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THE CHIRAL ANOMALY IN THE SCHWINGER MODEL

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu}^2 + \bar{\psi} i \not{D} \psi$$

The Schwinger model is massless QED with $d=2$. In two dimensions the Dirac algebra can be realized with 2x2 matrices such that

$$\gamma^0 = \sigma_2 \qquad \gamma^1 = i\sigma_1 \qquad \gamma^5 = \sigma_3 \qquad \Rightarrow \quad \gamma^0 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} & i \\ i & \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

If we write $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ we have $\gamma_5 \psi_+ = \psi_+ \qquad \gamma_5 \psi_- = -\psi_-$

Although there are many simplifications with respect to $d=4$, in $d=2$ the photon does not have transverse degrees of freedom and the interaction reduces itself to the Coulomb interaction, which grows with distance, giving rise to confinement. For this reason the model has been studied as a prototype for confinement in QCD.

To simplify the situation, we may consider the system on a interval of lenght L. If L is small the Coulomb interaction can be assumed to be small.

By choosing periodic boundary conditions we can expand the fields in their Fourier modes. We note that gauge invariance

$$\psi \to e^{i\alpha(x,t)}\psi \qquad\qquad A_\mu \to A_\mu - \partial_\mu \alpha$$

allows us to eliminate all the modes for the field A_1 except the zero mode $(m=0)$. In fact, a term $a(t)\, e^{i\frac{2\pi}{L}mx}$ can be eliminated with $\alpha = +\left(i\frac{2\pi m}{L}\right)^{-1} a(t)\, e^{i\frac{2\pi}{L}mx}$

We thus treat A_1 as a constant independent on x (and neglect A_0 by considering it as a small perturbation). As an alternative, we can consider the fermion field in a A_1 background. Let us now consider the Hamiltonian for this model:

$$\frac{\partial \mathcal{L}}{\partial(\partial_0\psi)} = \bar{\psi} i \gamma^0 \quad \Rightarrow \quad \mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)}\partial_0\psi - \mathcal{L} = \bar{\psi} i \gamma^0 \partial_0 \psi - \bar{\psi} i \not{D} \psi$$

$$= -\bar{\psi} i \gamma^1 D_1 \psi = -\psi^\dagger i \gamma^0 \gamma^1 D_1 \psi$$

$$= -\psi^\dagger i \gamma^5 D_1 \psi$$

②

$$= -\psi_+^\dagger i(\partial_1 + iA_1)\psi_+ + \psi_-^\dagger i(\partial_1 + iA_1)\psi_-$$

In a free theory the Dirac equation tell us

$$(\gamma^0\partial_0 + \gamma^1\partial_1)\psi = 0 \qquad \gamma^0 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} & i \\ i & \end{pmatrix}$$

$$\begin{cases} i(-\partial_0 + \partial_1)\psi_- = 0 \\ i(\partial_0 + \partial_1)\psi_+ = 0 \end{cases} \qquad i\left[\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\partial_0 + \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\partial_1\right]\begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} = 0$$

ψ_- : left moving $\qquad \psi_- \sim e^{ik_m(x+t)}$ $\qquad \frac{1-\gamma_5}{2}\psi = \psi_L = \begin{pmatrix}0 \\ \psi_-\end{pmatrix}$ $\qquad \frac{1+\gamma_5}{2}\psi = \psi_R = \begin{pmatrix}\psi_+ \\ 0\end{pmatrix}$

ψ_+ : right moving $\qquad \psi_+ \sim e^{ik_m(x-t)}$

The single particle eigenstates of $H = \int d^3x\, \mathcal{H}$ have energies

$$\psi_+ \qquad E_m = k_m + A_1$$

$$k_m = \frac{2\pi}{L}\cdot m \qquad m = \pm 1, \pm 2 \ldots$$

$$\psi_- \qquad E_m = -k_m - A_1$$

When $A_1 = 0$ the energy levels of the ψ_+ and ψ_- fermions are degenerate

If we adiabatically increase A_1 to $A_1 > 0$ the degeneracy is lifted : the ψ_+ levels move up and the ψ_- levels move down. When $A_1 = \frac{2\pi}{L}$ the structure of the levels is the same as before. It is this restructuring of the energy levels that produces the anomaly

We can see this through the construction of the Dirac sea : we start with $A_1 = 0$ and fill out all the negative energy states. When A_1 moves from 0 to $\frac{2\pi}{L}$ a plus (right moving) particle is created and a minus (left-moving) hole is also created. In this process the charge is conserved but the CHIRALITY is not!

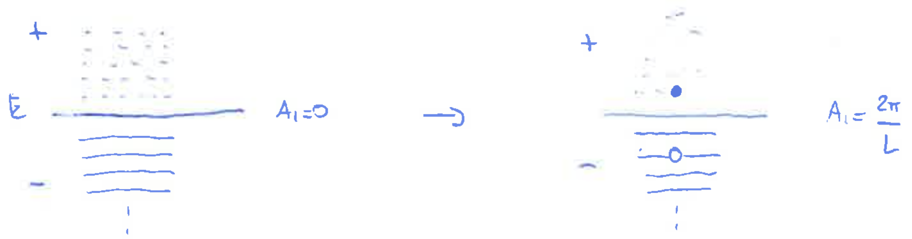
$$\Delta Q_5 = 2 \quad \Rightarrow \quad \Delta Q_5 = \frac{2\pi}{L}\frac{L}{\pi} = \frac{L}{\pi}\Delta A_1$$

Dividing by Δt we get $\qquad \dot{Q}_5 = \frac{L}{\pi}\dot{A}_1$

③ The conserved quantity is thus $\tilde{J}^{5\,0} = J^{5\,0} - \frac{1}{\pi} A_1$

$$\Rightarrow \quad \tilde{J}^{5\mu} = J^{5\mu} - \frac{1}{\pi}\varepsilon^{\mu\nu} A_\nu \qquad\qquad \varepsilon^{01} = -\varepsilon^{10} = 1$$

$$\boxed{\partial_\mu J^{5\mu} = \frac{1}{\pi}\varepsilon^{\mu\nu}\partial_\mu A_\nu = \frac{1}{2\pi}\varepsilon^{\mu\nu} F_{\mu\nu}}$$



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ANOMALY CANCELLATION

We have seen that the chiral anomaly receives contribution from two diagrams

In the case in which the symmetry group is non trivial the anomaly will be proportional to $Tr\left[T^a\{T^b,T^c\}\right]$ where the generators T are in the representation of the symmetry group to which the fermion belong.

Let us consider the case in which the currents are all gauge currents

$\Rightarrow$ then the anomaly must cancel out to preserve the consistency of the theory

We distinguish two cases : the first case is the one of SAFE GROUPS (e.g. $SO(2n+1)$, $SU(2)$, E_7, E_8 ....) for which $d^{abc} \equiv Tr\left[T^a\{T^b,T^c\}\right]=0$ for ANY representation. The second case is the one in which the group is NOT SAFE but the fermions live in a representation for which $d^{abc}=0$

Let us recall a few aspects of group representations. Given the Lie algebra

$$[T^a,T^b]=if^{abc}T^c$$

a representation of the group is specified by a set of $D(R)\times D(R)$ hermitian matrices such that

$$[T^a_R,T^b_R]=if^{abc}T^c_R$$

For $SU(N)$, the original set of $N\times N$ matrices correspond to the FUNDAMENTAL REP. By taking the complex conjugate of the above equation we see that if T_R provide a representation, also $-T_R^*$ give a representation

$$[T^{a*}_R,T^{*b}_R]=-if^{abc}T^{*c}_R$$

$$\Rightarrow \quad [-T^{*a}_R,-T^{*b}_R]=if^{abc}(-T^*_R)^c$$

② The $-T_R^*$ define the <u>COMPLEX CONJUGATE</u> representations.

We now show that, if $-T_R^*$ and T_R are EQUIVALENT, i.e. if there is a matrix V such that $(-T_R^a)^* = V^{-1} T_R^a V \Rightarrow d_{abc} = 0$

$$d_{abc} = \mathrm{Tr}\left[T_R^a \{T_R^b, T_R^c\}\right] = \mathrm{Tr}\left[T_R^a T_R^b T_R^c\right] + \mathrm{Tr}\left[T_R^a T_R^c T_R^b\right]$$

$$= \mathrm{Tr}\left[V^{-1}T_R^a V V^{-1} T_R^b V V^{-1} T_R^c V\right] + \mathrm{Tr}\left[V^{-1}T_R^a V V^{-1} T_R^c V V^{-1} T_R^b V\right]$$

$$= \mathrm{Tr}\left[(-T_R^a)^*(-T_R^b)^*(-T_R^c)^*\right] + \mathrm{Tr}\left[(-T_R^a)^*(-T_R^c)^*(-T_R^b)^*\right]$$

$$= -\mathrm{Tr}\left[T^a_{R\,ji}\, T^b_{R\,kj}\, T^c_{R\,ek}\right] - \mathrm{Tr}\left[T^a_{R\,ji}\, T^c_{R\,kj}\, T^b_{R\,ek}\right]$$

because $T^+ = (T^T)^* = T$

$$= -\mathrm{Tr}\left[T^c T^b T^a\right] - \mathrm{Tr}\left[T^b T^c T^a\right] = -d_{abc}$$

$\Rightarrow \quad d_{abc} = 0$

If there exists a matrix V such that $(-T_R^a)^* = V^{-1} T_R^a V$ we may distinguish two cases: the representation R is

- <u>REAL</u> if there is a unitary transformation U such that $T_R \to U^+ T_R U$ makes $-T_R^* = T_R$ (or if a basis exists such that $-T_R^* = T_R$)

- <u>PSEUDOREAL</u> if it is not real but still $(-T_R)^* = V^{-1} T_R^a V$

The fundamental representation of $SO(N)$ is REAL. The fundamental representation of $SU(2)$ is PSEUDOREAL:

$$\left(-\tfrac{1}{2}\sigma^a\right)^* = V^{-1} \frac{\sigma^a}{2} V \qquad \text{with } V = \sigma_2 \qquad \left(\text{remember that } \sigma_2^+ = \sigma_2 = \sigma_2^{-1}\right)$$

The three dimensional representation of $SU(2)$ is REAL (the $SU(2)$ representations are either real or pseudoreal).

Note that if the representation is REAL we can find a basis such that

$$-T_R^* = T_R$$

(3) However, since $T_R^\dagger = T_R$ we conclude that T_R must be IMAGINARY and ANTISYMMETRIC

If the representation R is neither real nor pseudoreal, then it is COMPLEX
(e.g. the fundamental representation of $SU(N)$, $N>2$)

The ADJOINT representation A is defined by the structure constants of the group

$$(T_A^a)_{bc} = i f_{abc}$$

and it is REAL $\left(-(T_A^a)^* = T_A^a \right)$

The above discussion provides us with a GUIDING PRINCIPLE to construct a consistent gauge theory: either we use SAFE GROUPS, we have to make sure that the fermions are such that $d_{abc} = 0$.

If a gauge theory is such that the left and right handed fermions transform in the same way under the gauge group, the theory is said to be a VECTOR theory.
If the left and right handed fermions transform differently under the gauge group the theory is said to be CHIRAL. QCD with $G = SU(3)_c$ is a VECTOR theory
The EW standard theory is CHIRAL, because $G = SU(2)_L \otimes U(1)_Y$.

Let us consider a theory with massless Dirac fermions and write the Lagrangian in the helicity basis

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} I & \\ & -I \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} & I \\ I & \end{pmatrix} \qquad \gamma^i = \gamma^0 \alpha^i$$

$$\alpha^i = \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix}$$

$$\sigma^\mu = (1, \sigma^i) \qquad \bar{\sigma}^\mu = (1, -\sigma^i)$$

The ψ_L and ψ_R spinors are 2-component Weyl spinors that transform under the $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ representations of the Lorentz group.

(4) In this representation the kinetic term for the fermions can be written as

$$\mathcal{L} = \psi_L^+ i \bar{\sigma} \cdot \partial \psi_L + \psi_R^+ i \sigma \cdot \partial \psi_R$$

In a chiral theory the ψ_L and ψ_R transform under different representations.

We now observe that the right-handed components can be rewritten as new left-handed fields. In fact $\psi_L' = \sigma^2 \psi_R^*$

transform like left-handed fields. By using the identity $\sigma^2 \underline{\sigma}^* = -\underline{\sigma} \sigma^2$

we can write

$$\int d^4x \; \psi_R^+ i \sigma \cdot \partial \psi_R = \int d^4x \; \psi_L'^+ i \bar{\sigma} \cdot \partial \psi_L'$$ integrating by parts

If the fermions are coupled to the gauge fields this manipulation leads to

$$\psi_R^+ i \sigma \cdot D \psi_R = \psi_R^+ i \sigma (\partial - i g A^a t_R^a) \psi_R$$

$$\rightarrow \psi_L^{+\prime} i \bar{\sigma} \cdot (\partial + i g A^a (t_R^a)^T) \psi_L'$$

$$= \psi_L'^+ i \bar{\sigma} \cdot (\partial - i g A^a t_{\bar{R}}^a) \psi_L'$$

$\Rightarrow$ if the ψ_R belong to the representation R, the new left-handed fields belong to the complex-conjugate representation $\bar{R}$

In this way QCD with M_F massless flavors can be rewritten as a $SU(3)$ gauge theory with M_F massless flavors in the 3 and M_F in the $\bar{3}$

More generally, in a vector theory we can consider the fermions in a $R + \bar{R}$ representation, which is real

$\Rightarrow$ VECTOR THEORIES ARE ANOMALY FREE

For CHIRAL theories, we have to check explicitly that $d^{abc} = 0$

(5)

ANOMALY CANCELLATION IN THE SM

In the $SU(2)_L \otimes U(1)_Y$ theory the fermions are either $SU(2)_L$ doublet or singlets. It is true that $SU(2)$ is anomaly free but $U(1)$ is not and we have to check all the possibilities.

When evaluating the d_{abc}, the generators will be either the Pauli matrices τ^a or the $U(1)$ hypercharge. The quantum numbers are assigned by using the relation

$$Q = T_3 + \frac{Y}{2} \qquad T_3 = \frac{\tau_3}{2}$$

For example, the lepton doublet is $\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ and it must have hypercharge -1 in order to reproduce the charges of the electron and of the neutrino.

Let us discuss the various possible cases in turn:

SU(2) - SU(2) - SU(2) $\qquad Tr\left(\tau^a \{\tau^b, \tau^c\}\right) = 0 \qquad$ because SU(2) is SAFE!

SU(2) - U(1) - U(1) $\qquad Tr\left(\tau, YY\right) \sim Tr(\tau) = 0$

this is because each member of a SU(2) doublet has the same hypercharge

SU(2) - SU(2) - U(1)

$$Tr\left(\tau^a \tau^b Y\right) \sim 2\delta^{ab}\, Tr(Y)$$

The anomaly here is proportional to $Tr(Y)$

(5) $\underline{U(1)-U(1)-U(1)}$

$$Tr(YYY) = 8\,Tr\left((Q-T_3)^3\right) = 8\,Tr\left(Q^3 - T_3^3 - 3Q^2T_3 + 3QT_3^2\right)$$

vanishes because the VVV fermion loop is anomaly free

vanishes because $T_3^3 \propto T_3$ and $Tr[T_3]=0$

$$\Rightarrow Tr(YYY) \sim Tr\left(Q^2T_3 - QT_3^2\right)$$

$$Tr(T_3Q^2) = Tr\left(T_3\left(T_3+\frac{Y}{2}\right)^2\right)$$

$$= Tr\left(\cancel{T_3^3}\right) + \frac{1}{4}Tr\left(\cancel{T_3Y^2}\right) + Tr\left(T_3^2Y\right) \sim Tr(Y)$$

↳ because $Tr(T_3)=0$

$$Tr(QT_3^2) \sim Tr\,Q$$

But since $Q = T_3 + \frac{Y}{2}$ and $Tr(T_3)=0$ $\Rightarrow$ $Tr(Y) \sim Tr\,Q$

$\Rightarrow$ We conclude that the anomaly cancels if $Tr\,Q=0$

In each family we have 3 quark doublets and 1 lepton doublet

$$\begin{array}{c} u \\ d \end{array}\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \times 3$$

↳ number of colors

$$\begin{array}{c} \nu \\ e^- \end{array}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$Tr\,Q=0$

$\Rightarrow$ The anomaly cancels out because the total charge of the fermions in each family vanishes

Note that to achieve the cancellation it is necessary to combine the contributions of quarks and leptons

(an $SU(2)_L \otimes U(1)_Y$ model describing only quarks or only leptons would not be anomaly free)