$\oint_C (u + iv)\,(dx + idy) = 0.$$

Show that Cauchy's theorem is a special case of Green's theorem in the plane.

Hint: Try separating out the real and imaginary parts of the integral. [12]

3. Show that the following are all zero.

(a) $\oint \mathbf{r} \cdot d\mathbf{r}$ over an arbitrary closed path.

(b) $\oint \mathbf{r} \times d\mathbf{S}$ over an arbitrary closed surface. [*Hint:* Try taking the $\mathbf{k}$ component.]

(c) $\oint d\mathbf{S}$ over an arbitrary closed surface. [28]

4. Write down the explicit form of

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\,\hat{\mathbf{e}}_1 & h_2\,\hat{\mathbf{e}}_2 & h_3\,\hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

in spherical polar coordinates

Calculate $\nabla \times \hat{\mathbf{e}}_r$ using that explicit form.

Can you get the value of $\nabla \times \hat{\mathbf{e}}_r$ *without* using the determinant? [*Hint:* Can you express $\hat{\mathbf{e}}_r$ as the gradient of something?] [16]

5. Compute $\nabla^2 \psi$ where

$$\psi(r) = \frac{1}{n-1}\left[\left(1 + \frac{1}{r}\right)^{1-n} - 1\right].$$

Simplify ψ and $\nabla^2 \psi$ for $n = 1$. [16]

6. Prove that divergence of a curl is identically zero, *without* doing any coordinate-dependent calculation, in the following way.

First consider two surfaces enclosed by the same closed curve (for example, the northern and southern hemispheres enclosed by the equator). Then invoke the divergence and Stokes' theorems using suitable arguments. You may wish to use a sketch. [20]

Problem Set 9

1. In section 9.6 of the notes, if we had chosen k^2 instead of $-k^2$ for the constant, what would have been different? [10]

2. Consider the square $x, y \in [0, \pi]$, and an integer n. What are the values of the following on the boundaries of the square?

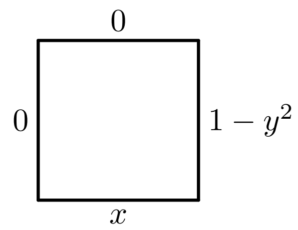
(a) $\sin(nx)\,\sinh(ny)$

(b) $\sinh(nx)\,\sin(ny)$

(c) $\sin(nx)\,\sinh(n(\pi - y))$

(d) $\sinh(n(\pi - x))\,\sin(ny)$ [15]

3. A function $F(x, y)$ satisfies Laplace's equation in the square $x, y \in [0, 1]$ with boundary values as indicated below.



This $F(x, y)$ can be split into two functions

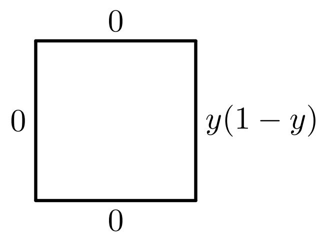
$$F(x, y) = f(x, y) + g(x, y)$$

such that $f(x, y) = 0$ at the corners and $g(x, y) = F(x, y)$ at the corners.

(a) Find $g(x, y)$.

(b) Find $f(x, y)$ on the sides. [The full solution for $f(x, y)$ is not required for this problem.] [10]

4. Solve Laplace's equation for $f(x, y)$ in the square $x, y \in [0, 1]$ with boundary values as indicated below.



[20]

5. A relative of the Laplace equation is the Helmholtz equation

$$\nabla^2 \psi + \psi = 0\,.$$

Suppose this equation has solutions of the form

$$\psi(\rho, \phi) = J_m(\rho)\, e^{-im\phi}$$

what differential equation must $J_m(\rho)$ satisfy? [20]

6. A function $\psi(x, y)$ satisfies Laplace's equation and is finite and single-valued in the circle $x^2 + y^2$. Write down the form of ψ in cylindrical coordinates, with a brief explanation. [You may assume the general solution in cylindrical coordinates.] Hence show that the average of ψ over the circle equals the value at the centre. [15]

7. Legendre polynomials satisfy many identities, one of which is

$$\frac{1}{\sqrt{1 - 2hx + h^2}} = \sum_{l=0}^{\infty} h^l P_l(x)$$

where h is a parameter independent of x.

Expand the LHS to order h^2 and verify that the expansion coefficients are indeed the the first three Legendre polynomials. [10]