INSTANTONS AND THE U(1) PROBLEM

In the seventies strong interactions had a puzzling problem, which became particularly clear with the advent of QCD. The QCD lagrangian with N_F flavors has a large global symmetry (chiral symmetry) in the limit of vanishing quark masses: $SU(N_F)_L \otimes SU(N_F)_R$. Since $m_u, m_d \ll \Lambda_{QCD}$ we know that at least for these quarks the approximation should be good, thus we would expect it to be realized in the particle spectrum. We have seen that this is not the case and that there is evidence that the chiral symmetry is spontaneously broken. The pions are indeed the (pseudo) Goldstone bosons of this spontaneous symmetry breaking. Besides chiral symmetry, the QCD lagrangian fulfill another symmetry: the $U(1)_V$ and $U(1)_A$. The $U(1)_V$ is indeed realized and corresponds to the conservation of the baryon number. What about $U(1)_A$? This symmetry is not realized in the particle spectrum, so we would expect it to be spontaneously broken. However, besides the pions, there are no signs of other light states in the hadronic spectrum. More precisely, Weinberg has shown that if $U(1)_A$ is spontaneously broken, the corresponding Goldstone boson should have mass $m_0 \leq \sqrt{3}\, m_\pi$. Moreover, the η is considerably heavier than the pion, and is well understood as a Goldstone boson in the SSB of $SU(3)_L \otimes SU(3)_R \to SU(3)_V$.
This problem goes under the name of U(1)$_A$ problem. A possible solution seems to be provided by the anomaly. The axial current

$$J_5^\mu = \sum_{f=1}^{N_F} \bar{q}_f \gamma^\mu \gamma_5 q_f$$

is not conserved since

$$\partial_\mu J_5^\mu = \frac{N_F g^2}{16\pi^2} F\tilde{F}$$

However $F\tilde{F} = \partial_\mu K^\mu$ where K^μ is the Chern-Simons current

$$K^\mu = \varepsilon^{\mu\nu\rho\sigma}\left(A_\nu^a F_{\rho\sigma}^a + \frac{8}{3} f_{abc} A_\nu^a A_\rho^b A_\sigma^c\right)$$

and thus

$$\int d^4x\, \partial_\mu J_5^\mu = \frac{N_f g^2}{16\pi^2}\int d^4x\, F\tilde{F} = \frac{N_f g^2}{16\pi^2}\int d^4x\, \partial_\mu K^\mu$$

$$= \frac{N_f g^2}{16\pi^2}\int_S d\sigma_\mu K^\mu$$

Hence, using naive boundary conditions that $A_\mu^a = 0$ at spatial infinity, the integral on the surface vanishes, and $U(1)_A$ appears to be a symmetry again.

Equivalently, we may define a new current $J_\mu^{5\,\prime} = J_\mu^5 - K_\mu \frac{N_f g^2}{16\pi^2}$ such that $\partial_\mu J^{5\mu\prime} = 0$ but this current would be gauge dependent.

What 't Hooft has shown, is that INSTANTONS provide a solution in which A_μ is indeed a pure gauge at infinity. With these boundary conditions there are gauge configurations for which $\int d\sigma_\mu K^\mu \neq 0$ and thus $U(1)_A$ is not a true symmetry of the theory. So we can say that <u>INSTANTONS SOLVE THE QCD $U(1)_A$ PROBLEM</u>

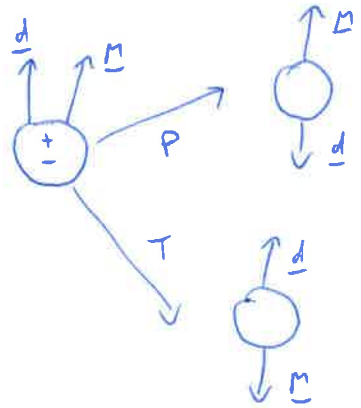
STRONG CP PROBLEM

We have seen that the analysis of the vacuum structure in the presence of instantons forces us to add a new term in the QCD lagrangian

$$\mathcal{L}_{QCD} \to \mathcal{L}_{QCD} + \frac{\theta g^2}{32\pi^2} \mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$$

θ new parameter of the theory

This new term violates P and T and thus CP and would induce a non-zero electric dipole moment for the neutron. The neutron electric dipole moment (nEDM)



is a measure of the distribution of positive and negative electric charge in the neutron. A finite EDM can exist only if the centers of positive and negative electric charges is not the same.

Current measurements indicate that, if it exist, the nEDM must be extremely small, $d_n < 2.9 \cdot 10^{-26}\, e\, \mathrm{cm}$.

From dimensional arguments one can show that $d_n \sim \theta e \frac{m_\pi^2}{m_N^3}$ $\Rightarrow$ this implies that $\theta < 10^{-9}$. However we would expect the θ parameter to be $O(1)$, so why is it so small? This is the <u>strong CP problem</u>

We note that a chiral phase transformation in the fermion fields $\psi \to e^{i\alpha\gamma_5}\psi$ changes the fermion integration measure as

$$[d\psi][d\bar{\psi}] \to \exp\left\{ -i \frac{g^2}{16\pi^2} \alpha \int d^4x\, F\tilde{F} \right\} [d\psi][d\bar{\psi}]$$

The θ term in the lagrangian can thus be eliminated but the mass terms for the quarks are not invariant!

Let us write the mass term for the fermions as

$$\mathcal{L}_m = -\sum_f m_f \bar{\psi}_f \psi_f \to -\frac{1}{2}\sum_f m_f \bar{\psi}_f (1+\gamma_5)\psi_f - \frac{1}{2}\sum_f m_f^* \bar{\psi}_f (1-\gamma_5)\psi_f$$

where we have assumed the possibility that m_q is complex.

The transformation $\psi_q \to e^{i\alpha_q \gamma_5}\psi_q$ implies $m_q \to e^{i2\alpha_q} m_q$

(write $\gamma_5 = \frac{1}{2}(1+\gamma_5) - \frac{1}{2}(1-\gamma_5)$:

in the first term in the previous equation we have $\gamma_5 \to 1$, in the second $\gamma_5 \to -1$

However, if one of the quarks happen to be exactly massless, we could limit ourselves to rotate only that field and thus the ϑ term could be eliminated!

But this does not seem to be a possibility, so we can still eliminate the ϑ term, but at the price of introducing complex phases in the quark mass matrix (CP violation).

THE AXION (PECCEI, QUINN, 1977; WEINBERG, 1978)

Suppose that we have some $U(1)$ symmetry that is spontaneously broken at energies much larger than those associated to QCD, and that a is the Goldstone boson associated to this SSB.

$$\mathcal{L}_{AXION} = \frac{1}{2}(\partial_\mu a)^2 + \frac{\left(\vartheta + \frac{a}{M}\right)}{32\pi^2} F\tilde{F}$$

axion

We assume that we have chosen the phases in the fermion fields to make the mass matrix real. For a constant a all observables will be a function of $\vartheta + a/M$.

If everything in the theory except the ϑ term and the $aF\tilde{F}$ interaction conserves CP, then the effective potential will be even in $\vartheta + \frac{a}{M}$ $\Rightarrow$ it will have a stationary point at $\vartheta + \frac{a}{M} = 0$ $\Rightarrow$ WITH A SIMPLE SHIFT IN a WE CAN ELIMINATE ϑ!

Without making further hypothesis, we can write the general form of the effective lagrangian for the axion up to $O\left(\frac{1}{M}\right)$

$$\mathcal{L}_{AXION} = \frac{1}{2}(\partial_\mu a)^2 + \frac{1}{32\pi^2}\left(\vartheta + \frac{a}{M}\right) F\tilde{F} - \frac{i f_u}{M}(\partial_\mu a)\bar{u}\gamma_5\gamma^\mu u - \frac{i f_d}{M}(\partial_\mu a)\bar{d}\gamma_5\gamma^\mu d$$

By performing a rotation of the quark fields $u \to e^{i\alpha_u \gamma_5} u$, $d \to e^{i\alpha_d \gamma_5} d$

we have

$$\mathcal{L}_{AXION} \to \mathcal{L}_{AXION} - \frac{\alpha_u + \alpha_d}{16\pi^2} F\tilde{F} \quad \Rightarrow \quad \frac{a}{M} + \vartheta \to \frac{a}{M} + \vartheta - 2(\alpha_u + \alpha_d)$$

If we choose $\alpha_q = \left(\frac{a}{M} + \vartheta\right)\frac{C_q}{2}$ such that $C_u + C_d = 1$ the term $F\tilde{F}$ is eliminated

The mass term in the QCD lagrangian becomes

$$\mathcal{L}_m = -m_u \bar{u}\, e^{iC_u\left(\vartheta + \frac{a}{M}\right)\gamma_5} u \; - m_d \bar{d}\, e^{iC_d\left(\vartheta + \frac{a}{M}\right)\gamma_5} d$$

and, in addition, we also pick up derivative terms from the kinetic term of the quark fields

$$\bar{u}\, i\gamma^\mu i \frac{C_u}{2} \frac{\partial_\mu a}{M} \gamma_5 u \; + \; u \to d .$$

These terms can be combined with the similar terms in f_u, f_d we have already by defining new parameters f_u', f_d'.

We can now study the mass matrix for the low energy pion-axion lagrangian.

In the lecture on chiral perturbation theory we have seen that a mass term in the chiral lagrangian is obtained by

$$\frac{v^2}{2} B \,\mathrm{Tr}\left(m^+ U + m U^+\right) \qquad \text{with } U = e^{\frac{i \underline{\pi}\cdot\underline{\tau}}{v}} \qquad \left(\begin{matrix}\text{we limit}\\ \text{to } SU(2)\end{matrix}\right.$$

and where

$$v^2 B = -\langle \bar{\psi}\psi\rangle \quad , \text{the quark condensate}$$

The ensuing pion mass is $\quad m_\pi^2 = (m_u + m_d)\dfrac{\langle\bar{\psi}\psi\rangle}{v^2}$

In a similar way, we can write the mass term in the low energy pion-axion lagrangian as

$$\frac{v^2}{2} B \,\mathrm{Tr}\left(m^+ e^{i\underline{\pi}\cdot\underline{\tau}/v} e^{iCa/M} + m\, e^{-i\underline{\pi}\cdot\underline{\tau}/v} e^{-iCa/M}\right)$$

$$C = \begin{pmatrix} C_u & \\ & C_d \end{pmatrix} \qquad \left(\text{we have replaced } \frac{a}{M} + \vartheta \text{ with } \frac{a}{M}\right)$$

If we limit ourselves to the neutral sector we have

$$\frac{v^2}{2}B\left(m_u e^{i\pi_0/v} e^{iC_u\frac{a}{M}} + m_d e^{-i\pi_0/v} e^{iC_d\frac{a}{M}} + m_u e^{-i\pi_0/v} e^{-iC_u\frac{a}{M}} + m_d e^{i\pi_0/v} e^{-iC_d a/M}\right)$$

$$= \frac{v^2}{2}B\left(2m_u\left(1-\frac{\pi_0^2}{2v^2} - C_u^2\frac{a^2}{2M^2} - \frac{C_u a\pi_0}{vM}\right) + 2m_d\left(1-\frac{\pi_0^2}{2v^2} - C_d^2\frac{a^2}{2M^2} + \frac{C_d a\pi_0}{vM}\right)\right)$$

The pion-axion mass matrix is

$$\frac{1}{2}\times 2\ \underbrace{v^2B}_{\langle\bar\psi\psi\rangle}\begin{pmatrix} \frac{1}{2v^2}(m_u+m_d) & \frac{-C_u m_u + C_d m_d}{2vM} \\ \frac{-C_u m_u + C_d m_d}{2vM} & \frac{C_u^2 m_u + C_d^2 m_d}{2M^2}\end{pmatrix}$$

overall factor for scalars $\frac{1}{2}m^2\phi^2$

In the large M limit the eigenvalues are $m_\pi^2 \simeq v^2B\,\frac{m_u+m_d}{v^2}$

$m_a^2 \simeq \frac{v^2B}{M^2}\,\frac{m_u m_d}{m_u+m_d}$

Solve eigenvalue equation for the matrix

$$\begin{pmatrix} \alpha & \beta/M \\ \beta/M & \frac{\gamma}{M^2}\end{pmatrix} \Rightarrow \begin{aligned}\lambda_1 &\approx \alpha \\ \lambda_2 &\approx \frac{1}{M^2}\left(\gamma - \frac{\beta^2}{\alpha}\right)\end{aligned}$$

in the large M limit

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using the values for m_u, m_d we get $m_a \approx \frac{13\,\text{MeV}}{M(\text{GeV})}$

We can also use this framework to say something about the interactions of axions with hadrons. The eigenvector of the mass matrix corresponding to the axion has a component along the original π^0 $\Rightarrow$ the ratio of axion to pion production amplitudes will be $O(v/M)$

It is possible to explain why axions have not been observed yet by taking M very large

(invisible AXION)

BICEP2 data $\Rightarrow$ $M \approx 10^{11}$ GeV ?