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CHIRAL SYMMETRY BREAKING

Let us consider the QCD lagrangian with two flavors of quarks u and d.

The masses of the light quarks are small and we neglect them to a first approximation.

By collecting the two quarks into a doublet $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$ the lagrangian can be written as

$$\mathcal{L} = \bar{\psi} \begin{pmatrix} i\not{D} & \\ & i\not{D} \end{pmatrix} \psi$$

We have seen that the QED lagrangian for a massless fermion is invariant under $U(1)_V$ and $U(1)_A$ phase rotations.

Analogously, the lagrangian of two massless quarks is invariant under the transformations

$$V: \quad \psi \to e^{i\theta_V^a \frac{\tau^a}{2}} \psi \qquad \bar{\psi} \to \bar{\psi}\, e^{-i\theta_V^a \frac{\tau^a}{2}}$$

$$A: \quad \psi \to e^{i\theta_A^a \frac{\tau^a}{2}\gamma_5} \psi \qquad \bar{\psi} \to \bar{\psi}\, e^{i\theta_A^a \frac{\tau^a}{2}\gamma_5}$$

where $\frac{\tau_a}{2}$ are the generators of $SU(2)$. The corresponding conserved currents are

$$J^a_\mu = \bar{\psi}\gamma_\mu \frac{\tau^a}{2}\psi \qquad J^{a5}_\mu = \bar{\psi}\gamma_\mu\gamma_5\frac{\tau^a}{2}\psi$$

Equivalently, we can decompose the field $\psi = \psi_L + \psi_R$ and perform independent $SU(2)$ transformations on the left-handed and right-handed fields.

$$\mathcal{L} = \bar{\psi}_L i\not{D}\psi_L + \bar{\psi}_R i\not{D}\psi_R$$

The CHIRAL SYMMETRY group is thus $SU(2)_L \times SU(2)_R$.

The question we should now address is the following: is this symmetry a good symmetry for nature? If this symmetry were exact, we should expect for every hadron in the spectrum which is made of up and down quarks another hadron with the same quantum numbers but opposite parity. THIS DOES NOT HAPPEN!

$\Rightarrow$ The (approximate) $SU(2)_L \otimes SU(2)_R$ symmetry must be broken

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We have assumed that $SU(2)_L \otimes SU(2)_R$ is a good symmetry. According to a theorem due to Coleman, the way in which a classical symmetry is implemented at quantum level is dictated by the behavior of the VACUUM:

Coleman theorem
$$\begin{cases} Q|0\rangle = 0 & \text{WIGNER} \\ Q|0\rangle \neq 0 & \text{GOLDSTONE} \end{cases}$$

- if the conserved charge Q annihilates the vacuum, we have an implementation a la Wigner: the physical states can be classified according to the irreducible representations of the symmetry group.
- if the conserved charge Q does not annihilate the vacuum, then the symmetry is still a symmetry of the Lagrangian, but it is not implemented in the physical states of the theory: it is said to be implemented a la Goldstone. One then talks about SPONTANEOUS SYMMETRY BREAKING (SSB)

The most important consequence of SSB is the GOLDSTONE THEOREM: if a theory has a global continuous symmetry of the Lagrangian which is not a symmetry of the vacuum there must be a massless (GOLDSTONE) boson for each generator of the broken symmetry.

Very good candidates of Goldstone bosons in strong interactions are the PIONS: they are not exactly massless, but they have the correct quantum numbers. The fact that they are not exactly massless is a consequence of the fact that the u and d quarks are not exactly massless.

Let us thus assume that the $SU(2)_L \otimes SU(2)_R$ is spontaneously broken to $SU(2)_V$

$\Rightarrow$ there are three Goldstone bosons π^a, one for each generator of the broken symmetry

The matrix element

$$\langle 0| J^{5a}_\mu(x) |\pi^b(p)\rangle = e^{-ipx} \langle 0| J^{5a}_\mu(0) |\pi^b(p)\rangle \qquad \text{by Lorentz invariance}$$

(3) Now since the axial charge does not annihilate the vacuum we have

$$\langle 0|\, J_\mu^{5a}(0)\,|\, \pi^b(p)\rangle \neq 0$$

What is the most general parametrization for this matrix element?

Since the pion is a scalar, we must have

$$\langle 0|\, J_\mu^{5a}(0)\,|\, \pi^b(p)\rangle = i f_\pi^{ab}\, p_\mu$$

We now want to show that since the currents J_μ^{5a} and the Goldstone bosons form an irreducible representation of the unbroken subgroup $H = SU(2)_V$ *1

$$\Rightarrow \quad f_\pi^{ab} = f_\pi\, \delta^{ab}$$

We consider the matrix element

$$\langle 0|\, [Q_a^V, J_\mu^{5b}]\,|\,\pi_c\rangle = \langle 0|\, J_\mu^{5d}\,|\,\pi_c\rangle\, i f_{abd}$$

since J_μ^5 lives in the adjoint representation of H

but

$$\langle 0|\, [Q_a^V, J_\mu^{5b}]\,|\,\pi_c\rangle = \cancel{\langle 0|\, Q_a^V J_\mu^{5b}\,|\,\pi_c\rangle} - \langle 0|\, J_\mu^{5b} Q_a^V\,|\,\pi_c\rangle$$

since Q_a^V annihilates the vacuum

$$= -\langle 0|\, J_\mu^{5b}\,|\,\pi_d\rangle\, i f_{acd}$$

We conclude that

$$\langle 0|\, J_\mu^{5d}\,|\,\pi_c\rangle\, f_{abd} = -\langle 0|\, J_\mu^{5b}\,|\,\pi_d\rangle\, f_{acd}$$

$$f_\pi^{dc}\, f_{abd} = -f_\pi^{bd}\, f_{acd}$$ and using $(T^a)_{bc} = i f_{abc}$

we find

$$[f_\pi, T^a] = 0$$ $\Rightarrow$ FOR THE SCHUR LEMMA*2, SINCE f_π COMMUTES WITH ALL THE GENERATORS OF H, IT MUST BE PROPORTIONAL TO THE IDENTITY $\boxed{f_\pi^{ab} = f_\pi\, \delta^{ab}}$

(x)1 $[Q_L^a, Q_L^b] = i\varepsilon^{abc} Q_L^c$

$[Q_R^a, Q_R^b] = i\varepsilon^{abc} Q_R^c$

$Q_L = \frac{1}{2}(Q_V - Q_A)$

$Q_R = \frac{1}{2}(Q_V + Q_A)$

$\Rightarrow [Q_V^a, Q_V^b] = i\varepsilon_{abc} Q_V^c$

$[Q_V^a, Q_A^b] = i\varepsilon_{abc} Q_A^c$

(x)2

SCHUR'S LEMMA 1

Let R be an irreducible representation of the group G on the vector space V and A be an arbitrary operator on V. If A commutes with all the group elements in the representation R then A must be a multiple of the identity operator.

We thus have

$$\langle 0|\, J_\mu^{5a}(0)\,|\pi^b(p)\rangle = i f_\pi\, p_\mu\, \delta^{ab}$$

↳ Pion decay constant $f_\pi \approx 92$ MeV

The identification of the pions as Goldstone bosons associated to the spontaneous breaking of the chiral symmetry has important consequences.

Let us consider the matrix element of the axial isospin current J_μ^{5a} between nucleon states, which is relevant for the theory of β decay

$$\langle N_1 |\, J_\mu^{5a}\, | N_2\rangle = \bar{N}_1 \left[\gamma_\mu \gamma_5 F_1^5(q^2) + \frac{i\,\sigma_{\mu\nu} q^\nu}{2m_N} \gamma_5 F_2^5(q^2) + q_\mu \gamma_5 F_3^5(q^2) \right] \tau^a N_2$$

Conventionally we can define $g_A = F_1^5(0)$

If we neglect quark masses the axial current is conserved and we can write

$$0 = \bar{N}_1 \left[\not{q}\, \gamma_5 F_1^5(q^2) + q^2 \gamma_5 F_3^5(q^2) \right] \tau^a N_2 \qquad \sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

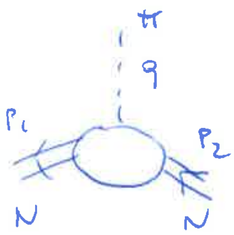
$$= \bar{N}_1 \left[(\not{p}_1 - \not{p}_2)\gamma_5 F_1^5(q^2) + q^2 \gamma_5 F_3^5(q^2) \right] \tau^a N_2 \qquad N = \begin{pmatrix} p \\ n \end{pmatrix}$$

$$= \bar{N}_1 \left[-2m_N \gamma_5 F_1^5(q^2) + q^2 \gamma_5 F_3^5(q^2) \right] \tau^a N_2$$

$$\Rightarrow \quad g_A = \lim_{q^2 \to 0} \frac{q^2 F_3^5(q^2)}{2m_N}$$

This means that g_A vanishes unless F_3^5 has a pole $\frac{1}{q^2}$ as $q^2 \to 0$

This is indeed the PION POLE associated to the diagram

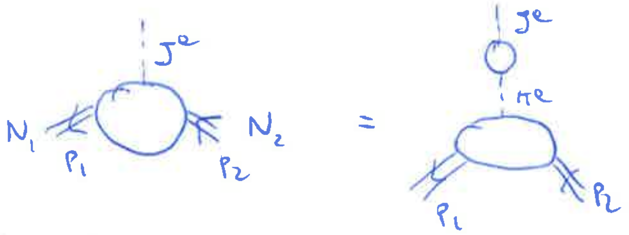


The pole term can be computed by using the Lagrangian

$$\Delta\mathcal{L} = 2 i\, g_{\pi NN}\, \pi^a\, \bar{N} \gamma^5 \tau^a N$$

which describes the pion-nucleon interaction

We thus pictorially have



$$\langle N_1 | J_\mu^{5e} | N_2 \rangle \underset{q^2\to 0}{\sim} \bar{N}_1 \left[q_\mu \gamma_5 F_3^5(q^2) \right] \tau^e N_2$$

and using the pion-nucleon vertex

contribution from the $\langle 0 | J | \pi \rangle$ matrix element

$$\langle N_1 | J_\mu^{5e} | N_2 \rangle \underset{q^2\to 0}{\sim} - g_{\pi NN} \bar{N}_1 \gamma^5 2\tau^e N_2 \frac{i}{q^2} (i q_\mu f_\pi)$$

pion propagator

By comparing these two equations we conclude that

$$2 m_N g_A = 2 g_{\pi NN} f_\pi$$

$$\boxed{g_A = \frac{g_{\pi NN} f_\pi}{m_N}}$$ GOLDBERGER - TREIMAN RELATION

This relation, which simply follows from the assumption of the spontaneous breaking of chiral symmetry in the strong interactions, is verified rather well

Taking $m_N = \frac{m_p + m_n}{2} = 938.9$ MeV, $g_A = 1.257$, $f_\pi = 92$ MeV

we get $g_{\pi NN} = 12.8$, in good agreement with the measured $g_{\pi NN} = 13.5$ (6% difference)