1 Effective field theories

Let us consider a theory with a characteristic large-energy scale M and suppose that we are only interested in physics at energies $E \ll M$. The idea is that at energies $E \gtrsim M$ the theory is some sort of fundamental theory, not necessarily a field theory, possibly a string theory, that answers all the fundamental questions.
At energies $E \ll M$ one can use an <u>effective lagrangian</u>, constructed only using the low energy degrees of freedom. Assuming that the full theory is a quantum field theory, it will be defined by a path integral, and all what we need can be obtained by calculating

$$\langle 0 | T\,\phi(x_1)\cdots\phi(x_n)|0\rangle = \left.\frac{\delta^n Z(J)}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0} (-i)^n$$

where

$$Z(J) = \int D\phi\, e^{iS[\phi] + i\int J\phi}$$

To obtain the low-energy effective action, we can split the field as $\phi = \phi_L + \phi_H$ where ϕ_H contains all the Fourier modes with $\omega > \Lambda$ and ϕ_L the modes with $\omega < \Lambda$.
Since we are interested in low-energy phenomena, we can limit ourselves to consider correlation functions that involve only the light fields

$$\langle 0 | T\,\phi_L(x_1)\cdots\phi_L(x_n)|0\rangle = \int D\phi_L \int D\phi_H\, e^{iS[\phi_L+\phi_H]}\,\phi_L(x_1)\cdots\phi_L(x_n)$$

$$= \int D\phi_L\, e^{iS_\Lambda(\phi_L)}\,\phi_L(x_1)\cdots\phi_L(x_n)$$

where we have chosen $\Lambda \lesssim M$ to integrate out the physics associated to large energy scales. $S_\Lambda(\phi_L)$ is called the "WILSON EFFECTIVE ACTION" and it is non local at scales $\Delta x \sim \frac{1}{\Lambda}$ because the high energy (small distance) fluctuations have been integrated out.

② Having integrated out the heavy degrees of freedom we now would like to expand $S_\Lambda(\phi_L)$ in series of local operators. We can write

$$S_\Lambda(\phi_L) = \int d^Dx \, \mathcal{L}_\Lambda^{eff}$$

with

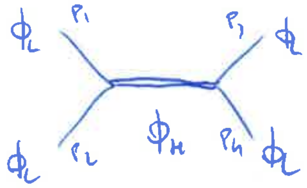
$$\mathcal{L}_\Lambda^{eff} = \sum_i g_i O_i$$

$\mathcal{L}_\Lambda^{eff}$ is the <u>effective lagrangian</u> corresponding to the Wilson effective action.

The coefficients g_i are sometimes called <u>Wilson coefficients</u>.

<u>EXAMPLE</u>

Let us assume that we have a theory with two scalar fields, one heavy, with mass M and one light. A scattering process of the light scalar particles can contain the diagram



$$\sim \frac{1}{p^2 - M^2}$$

Since the momenta are much smaller than M, we can perform the expansion

$$\frac{1}{p^2-M^2} = -\frac{1}{M^2} - \frac{p^2}{M^4} + \cdots$$

This means that $\mathcal{L}_\Lambda^{eff}$ will contain terms like ϕ_L^4 and $\phi^2(\partial_\mu\phi)(\partial^\mu\phi)$.

The question is: how can we identify the operators O_i and the corresponding coefficients?

To this purpose we can use dimensional analysis.

Let us assume that $[g_i] = -\gamma_i$ is the mass dimension of g_i.

$$\Rightarrow \quad g_i = c_i M^{-\gamma_i}$$

Since the coefficients g_i are generated after integration of the heavy degrees of freedom, it is natural to assume $c_i \sim 1$

③ At low energies, the contribution of $\mathcal{O}_i$ to a dimensionless observable scales as

$$\left(\frac{E}{M}\right)^{\gamma_i} = \begin{cases} O(1) & \text{if } \gamma_i = 0 \\ \gg 1 & \text{if } \gamma_i < 0 \\ \ll 1 & \text{if } \gamma_i > 0 \end{cases}$$

We thus conclude that only the operators with $\gamma_i \leq 0$ are important at low energies.

Since the action is dimensionless, it follows that in D dimensions an operator $\mathcal{O}_i$ with mass dimensions d_i has $\gamma_i = \delta_i - D$

Here are some examples

	δ_i	γ_i	g_i
$\partial_\mu\phi\,\partial^\mu\phi$	D	0	1
ϕ^2	$D-2$	-2	M^2
ϕ^4	$2(D-2)$	$D-4$	M^{4-D}
ϕ^6	$3(D-2)$	$2D-6$	M^{6-2D}

So only few operators have $\gamma_i \leq 0$ (unless $D \leq 2$)

We define the various operators as

DIM		behaviour $E \to 0$	
$\delta_i < D$	$\gamma_i < 0$	grows	RELEVANT (SUPERRENORMALIZABLE)
$\delta_i = D$	$\gamma_i = 0$	constant	MARGINAL (RENORMALIZABLE)
$\delta_i > D$	$\gamma_i > 0$	decreases	IRRELEVANT (NON RENORMALIZABLE)

4 The terminology we used here is commonly used, but it is MISLEADING.
For example, the IRRELEVANT OPERATORS, since they provide information on the physics at very high scale, are actually rather important.
Usually one avoids irrelevant operators because they make a theory non renormalizable. However, once we admit that a theory is valid only up to some scale Λ, it is clear that also the irrelevant operators have to be considered. Renormalizable Lagrangians are so successful in our understanding of particle physics at relatively low energies because the relevant and marginal operators are the most important at low energy

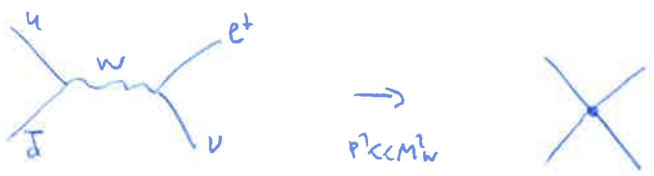
EXAMPLE: FERMI THEORY

The effective lagrangian describing 4-fermion interactions at low energy is

$$\mathcal{L} = \frac{G_F}{\sqrt{2}}\, \bar{u}\gamma_\mu(1-\gamma_5)d \;\bar{\ell}\gamma^\mu(1-\gamma_5)\nu$$

Since the dimension of fermion fields is $\frac{D-1}{2} = \frac{3}{2}$ the effective operator has dimension 6 and the constant G_F has dimension -2

$\Rightarrow$ $G_F \sim \frac{1}{M_W^2}$ where the mass M_W is the mass of the carrier of the electroweak interaction



We said that irrelevant operators are perfectly natural, and they give us insight on the underlying theory. On the contrary, relevant operators can be problematic. Consider for example the ϕ^2 operator in $\lambda\phi^4$ theory.
Even if we start from a mass $m \ll \Lambda$, there is no symmetry that prevents this mass to become $O(\Lambda^2)$ when quantum corrections are considered

⑤ So the natural value for m is $m \sim \Lambda$. But this is a contradiction: if $m \sim \Lambda$ we should have integrated out the corresponding field ϕ.

Looking at the Standard Model as an effective field theory up to a scale Λ (we know that at the Planck scale gravitational interactions have to enter anyway) the mass terms for the fermions and the gauge bosons do not give problems, since they are generated by the Higgs mechanism. The problem comes from the mass term for the Higgs field, which is not protected by any symmetry. Following the same line of reasoning as above, we are led to conclude that the Higgs mass m_H should be of order Λ. Since now we know that $m_H \sim 125\,GeV$ either Λ is very close and new physics is behind the corner (but this picture is challenged by the lack of any evidence for new physics signals from the LHC) or a naturalness problem is there.

Supersymmetry, by introducing a symmetry between fermions and scalars, induces a protection on the Higgs mass, that can thus be of the order of the electroweak scale. However, no superpartners have been found yet.

An intriguing possibility is that the smallness of m_H is due to some accidental cancellation. In this case some people invoke the anthropic principle (we live in the only universe in which indeed life is possible).

⑥ EULER – HEISENBERG LAGRANGIAN

Let us consider QED and suppose that we have a laboratory in which the available energies are much smaller than the electron mass m_e, and that the only available particles are the photons. We are thus limited to study phenomena like, for example, light-light scattering. We want to find the effective lagrangian able to describe these phenomena. To do so we have to integrate out the electron field. Let us start from the QED lagrangian in a covariant gauge:

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m_e)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial^\mu A_\mu)^2$$

The generating functional is

$$Z(J,\eta,\bar{\eta}) = \int D\psi D\bar{\psi} DA_\mu \exp\left\{ i\int d^4x \left(\mathcal{L}_{QED} + \bar{\eta}\psi + \bar{\psi}\eta + J_\mu A^\mu \right)\right\}$$

where J and η are external sources for the photon and electron fields.
The Green functions can be obtained in the usual way by differentiating the generating functional with respect to the sources. However, since we are interested in processes where only photons are the external particles, we can use

$$Z(J) = Z(J,\eta,\bar{\eta})\Big|_{\eta=\bar{\eta}=0} = \int D\psi D\bar{\psi} DA_\mu \exp\left\{ i\int d^4x \left(\mathcal{L}_{QED} + J_\mu A^\mu \right)\right\}$$

$$= \int DA_\mu \exp\left\{ i\int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial^\mu A_\mu)^2 + \mathcal{L}_{eff}(A_\mu) + J_\mu A^\mu \right)\right\}$$

where

$$\exp\left\{ i\int d^4x\, \mathcal{L}_{eff}(A_\mu)\right\} = \int D\psi D\bar{\psi} \exp\left\{ i\int d^4x\, \bar{\psi}(i\not{D} - m_e)\psi\right\}$$

Let us now notice that the integrand of the above equation is gauge invariant

$$\psi \to e^{ie\alpha(x)}\psi \qquad \bar{\psi} \to e^{-ie\alpha(x)}\bar{\psi} \qquad A_\mu \to A_\mu + \partial_\mu\alpha$$

⑦ As such, assuming that the path integral measure is also invariant (this is indeed the case for a vector gauge symmetry) also the effective action is gauge invariant

$$\exp\left\{ i \int d^4x\, \mathcal{L}_{eff}(A_\mu) \right\} = \exp\left\{ i \int d^4x\, \mathcal{L}_{eff}(A_\mu + \partial_\mu \alpha) \right\}$$

$\Rightarrow$ this means that the effective Lagrangian depends only on the gauge invariant field tensor $F_{\mu\nu}$

$$\mathcal{L}_{eff}(A_\mu) \to \mathcal{L}_{eff}(F_{\mu\nu})$$

The additional requirements of Lorentz and parity invariance allow us to write

$$\mathcal{L}_{eff} = -\frac{c_0}{4} F_{\mu\nu}^2 + \frac{c_1}{m_e^2} F_{\mu\nu} \Box F^{\mu\nu} + \frac{1}{m_e^4}\Big[\alpha^2 c_2^1 (F_{\mu\nu})^4 + \alpha^2 c_2^2 (F_{\mu\nu}\tilde{F}^{\mu\nu})^2$$
$$+ c_2^3 F_{\mu\nu} \Box^2 F^{\mu\nu} \Big] + \dots$$

where $\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$.

If we are interested only in photon interactions, we can limit ourselves to consider the operators $(F_{\mu\nu})^4$ and $(F\tilde{F})^2$ and write

$$\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{\alpha^2}{m_e^4}\left[c_2^1 (F_{\mu\nu})^4 + c_2^2 (F_{\mu\nu}\tilde{F}^{\mu\nu})^2 \right] + \dots$$

This Lagrangian was written by Euler in 1936 using the results obtained by Heisenberg on light-light scattering and goes under the name of <u>EULER-HEISENBERG</u> Lagrangian.

The power $\frac{1}{m_e^4}$ is added on purely dimensional grounds and the power of α^2 is there because we want to describe four photons. From the above form of the Euler-Heisenberg Lagrangian we can write the form of the cross-section for light-light scattering at low energy

$$\sigma_{\gamma\gamma} \sim \frac{\alpha^4 E^6}{m_e^8}$$

where the power E^6 is again added from dimensional analysis.

We now want to sketch the calculation that leads to fix the coefficients of the Euler-Heisenberg lagrangian.

The effective lagrangian can be obtained by using the well known relation, valid in the case of integration of fermionic fields

$$\int D\psi D\bar{\psi} \ \exp\left\{ i\int d^4x\, \bar{\Psi}(i\not{D}-m)\psi \right\} = \det(i\not{D}-m)$$

this formula should be contrasted with the analogous one for bosons

$$\int D\phi\, e^{iS_0[\phi]} \sim \left(\det(\Box+m^2)\right)^{-\frac{1}{2}} \qquad S_0 = -\frac{1}{2}\int d^4x\, \phi(\Box+m^2)\phi$$

which is reminiscent of $\left(\prod_k \int d\xi_k\right) e^{-i\xi_i B_{ij}\xi_j} = \prod_i \sqrt{\frac{\pi}{b_i}} \sim (\det B)^{-\frac{1}{2}}$

↖ eigenvalues of B

We thus can write

$$\int \mathcal{L}_{eff}(A)\, d^4x = -i \ln \det(i\not{D}-m)$$

The key observation that simplifies the calculation is that in order to find the coefficients it is enough to consider constant fields $\underline{E}$ and $\underline{B}$. If we define

$$a^2-b^2 = \underline{E}^2-\underline{B}^2 = -\frac{1}{2}F_{\mu\nu}^2 \qquad ab = \underline{E}\cdot\underline{B} = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}$$

it can be shown that

$$-i\ln\det(i\not{D}-m) = \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s}\, e^{-is(m^2-i\varepsilon)}\left[e^2ab\,\frac{\cosh(eas)\cos(ebs)}{\sinh(eas)\sin(ebs)} - \frac{1}{s^2}\right]$$

expanding in powers of a and b we get

$$e^2ab\,\frac{\cosh(eas)\cos(ebs)}{\sinh(eas)\sin(ebs)} - \frac{1}{s^2} \approx \frac{1}{3}e^2(a^2-b^2) - \frac{1}{45}\left(a^4+5a^2b^2+b^4\right)e^4s^2 + O(e^6)$$

$$= \frac{1}{3} e^2 \left(\underline{E}^2 - \underline{B}^2 \right) - \frac{1}{45} e^4 s^2 \left(\left(\underline{E}^2 - \underline{B}^2 \right)^2 + 7 \left(\underline{E} \cdot \underline{B} \right)^2 \right) + \dots$$

The first term, after integration over s, leads to an UV divergence that can be removed by the renormalization of the free photon $F_{\mu\nu}^2$ term in the lagrangian. The second term leads to a finite contribution and we get

$$-i \ln \det \left(i\not{\partial} - m \right) = C F_{\mu\nu} F^{\mu\nu} + \frac{\alpha^2}{90 m_e^4} F_{\mu\nu}^4 + \frac{7\alpha^2}{360 m_e^4} \left(F_{\mu\nu} \tilde{F}_{\mu\nu} \right)^2 + \dots$$

We add few comments on the usefulness of effective lagrangians.
Let us suppose that we don't know the fundamental theory behind photon interactions. Then the parameters c_i^1 and c_i^2 are arbitrary, but experimental measurements could give us important information on the fundamental theory, for example the ratio α/m_e. The effective field theory is not renormalizable but the divergences can be absorbed in the coefficients of the same lagrangian. In principle such process requires an arbitrary number of coefficients, with apparent loss of predictivity. However, at any loop order there will be an additional factor W/m_e, and thus at any finite order in such ratio, only a finite number of terms is required. More importantly, the validity of the effective lagrangian is not limited to the case in which the fundamental theory has, like QED, a small expansion parameter. From the previous arguments we conclude in fact that the effective expansion parameter is $\alpha W/m_e$. Even if α is large, we could not compute the coefficients from the fundamental theory, but we could still use our effective lagrangian as long as $\alpha W/m_e \ll 1$.