Short summary

YM theory with two massless fermions:

$$\mathcal{L} = (\bar{\psi}_1 \; \bar{\psi}_2) \begin{pmatrix} i\not{D} & \\ & i\not{D} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - \frac{1}{2} \mathrm{Tr}[F_{\mu\nu}^2]$$

classically: $U(1)_V \times SU(2)_V \times U(1)_A \times SU(2)_A$

$$\psi \to e^{i\vartheta_V^a \frac{\tau^a}{2}} \psi \quad ; \quad \psi \to e^{i\vartheta_A^a \frac{\tau^a}{2}\gamma_5} \psi$$

$$\tau^0 = \mathbb{1} \,, \quad \tau^i = \sigma^i , \quad i = 1,2,3$$

symmetry	status
$U(1)_V$	preserved in QED/QCD lepton/baryon number conservation
$SU(2)_V$	preserved in QED/QCD isospin conservation
$U(1)_A$	anomalous in QED/QCD
$SU(2)_A$	anomalous in QED spont. broken in QCD

electroweak model:

$U(1)_Y \times SU(2)_L$ is gauge group, a priori anomalous, but (nontrivial!) anomaly cancellation in the SM.

$\pi^0 \to \gamma\gamma$

We want to study the neutral π decay $\pi^0 \to \gamma\gamma$.

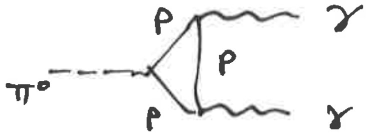
effective Lagrangian:

$$\mathcal{L}_{\pi\gamma\gamma} = g\, \pi^0\, \varepsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}$$

with (naively) $\quad g \approx \frac{1}{8\pi^2}\, e^2 \frac{1}{2F_\pi} = \alpha \frac{1}{4\pi F_\pi}$,

where $F_\pi \simeq 92\, MeV$.

Consider for example the diagram in Yukawa theory:



Steinberger (1949): $\quad g_\Delta = \alpha \frac{G_{\pi N}}{8\pi m_N} = \alpha \frac{g_A}{4\pi F_\pi}$,

$$g_A \simeq 1.257$$

$\leadsto \quad \Gamma(\pi^0 \to \gamma\gamma) \approx m_\pi^3 \frac{\alpha^2}{16\pi^3 F_\pi^2} \approx 4.4 \cdot 10^{16}\, s^{-1}$

exp.: $\quad \Gamma(\pi^0 \to \gamma\gamma) = (1.19 \pm 0.08) \cdot 10^{16}\, s^{-1}$

However: additional constraint from approximate $SU(2)_A$ symmetry!

Can show (Sutherland, Veltman) that $g = 0$ if $m_\pi = 0$ (i.e. symmetry is exact classically)

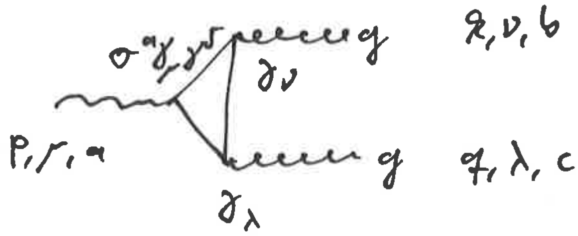
$\Rightarrow$ expect additional suppression of $\mathcal{L}_{\pi\gamma\gamma}$ by factor of $\frac{m_\pi^2}{m_N^2} \approx \frac{1}{50}$

$$\Rightarrow \quad \Gamma(\pi^0 \to \gamma\gamma) \approx \frac{m_\pi^7}{m_N^4} \frac{\alpha^2}{16\pi^3 F_\pi^2} \approx 1.9 \cdot 10^{13}\, s^{-1}$$

What did we miss?

There is no $SU(2)_A$ anomaly in pure QCD

p_μ ·



$$\sim \mathrm{Tr}\left[\sigma^a \underset{\substack{\uparrow \\ \text{color matrices}}}{t^b t^c}\right]$$

$$= \underbrace{\mathrm{Tr}[\sigma^a]}_{=0} \mathrm{Tr}[t^b t^c]$$

but we know there is an axial anomaly in QED, and we <u>do</u> couple photons. Expect

$$\mathrm{Tr}\left[\sigma^3 Q^2\right] = N_c e^2 \,\mathrm{Tr}\left[\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} (\frac{2}{3})^2 & \\ & (-\frac{1}{3})^2 \end{pmatrix}\right]$$

$$= \frac{N_c}{3} e^2 \neq 0 .$$

How can we make this more explicit? Consider

$$\int d^4x \; e^{-iqx} \langle \gamma(k_1) \gamma(k_2) | T S \, j_\mu^{53}(x) | 0 \rangle$$

$$\overset{LSZ}{=} \int d^4x_1 \, d^4x_2 \, d^4x \; e^{ik_1 x_1} e^{ik_2 x_2} e^{-iqx}$$

$$\times \int d^4y_1 \, d^4y_2 \; G^{-1}(x_1 - y_1) \, G^{-1}(x_2 - y_2)$$

$$\times \langle 0 | T \, \varepsilon_1 \cdot A(y_1) \, \varepsilon_2 \cdot A(y_2) \, j_\mu^{53}(x) | 0 \rangle$$

G is the Green's fct. of QCD, i.e.

$$(g_{\mu\nu} \Box - \partial_\mu \partial_\nu) \, G(x) = \delta(x)$$

$$\Rightarrow \int d^4y \; G^{-1}(x-y) \, A_\mu(y) = j_\mu(x)$$

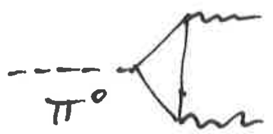
$$\Rightarrow \int d^4x \; e^{-iqx} \langle \gamma(k_1)\gamma(k_2)| T S j_\mu^{53}(x)|0\rangle$$

$$= \int d^4x_1\, d^4x_2\, d^4x \; e^{ik_1x_1} e^{ik_2x_2} e^{-iqx}$$

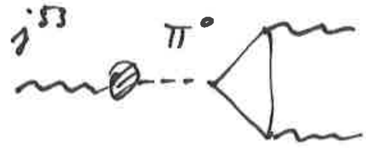
$$\times\; \varepsilon_1^\nu \varepsilon_2^\delta \; \langle 0| T j_\nu(x_1) j_\delta(x_2) j_\mu^{53}(x)|0\rangle$$

$$\equiv \varepsilon_1^\nu \varepsilon_2^\delta \; \Gamma^5_{\mu\nu\delta}(q, k_1, k_2).$$

On the other hand, from the last lecture we have



$\simeq$



pion pole approx.

$$\rightsquigarrow \int d^4x \; e^{-ipx} \langle \gamma(k_1)\gamma(k_2)| T S j_\mu^{53}(x)|0\rangle$$

$$= F_\pi \frac{p_\mu}{p^2} \langle \gamma(k_1)\gamma(k_2)| S |\pi^0\rangle + \underbrace{\mathcal{O}(k_1, k_2)}_{\text{no pole in } p^2}$$

$$\Rightarrow \langle \gamma(k_1)\gamma(k_2)|S|\pi^0\rangle$$

$$\simeq \frac{1}{F_\pi} p^\mu \; \Gamma^5_{\mu\nu\delta}(p, k_1, k_2)\; \varepsilon_1^\nu \varepsilon_2^\delta$$

$$= \frac{1}{F_\pi} \frac{i}{2\pi^2} \varepsilon_{\nu\delta\beta\alpha} k_1^\beta k_2^\alpha \varepsilon_1^\nu \varepsilon_2^\delta \; \mathrm{Tr}\left[\tfrac{1}{2}\sigma^3 Q^2\right]$$

$$= \frac{1}{F_\pi} \frac{i}{2\pi^2} \varepsilon_{\nu\delta\beta\alpha} k_1^\beta k_2^\alpha \varepsilon_1^\nu \varepsilon_2^\delta \; \tfrac{1}{2} e^2$$

from Lorentz and parity invariance:

$$\langle \gamma(k_1)\gamma(k_2)|S|\pi^0\rangle = i\, f(p^2)\, \varepsilon_{\nu\delta\beta\alpha} k_1^\beta k_2^\alpha \varepsilon_1^\nu \varepsilon_2^\delta$$

$$\Rightarrow f(0) = \frac{1}{F_\pi}\frac{e^2}{4\pi^2} = \frac{\alpha}{\pi F_\pi}.$$

The decay rate is given by

$$\Gamma(\pi^0 \to \gamma\gamma) = \underbrace{\frac{1}{2}}_{\substack{\uparrow \\ \text{symmetry} \\ \text{factor}}} \frac{1}{2m_\pi} \frac{1}{8\pi} \sum_{\text{pol.}} \left| \langle \gamma(k_1)\gamma(k_2) | S | \pi^0 \rangle \right|^2$$

$$\sum_{\text{pol.}} \left| \langle \gamma(k_1)\gamma(k_2) | S | \pi^0 \rangle \right|^2 = \sum_{\varepsilon_1, \varepsilon_2} |f(0)|^2 \left| \varepsilon_{\nu\delta\beta\kappa} k_1^\beta k_2^\kappa \varepsilon_1^\nu \varepsilon_2^\delta \right|^2$$

$$\sum_{\varepsilon} \varepsilon^\mu(k) \varepsilon^\nu(k) = -g^{\mu\nu} \quad \text{for massless vector bosons}$$

$$\rightsquigarrow \varepsilon_{\nu\delta\beta_1\alpha_1} \varepsilon^{\nu\delta\beta_2\alpha_2} k_1^{\beta_1} k_2^{\alpha_1} k_{1\beta_2} k_{2\alpha_2}$$

$$= 2\left(\delta_{\beta_1}^{\alpha_2} \delta_{\alpha_1}^{\beta_2} - \delta_{\alpha_1}^{\alpha_2} \delta_{\beta_1}^{\beta_2}\right) k_1^{\beta_1} k_2^{\alpha_1} k_{1\beta_2} k_{2\alpha_2}$$

$$= 2(k_1 \cdot k_2)^2 = \frac{1}{2}(p^2)^2 = \frac{m_\pi^4}{2}$$

$$\Rightarrow \Gamma(\pi^0 \to \gamma\gamma) = \frac{\alpha^2}{64\pi^3} \frac{m_\pi^3}{F_\pi^2} \approx 7.63\,\text{eV} \approx 1.16 \cdot 10^{16}\,\text{s}^{-1}$$

exp.: $(1.19 \pm 0.08) \cdot 10^{16}\,\text{s}^{-1}$

without axial anomaly: $f(0) = 0$ ↯

with explicit factor of N_c:

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{N_c^2}{9} \frac{\alpha^2}{64\pi^3} \frac{m_\pi^3}{F_\pi^2}$$

$\Rightarrow$ confirms $N_c = 3$.

But note

$$\mathrm{Tr}\left[\frac{1}{2}\sigma^3 Q^2\right] = \frac{1}{2}e^2\, \mathrm{Tr}\left[\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}^2\right] = \frac{1}{2}e^2$$

as well in Yukawa theory.