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The σ model

In the previous lectures we were able to use the axial current conservation and the Goldstone theorems to derive important results like the Goldberger-Treiman relation. The situation becomes more complicated when we want to study processes where more than one pions are involved. In this case we should apply the condition of current conservation on matrix elements like

$$\langle \alpha |\, J^{5a}(x_1)\, J^{5b}(x_2)\, | \beta \rangle$$

where α and β denote the other particles, besides the Goldstone bosons, involved in the scattering process. Taking the derivative we get commutators whose value depend on the group algebra. Studying these processes is thus particularly interesting, because it can tell us something on the nature of the symmetry group.
Because of the appearance of such commutators, this approach is known as the method of CURRENT ALGEBRA.
These calculations are difficult and tedious, so one can try to use instead a simplified method, in which Goldstone boson amplitudes are computed in perturbation theory by using an effective Lagrangian.

The linear σ-model (Gell-Mann-Levy, 1960)

We consider the Lagrangian

$$\mathcal{L} = \bar{\psi} i \not\partial \psi + \frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma + \frac{1}{2}(\partial_\mu \underline{\pi})(\partial^\mu \underline{\pi}) - \frac{\lambda}{4}\left(\sigma^2+\underline{\pi}^2\right)^2 - \frac{\mu^2}{2}\left(\sigma^2+\pi^2\right) - g\bar{\psi}\left(\sigma + i\underline{\tau}\cdot\underline{\pi}\gamma_5\right)\psi$$

The fermion field ψ is the nucleon doublet $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$, $\underline{\pi}$ are the three pions and σ is a scalar field.

If we define $\quad \Sigma = \sigma \mathbb{1} + i\,\underline{\tau}\cdot\underline{\pi}$

Then we have

$$\sigma^2+\underline{\pi}^2 = \frac{1}{2}\,\mathrm{Tr}\left(\Sigma^\dagger\Sigma\right) \qquad \sigma = \frac{1}{2}\,\mathrm{Tr}\,\Sigma \qquad \text{(Pauli matrices are traceless)}$$

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$$\frac{1}{4}\mathrm{Tr}(\partial_\mu\Sigma^+\partial^\mu\Sigma) = \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\underline{\pi})^2$$

We can also write

$$-g\bar\psi(\sigma + i\underline{\tau}\cdot\underline{\pi}\gamma_5)\psi = -g\bar\psi\frac{\Sigma+\Sigma^+}{2}\psi - g\bar\psi\left(\Sigma\frac{\gamma_5}{2} - \Sigma^+\frac{\gamma_5}{2}\right)\psi$$

$$= -g\bar\psi\,\Sigma\,\frac{1+\gamma_5}{2}\psi - g\bar\psi\,\Sigma^+\frac{1-\gamma_5}{2}\psi$$

$$= -g\bar\psi_L\Sigma\psi_R - g\bar\psi_R\Sigma^+\psi_L$$

And thus the Lagrangian can be written as $\qquad v^2 = -\frac{\mu^2}{\lambda}$

$$\mathcal{L} = \bar\psi_L i\not\partial\psi_L + \bar\psi_R i\not\partial\psi_R + \frac{1}{4}\mathrm{Tr}(\partial_\mu\Sigma^+\partial^\mu\Sigma) - \frac{\lambda}{4}\left(\frac{1}{2}\mathrm{Tr}(\Sigma^+\Sigma) - v^2\right)^2$$

$$- g\bar\psi_R\Sigma^+\psi_L - g\bar\psi_L\Sigma\psi_R$$

and is invariant under $SU(2)_L\otimes SU(2)_R$ transformations

$$\psi_L \to U_L\psi \qquad \Sigma \to U_L\Sigma U_R^+$$
$$\psi_R \to U_R\psi$$

consider infinitesimal $SU(2)_L\otimes SU(2)_R$ transformations

$$U_L \approx 1 + i\theta_L^a\frac{\tau_a}{2} \qquad U_R \approx 1 + i\theta_R^a\frac{\tau_a}{2}$$

$$\Sigma \to \left(1 + i\theta_L^a\frac{\tau_a}{2}\right)(\sigma + i\underline{\tau}\cdot\underline{\pi})\left(1 - i\theta_R^a\frac{\tau_a}{2}\right)$$

$$\approx \sigma\left(1 + i(\theta_L^a - \theta_R^a)\frac{\tau^a}{2}\right) + i\pi^a\left[\tau^a + \frac{i}{2}\theta_L^b\tau_b\tau_a - \frac{i}{2}\theta_R^b\tau_a\tau_b\right]$$

<u>$SU(2)_V$ transf.</u>

$$\theta_L^a = \theta_R^a \quad \Rightarrow \quad \sigma\to\sigma$$
$$\pi^a \to \pi^a - \frac{1}{2}\varepsilon_{abc}\theta_V^b\pi^c$$

σ and π do not mix

<u>$SU(2)_A$ transf</u>

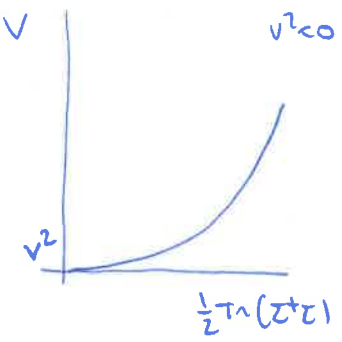
$$\theta_L^a = -\theta_R^a = \theta_A^a \quad \Rightarrow \quad \sigma \to \sigma - \pi^a\theta_A^a$$
$$\pi^a \to \pi^a + \sigma\theta_A^a$$

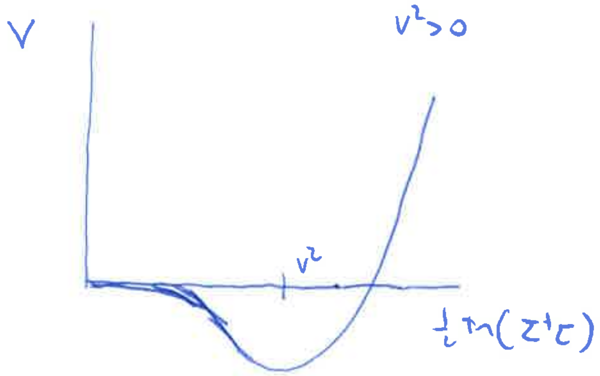
σ and π mix

③ We now discuss the structure of the potential $V\left\{Tr(\Sigma^\dagger\Sigma)\right\} = \frac{\lambda}{4}\left(\frac{1}{2}Tr(\Sigma^\dagger\Sigma)-v^2\right)^2$

When $v^2<0$ the minimum is at $\Sigma=0$, which implies that the perturbative expansion can be done around $\sigma=0$ and $\underline{\pi}=0$.

When $v^2>0$ the potential has a local minimum at $Tr(\Sigma^\dagger\Sigma)=2v^2$





This means that for $v^2>0$ the minimum corresponds to $\sigma^2+\underline{\pi}^2=v^2$

We choose $\langle\sigma\rangle = v$ and $\langle\underline{\pi}\rangle=0$ and define $\sigma' = \sigma - v$

(THIS IS BECAUSE WE WOULD LIKE TO PRESERVE THE $SU(2)_V$ SYMMETRY

We put

$$\mathcal{L} = \bar\psi i\partial\!\!\!/\psi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}(\partial_\mu\underline{\pi})(\partial^\mu\underline{\pi})$$

$$-g\bar\psi(\sigma'+v+i\underline{\tau}\cdot\underline{\pi}\gamma_5)\psi - \frac{\lambda}{4}\left((\sigma'+v)^2+\underline{\pi}^2-v^2\right)^2$$

$$= \bar\psi i\partial\!\!\!/\psi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}(\partial_\mu\underline{\pi})(\partial^\mu\underline{\pi}) - gv\bar\psi\psi$$

↑ mass term for ψ

$$-g\bar\psi(\sigma'+i\underline{\tau}\cdot\underline{\pi}\gamma_5)\psi$$

$$-\frac{\lambda}{4}\left(\sigma'^4 + 4\sigma'^2v^2 + \underline{\pi}^4 + 4\sigma'^3v + 2\sigma'^2\pi^2 + 4\sigma' v\underline{\pi}^2\right)$$

↑ mass term for the σ' field

$\Rightarrow$ The symmetry is spontaneously broken to $SU(2)_V$ and the SSB generates a mass term for the fermions, a mass term for the σ field and the remaining interaction terms.

④ Let us now compute the Noether current corresponding to the axial-vector rotation

$$\delta J_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)}\,\bar\delta\phi_n = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\,\bar\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \sigma)}\,\bar\delta\sigma + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \pi^a)}\,\bar\delta\pi^a$$

$$= \left(\bar\psi\, i\gamma_\mu\gamma_5 \frac{\tau_a}{2}\psi - \pi_a \partial_\mu\sigma + \sigma(\partial_\mu \pi_a)\right)\vartheta_A^a \equiv J_\mu^{5a}\,\vartheta_A^a$$

The matrix element of this current between the vacuum and the pion state is

$$\langle 0|J_\mu^{5a}(x)|\pi^b(p)\rangle = \langle 0|\sigma|0\rangle\langle 0|\partial_\mu \pi_a|\pi_b\rangle = i p_\mu \delta^{ab} e^{-ipx}\langle 0|\sigma|0\rangle$$

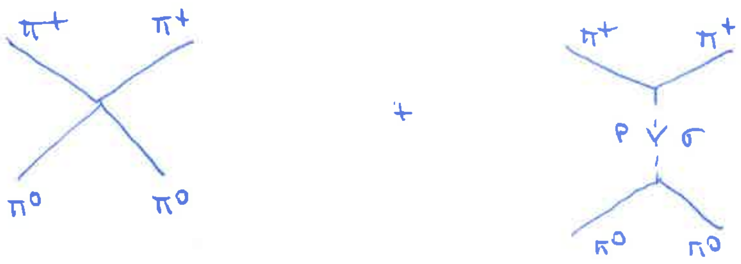
$\Rightarrow$ we conclude that in this model $f_\pi = \langle 0|\sigma|0\rangle = v$

Since $m_N = gv$ this relation turns out to be a simplified version of the Goldberger-Treiman relation (when $g_A \simeq 1$). This is a clear limitation of the model

We now use the lagrangian of the linear σ-model to study $\pi^+\pi^0$ scattering

The relevant part of the interaction lagrangian is $\Delta\mathcal{L}_{int} = -\frac{\lambda}{4}(\underline{\pi}^2)^2 - \lambda v \sigma \underline{\pi}^2$

The Feynman diagrams contributing to $\pi^+\pi^0$ scattering are



and the amplitude is*

$$\mathcal{M} = -2i\lambda + (-2i\lambda v)^2 \frac{i}{p^2 - m_\sigma^2} = -2i\lambda\left(1 + \frac{2\lambda v^2}{p^2 - 2\lambda v^2}\right) = \frac{i p^2}{v^2} + \dots$$

The couplings are non derivative but the constant term cancels out as $p^2 \to 0$ and we are left with an amplitude that vanishes: this is what we expect for Goldstone bosons, since they have derivative couplings!

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The non-linear σ-model

We now go back to the Lagrangian written in terms of Σ

$$\mathcal{L} = \bar{\psi}_L i \not{\partial} \psi_L + \bar{\psi}_R i \not{\partial} \psi_R + \frac{1}{4} \mathrm{Tr}(\partial_\mu \Sigma^+ \partial^\mu \Sigma) - \frac{\lambda}{4}\left(\frac{1}{2}\mathrm{Tr}(\Sigma^+\Sigma) - v^2\right)^2$$
$$- g\,\bar{\psi}_R \Sigma^+ \psi_L - g\,\bar{\psi}_L \Sigma \psi_R$$

which is explicitly invariant under $SU(2)_L \otimes SU(2)_R$. The linear version of the model is obtained by setting $\Sigma = \sigma + i\,\underline{\tau}\cdot\underline{\pi}$. We can instead set $\Sigma = SU$ where $U = \exp(i\,\underline{\tau}\cdot\underline{\xi}/v)$. By using the properties of the Pauli matrices we get

$$U = \cos\left(\frac{\xi}{v}\right) + i\,\frac{\underline{\tau}\cdot\underline{\xi}}{\xi}\sin\frac{\xi}{v} \qquad\qquad \xi = |\underline{\xi}| = \sqrt{\xi_1^2+\xi_2^2+\xi_3^2}$$

We thus have

$$\sigma + i\,\underline{\tau}\cdot\underline{\pi} = S\left(\cos\left(\frac{\xi}{v}\right) + i\,\frac{\underline{\tau}\cdot\underline{\xi}}{\xi}\sin\frac{\xi}{v}\right)$$

and

$$\sigma = S\cos\left(\frac{\xi}{v}\right) \qquad\qquad \pi^a = S\,\frac{\xi^a}{\xi}\sin\frac{\xi}{v}$$

Now since

$$\Sigma^+\Sigma = S^2\cdot\mathbb{1}$$

we have

$$S^2 = \frac{1}{2}\mathrm{Tr}(\Sigma^+\Sigma) \quad \Rightarrow \quad S \text{ is a SINGLET under } SU(2)_L \otimes SU(2)_R$$

For consistency with the linear representation we must require $\langle S\rangle = v$

By introducing the field $S' = S - v$ such that $\langle S'\rangle = 0$ the Lagrangian becomes

$$\mathcal{L} = \bar{\psi}_L i\not{\partial}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R + g(s'+v)\left[\bar{\psi}_R U^+\psi_L + \bar{\psi}_L U \psi_R\right]$$
$$+ \frac{1}{2}(\partial_\mu s')^2 + \frac{1}{4}(s'+v)^2\,\mathrm{Tr}(\partial_\mu U^+ \partial^\mu U) - \frac{\lambda}{4}(s'^2 + 2vs')^2$$
$$= \bar{\psi}_L i\not{\partial}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R - g(s'+v)\left[\bar{\psi}_R U^+\psi_L + \bar{\psi}_L U\psi_R\right]$$
$$+ \frac{1}{2}\left[(\partial_\mu s')^2 - 2\lambda v^2 s^2\right] + \frac{(v+s')^2}{4}\mathrm{Tr}\left[\partial_\mu U \partial^\mu U^+\right] - \lambda v s'^3 - \frac{\lambda}{4}s'^4$$

⑥ We see that in the new representation the Goldstone bosons ξ do not enter the expression for the potential. Moreover the new matrix U transform like Σ:

$$U \to U_L \, U \, U_R^\dagger$$

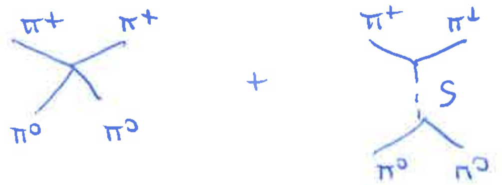
The new representation is just a redefinition of the fields and does not change the physics. There is a theorem (R. Haag (1958), S. Coleman (1968)) that establishes the INDEPENDENCE of the physical content of a theory from the choice of the interpolating fields, and can be phrased as follows:

THEOREM If a theory is defined by the lagrangian density $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ and the following local transformation $\phi = F(\phi')$ is performed then the transformed lagrangian

$$\mathcal{L}'(\phi', \partial_\mu \phi') = \mathcal{L}(F(\phi'), \partial_\mu F(\phi'))$$

defines in general a new theory. However, if the transformation $\phi = F(\phi')$ has a Jacobian which reduces to one in the origin, the S matrices of the two theories are the same.

We can check that the linear and non-linear versions of the σ-model are equivalent by recomputing the scattering amplitude for $\pi^+\pi^0 \to \pi^+\pi^0$ in the non linear version of the model



the second diagram gives now a contribution proportional to p^4 and can be neglected as $p^2 \to 0$

By expanding the $\mathrm{Tr}[\partial_\mu U^\dagger \partial^\mu U]$ term in the non-linear lagrangian one can indeed check that

$$\mathcal{M} \approx \frac{i p^2}{v^2} + O(p^4)$$

as we found in the linear representation (EXERCISE)

$$\frac{1}{4}(S'+v)^2 \mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U) \approx \frac{1}{6v^2}\left[(\xi \cdot \partial_\mu \xi)^2 - \xi^2 (\partial_\mu \xi \cdot \partial^\mu \xi)\right] + \frac{S}{v}(\partial_\mu \xi)^2 + \dots$$

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SUMMARY ON THE σ MODEL

- LINEAR FORMULATION

An isospin singlet σ plus a triplet $\underline{\pi}$ both transforming LINEARLY under the full chiral group

- NON LINEAR FORMULATION

A singlet S (under the full chiral group) plus three scalars ξ_a transforming LINEARLY under isospin ($SU(2)_V$) transformations but NON LINEARLY under the full chiral group (to compensate for the non transformation of S)

Fermions acquire mass $m_N = g V$ and a simplified Goldberger-Treiman relation holds (with $g_A = 1$)

- LINEAR FORMULATION

π^a and σ mix under chiral rotations

$\Rightarrow$ difficult to write an effective lagrangian involving the pions only

- NON LINEAR FORMULATION

Allows to decouple the singlet in a chiral-invariant way !

A low-energy effective lagrangian to describe pion interactions can be obtained by freezing S to its vacuum expectation value $S \to V$

$$\mathcal{L} = \frac{V^2}{4} \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^+\right)$$