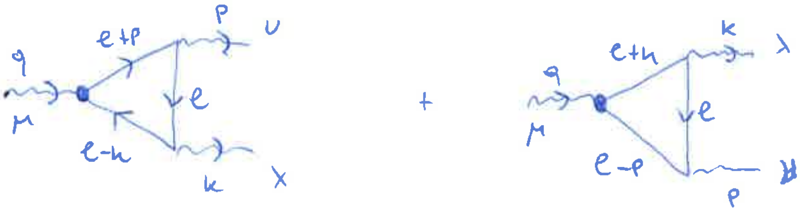
(1)

AXIAL ANOMALY :

CALCULATION OF TRIANGLE DIAGRAMS



The first triangular diagram gives

$$e^2\int\frac{d^4e}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma^\mu\gamma^5\,\frac{i(\not e-\not k)}{(e-k)^2}\,\gamma^\lambda\,\frac{i\not e}{e^2}\,\gamma^\nu\,\frac{i(\not e+\not p)}{(e+p)^2}\right]$$

while the second gives a similar contribution with $(p,\nu)\leftrightarrow(k,\lambda)$

If the axial current is conserved, we should get a vanishing contribution when contracting with q^μ. Let us multiply by iq_μ and use

$$q_\mu\gamma^\mu\gamma^5=(\not e+\not p-\not e+\not k)\gamma^5=(\not e+\not p)\gamma^5+\gamma^5(\not e-\not k)$$

In the above expression each momentum factor can be used to cancel a contribution of the denominator and we get

$$ie^2q_\mu\int\frac{d^4e}{(2\pi)^4}\,\mathrm{Tr}\left[\cdots\right]=i^2e^2\int\frac{d^4e}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma^5\,\frac{i(\not e-\not k)}{(e-k)^2}\,\gamma^\lambda\,\frac{i\not e}{e^2}\,\gamma^\nu\right.$$

$$\left.+\gamma^5\gamma^\lambda\,\frac{i\not e}{e^2}\,\gamma^\nu\,\frac{i(\not e+\not p)}{(e+p)^2}\right]$$

$$=i^2e^2\int\frac{d^4e}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma^5\,\frac{i(\not e-\not k)}{(e-k)^2}\,\gamma^\lambda\,\frac{i\not e}{e^2}\,\gamma^\nu-\gamma^\lambda\gamma_5\,\frac{i\not e}{e^2}\,\gamma^\nu\,\frac{i(\not e+\not p)}{(e+p)^2}\right]$$

if we now shift in the first term $e\to e+k$ we get

$$i^2e^2\int\frac{d^4e}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma^5\,i\,\frac{\not e}{e^2}\,\gamma^\lambda\,\frac{i(\not e+\not k)}{(e+k)^2}\,\gamma^\nu-\gamma_5\,\frac{i\not e}{e^2}\,\gamma^\nu\,\frac{i(\not e+\not p)}{(e+p)^2}\,\gamma^\lambda\right]$$

② We now see that the integral is ANTISYMMETRIC in the exchange $(p,\nu) \leftrightarrow (k,\lambda)$ so it will give a vanishing contribution when combined with the second diagram.

However this derivation required a shift of the integration. Is it legitimate?

The integral is divergent, so we have to be careful!

EXAMPLE

Consider the integral $\int_{-\infty}^{+\infty} dx\,(x+a) = \frac{x^2}{2} + a\,x \Big|_{-\infty}^{+\infty} = a\cdot\infty$

If we shift $x \to x-a$ we get $\int_{-\infty}^{+\infty} dx'\,x' = 0$?!

The computation of the anomaly can be carried out with the Pauli-Villars regularization and in this case the shift leaves a finite remainder. Here we will use dimensional regularization. We know that in d dimensions we can still define

$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ but that $\{\gamma_5, \gamma^\mu\} = 0$ holds only for $\mu = 0,1,2,3$ while γ_5 commutes with γ^μ for other values of μ

In the calculation that we have to perform all the momenta p, k, q live in 4 dimensions whereas ℓ can be written as $\ell = \ell_\parallel + \ell_\perp$

$\ell_\perp$: component in $d-4$ dimension

The identity that we have used before can thus be rewritten as

$$q_\mu \gamma^\mu \gamma^5 = (\not\ell + \not p - \not\ell + \not k)\gamma^5 = (\not\ell_\parallel + \not p - \not\ell_\parallel + \not k)\gamma^5 = (\not\ell_\parallel + \not p)\gamma^5 + \gamma^5(\not\ell_\parallel - \not k)$$

$$= (\not\ell + \not p)\gamma^5 + \gamma^5(\not\ell - \not k) - 2\not\ell_\perp \gamma^5$$

ADDITIONAL TERM

In dimensional regularization, we can use this modified identity and apply the momentum shift to cancel the two triangular diagrams, but we are left with

$$e^2 \int \frac{d^D\ell}{(2\pi)^D} \mathrm{Tr}\left[-2\gamma^5 \not\ell_\perp \frac{\not\ell - \not k}{(\ell-k)^2} \gamma^\lambda \frac{\not\ell}{\ell^2} \gamma^\nu \frac{\not\ell + \not p}{(\ell+p)^2} \right] + (p,\nu) \leftrightarrow (k,\lambda)$$

③

To evaluate this contribution we should use the Feynman parametrization

$$\frac{1}{A_1A_2A_3} = \int_0^1 dx_1 dx_2 dx_3 \; \delta(x_1+x_2+x_3-1) \frac{2}{(x_1A_1+x_2A_2+x_3A_3)^3}$$

to reduce the product of denominators to a single denominator, and then shift the loop momentum $\ell \to \ell' = \ell + P$ so as to have the denominator in the form $\ell'^2 + \Delta$

Since the momentum P is a function of the external momenta, and thus lives in four dimensions, we have $\ell'_\perp = \ell_\perp$. In the integral we can get a non vanishing contribution from terms with 2 and 4 ℓ. However, since

$$\int \frac{d^D\ell'}{(2\pi)^D} \frac{\ell'_\mu \ell'_\nu \ell'_\rho \ell'_\sigma}{(\ell'^2+\Delta)^m} \sim \delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}$$

these terms lead to a contribution containing $Tr[\gamma_5 \gamma^\lambda \gamma^\sigma]$ which vanishes.

$\Rightarrow$ the only non vanishing contribution is obtained from terms <u>quadratic</u> in the loop momentum

We are thus left to evaluate the integral

$$\int \frac{d^D\ell'}{(2\pi)^D} \frac{\not\ell'_\perp \not\ell'_\perp}{(\ell'^2-\Delta)^3}$$ but $\not\ell_\perp \not\ell_\perp = \ell_\perp^2 = \ell_\perp^\mu \ell_\perp^\nu g^T_{\mu\nu}$ (we use ℓ instead of ℓ' from now)

$\Rightarrow$ the integral of $\ell_\perp^\mu \ell_\perp^\nu$ will give a $g_T^{\mu\nu}$ $\quad g^\mu_{T\mu} = d-4$

If instead of $\not\ell_\perp \not\ell_\perp$ we had ℓ^2 we would have obtained $g^\mu{}_\mu = d$

$\Rightarrow$ $\boxed{\text{We can replace } \not\ell_\perp \not\ell_\perp \text{ with } \ell^2 \frac{d-4}{d}}$

By using the well known formula

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta)^3} = \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(3)} (\Delta)^{d/2-2}$$

(4) We can write

$$\int \frac{d^d \ell'}{(2\pi)^d} \frac{\ell'_\perp \ell'_\perp}{(\ell'^2-\Delta)^3} = \frac{i}{(4\pi)^{d/2}} \frac{\cancel{d}}{2} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(3)} (\Delta)^{\frac{d}{2}-2} \cdot \frac{d-4}{\cancel{d}} \xrightarrow[d\to 4]{} \boxed{\frac{-i}{2(4\pi)^2}}$$

We see that the divergence of the integral at $d\to 4$, which is contained in the factor

$\Gamma(2-\frac{d}{2}) \sim -\frac{2}{d-4}$ is cancelled by the factor $d-4$

For the first triangular diagram we thus get* $\left(Tr[\gamma^5 \gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\rho] = -4i\, \varepsilon^{\mu\lambda\nu\rho} \right)$

$$e^2 \left(\frac{-i}{2(4\pi)^2}\right) Tr\left[-2\gamma^5 (-\not{k}) \gamma^\lambda \gamma^\nu \not{p}\right] = \frac{e^2}{(2\pi)^2} \varepsilon^{\mu\lambda\rho\nu} k_\mu p_\rho$$

This expression is symmetric under the exchange $(p,\nu) \leftrightarrow (k,\lambda)$

$\Rightarrow$ The second diagram gives just an additional factor 2

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* Note that the integration on the Feynman parameters becomes trivial because there is no dependence on the x_i

$$2\int_0^1 dx_1 dx_2 dx_3\, \delta(1-x_1-x_2-x_3) = 1$$