INSTANTONS

We know that all fundamental interactions in nature are of gauge type. For more than 50 years the perturbative approach has proven to be very successful to calculate observable quantities within QED, QCD and the EW theory. However, it is clear that, contrary to what happens in QED, QCD is not exhausted by perturbation theory. The phenomenon of confinement and of the formation of bound states of colored objects is intrinsically non-perturbative. In 1975 one of the most beautiful phenomena in QCD was discovered, and it goes under the name of INSTANTONS, classical solutions of the field equations with non trivial topological properties. This is an important example of fluctuation of the gluon field which is not encompassed by perturbation theory.

Example : the double well potential

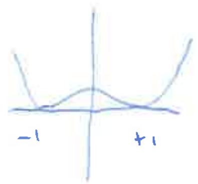
We consider the problem of the one-dimensional motion of a spinless particle in a potential $V(x)$

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 - V(x)$$

We are interested in the amplitude for going from the position x_i at the time $-t_0/2$ to the position x_f at the time $t_0/2$. According to the path integral formulation the amplitude is given by the sum of all path joining the points $(-t_0/2, x_i)$, $(t_0/2, x_f)$ with the weight e^{iS} with $S = \int_{-t_0/2}^{t_0/2} dt\, \mathcal{L}(x,\dot{x})$

$$\langle x_f | \underbrace{e^{-iHt_0}}_{\text{evolution operator}} | x_i \rangle = N \int Dx\, e^{iS[x(t)]}$$

We now specify to the case $V(x) = (x^2-1)^2$



Classically we know that the ground states correspond to $x = \pm 1$ (There is no path at $E=0$ connecting the points $x = \pm 1$)

In quantum mechanics, even if one starts with a wave function localized in $x=1$ or $x=-1$, the tunneling phenomenon occurs, and the correct ground state is a superposition of the wave functions in each well.

Let us now go in the Euclidean space time by setting $it = \tau$

$$\mathcal{L} \to -\frac{1}{2}\dot{x}^2 - (x^2-1)^2 = -\mathcal{L}_E$$

$$e^{iS} = e^{i\int dt\,\mathcal{L}} = e^{-\int d\tau\,\mathcal{L}_E} = e^{-S_E}$$

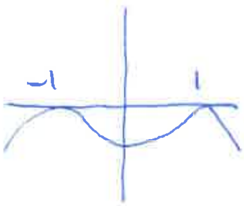
exponentially decreasing weight
classical contribution obtained when S_E is minimum

$$S_E = \int d\tau \left\{ \frac{1}{2}\dot{x}^2 + (1-x)^2 \right\}$$

classical eqs. of motion

$$\frac{\partial \mathcal{L}_E}{\partial x} - \frac{d}{d\tau}\frac{\partial \mathcal{L}_E}{\partial \dot{x}} = 0$$

In the Euclidean the potential is



$\Rightarrow$ now a classical trajectory exist

$$\boxed{\begin{array}{c} x(\tau) = \tanh(\sqrt{2}\,\tau) \\ (t_0 \to \infty) \end{array}}$$

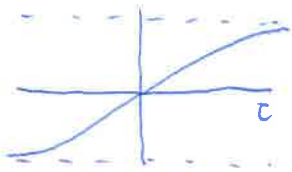
Instanton!

$$\tau = \frac{1}{\sqrt{2}}\,\mathrm{ArcTgh}(x) \qquad d\tau = \frac{1}{\sqrt{2}(1-x^2)}$$

$$S_E = \int d\tau\, 2(1-x^2)^2 = \dots \frac{4}{3}\sqrt{2}$$

↑ finite action

Instanton: classical solution of Euclidean eqs of motion interpolating between two vacua



How should we interpret all this?

It is known that there are quantum mechanical situations in which no classical trajectory exist. When the classical trajectory exists, then the semiclassical (WKB) approximation tells us that the amplitude is $\sim e^{iS_{cl}}$ where the classical action S_{cl} is the action evaluated for the classical trajectory, solution of the eqs of motion with Lagrangian $\mathcal{L} = \frac{1}{2}\dot{x}^2 - V(x)$. When the classical trajectory does not exist, one can still use the semiclassical (WKB) approximation to derive the tunneling probability. In this case the amplitude is e^{-S_E} where S_E is evaluated with a solution of the eqs of motion for $\mathcal{L}_E = \frac{1}{2}\dot{x}^2 + V(x)$

$\Rightarrow$ to obtain the tunneling amplitude we can use the same formula used to compute the amplitude for a normal transition, making only the formal replacement $\underline{it \to \tau}$

CLASSICAL SOLUTIONS OF EUCLIDEAN EQS. OF MOTION CORRESPOND TO TUNNELING PHENOMENA

HOMOTOPY CLASSES

X, Y topological spaces $\quad I = [0,1]$

$f_0 : X \to Y$
$f_1 : X \to Y$

are said to be homotopic if they are continuously deformable one into the other. In more formal words, they are homotopic if

$$\exists \underset{\text{continuous}}{F} : \{X, I\} \to Y \;\Big|\; F(x,0) = f_0(x) \;,\; F(x,1) = f_1(x)$$

The mappings $X \to Y$ can be divided into HOMOTOPY CLASSES. A group structure emerges.

S_1 unit circle with 0 and 2π identified. Consider the mappings from S_1 to a group manifold G. The group of homotopy classes of mappings from S_1 to G is $\pi_1(G)$

$\underline{S_1 \to S_1}$ $\qquad G = U(1) \qquad u = e^{i\alpha}$

$$\theta \to e^{i(m\theta + a)} \equiv f(\theta)$$

At fixed m, the mappings with different values of a, say a_1 and a_2 are continuously deformable one into the other

$$F(\theta, t) = e^{i(m\theta + a_1 t + a_2(1-t))}$$

One can think of these mappings as mappings from a circle into another circle such that m points of the first circle are mapped to one point of the second circle $\Rightarrow$ the integer m is called WINDING NUMBER

EXAMPLE: $m=3$

$$\frac{\pi}{3},\ \frac{5}{3}\pi,\ \pi \to e^{i\pi} \qquad \pi_1(U(1)) = \pi_1(S_1) = \mathbb{Z}$$

The winding number can be expressed as $\quad m = -i\int_0^{2\pi} \frac{d\theta}{2\pi}\left(\frac{1}{g}\frac{dg}{d\theta}\right)$

$\hookrightarrow$ stable upon deformations!

<u>$S_3 \to S_3$</u> $\qquad S_3$ sphere in 4-dimensional space (the dimensional space with points at infinity identified)

$$S_3 \leftrightarrow SU(2)$$

In fact $g \in SU(2)$ can be written as $g = e^{i\underline{\varepsilon}\cdot\underline{\tau}}$ but since the commutator of τ matrices gives a τ matrix the algebra is closed ($\tau_i^2 = 1$) $\Rightarrow g = u_0 + i\underline{u}\cdot\underline{\tau}$

Since $g^\dagger g = 1$ it follows that $u_0^2 + \underline{u}^2 = 1$

We now consider the Yang-Mills theory with $SU(2)$ gauge group

For the moment we use the notation

$$D_\mu = \partial_\mu + igA_\mu \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\,\varepsilon^{abc}A^b_\mu A^c_\nu \qquad \mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu}$$

A general gauge transformation is $\quad A'_\mu = UA_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger$

The gauge $A_0 = 0$ leaves the possibility to perform further gauge transformations with a matrix $U(\underline{x})$ depending only on space variables

$$A'_0 = A_0 \qquad A'_i = \frac{i}{g}(\partial_i U)U^\dagger$$

The gauge matrix $U(\underline{x})$ provides a <u>mapping from $\underline{x}$ to $SU(2)$</u>

(we have to impose the additional restriction $\lim_{|\underline{x}|\to\infty} U(\underline{x}) = U_\infty$ $\quad$ global, position independent matrix)

<u>$U(\underline{x})$ really defines a mapping $S_3 \to S_3$</u>

We now look for a solution of the Euclidean eqs. of motion with FINITE ACTION

Since the action has to be finite, we must have $\quad F_{\mu\nu} \underset{|\underline{x}|\to\infty}{\longrightarrow} 0$

The instanton solution was discovered by Belavin, Polyakov, Schwarz, Tyupkin (1975)

It corresponds to situations with non trivial winding number ↘ see exercise

$$m = \frac{g^2}{32\pi^2} \int d^4x \, \mathrm{Tr}\left(F_{\mu\nu} \tilde{F}^{\mu\nu}\right)$$

Typically we are used to represent gauge transformations by the exponential of the group generators controlled by "SMALL" parameters. This means that such gauge transformations can be continuously deformed into the identity. However in non abelian gauge theories there exist "LARGE" gauge transformations that cannot be obtained directly from the identity. This transformations correspond to non trivial winding numbers $m \neq 0$.

Let us consider the energy-momentum tensor of the pure Yang-Mills theory, which (see LECTURE 1), after Belinfante construction, can be written as

$$\theta^{\mu\nu}_{can} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a_\rho)} \partial^\nu A^\rho_a - g^{\mu\nu} \mathcal{L} \longrightarrow - F^{\mu\alpha}_a F^\nu_{a\alpha} + \frac{1}{4} g^{\mu\nu} F^2 = \theta^{\mu\nu}$$

The energy momentum tensor $\theta^{\mu\nu}$ can be rewritten as

$$\theta^{\mu\nu} = -\frac{1}{2} g_{\alpha\beta} F^{\mu\alpha}_a F^{\nu\beta}_a - \frac{1}{2} g_{\alpha\beta} \tilde{F}^{\mu\alpha}_a \tilde{F}^{\nu\beta}_a$$

thus

$$\theta^{00} = \frac{1}{2} \sum_k \left((F^{0k}_a)^2 + (\tilde{F}^{0k}_a)^2 \right)$$

and

$$\theta^{00} = 0 \quad \text{requires} \quad F = 0 \quad \text{if } F \text{ is real}$$

$\Rightarrow$ only zero field configurations can be identified with the vacuum

However, we can consider the case in which $F_{\mu\nu}$ is complex. Particularly important is the case in which F is real in the Euclidean space. According to the discussion on the double well potential, this situation will correspond to TUNNELING

This is the reason why we look for solutions of the QCD eqs. of motions in the Euclidean space!

In Minkowski space we have

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \qquad \tilde{\tilde{F}}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\frac{1}{2}\varepsilon_{\rho\sigma\alpha\beta}F^{\alpha\beta}$$

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\rho\sigma\alpha\beta} = -2\left(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}\right)$$

$$\Rightarrow \quad \tilde{\tilde{F}}^{\mu\nu} = -\frac{1}{2}\left(F_a^{\mu\nu} - F_a^{\nu\mu}\right) = -F_a^{\mu\nu}$$

So only $F=0$ can respect the duality condition $\tilde{F} = \pm F$ in Minkowski space

(suppose F exists such that $\tilde{F} = \pm F$
$\Rightarrow \tilde{\tilde{F}} = \pm\tilde{F}$ and $-F = \pm\tilde{F}$)

But in the Euclidean $\tilde{\tilde{F}} = F$ so solutions with $F \neq 0$ and $\tilde{F} = \pm F$ may exist!

In the Euclidean space the energy momentum tensor becomes

$$\theta_{\mu\nu} = -\frac{1}{2}\left(F^a_{\mu\lambda}F^a_{\nu\lambda} - \tilde{F}^a_{\mu\lambda}\tilde{F}^a_{\nu\lambda}\right)$$

$\Rightarrow$ $\underline{\theta_{\mu\nu} = 0 \text{ if } F \text{ is DUAL!}}$

Another property of dual configurations can be inferred by observing that

$$\mathrm{Tr}\int\left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right)^2 d^4x \geq 0$$

$$\left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right)^2 = 2\left(F_{\mu\nu}F_{\mu\nu} \pm F_{\mu\nu}\tilde{F}_{\mu\nu}\right) \quad \Rightarrow \quad \mathrm{Tr}\int F_{\mu\nu}F_{\mu\nu} \geq \left|\mathrm{Tr}\int F_{\mu\nu}\tilde{F}_{\mu\nu}\right|$$

$$= 32\pi^2 m / g^2$$

$\Rightarrow$ we see that DUAL SOLUTIONS $F = \pm\tilde{F}$ are such that the Euclidean action reaches its minimum

$$S_E(A) \geq \frac{8\pi^2 m}{g^2}$$

This inequality also tells us that the instanton phenomena are PURELY NON PERTURBATIVE (effect goes like $1/g^2$)

Let us consider the classical vacuum of the theory $\underline{F_{\mu\nu}=0}$ → classical vacuum in Minkowski space, A_μ pure gauge

If we choose the gauge $A_0=0$ this leaves the possibility to do time independent gauge transformations with $U(\underline{x})$

⇒ <u>we conclude that there exist a multiplicity of vacua each with its winding number</u>

$$A_i^\bullet(x_0=-\infty)=\frac{i}{g}\left(\partial_i U_m(\underline{x})\right)U_m^+$$

$$A_i(x_0=+\infty)=\frac{i}{g}\left(\partial_i U_{m+\nu}(\underline{x})\right)U_{m+\nu}^+$$

the instanton interpolates between these nontrivial vacua

Like in the double well!

Let us consider a gauge transformation with winding number ν

$$T_\nu|m\rangle=|m+\nu\rangle$$

↳ <u>SAME SITUATION SEEN IN THE SCHWINGER MODEL ON A CIRCLE!</u>

Gauge invariance implies $[T_\nu,H]=0$ ⇒ the true vacuum must be a superposition of $|m\rangle$ which is at least phase invariant with respect to T_ν

$$\Rightarrow\ |\vartheta\rangle=\sum_m e^{-im\vartheta}|m\rangle$$

$$T_\nu|\vartheta\rangle=\sum_m e^{-im\vartheta}|m+\nu\rangle=e^{i\nu\vartheta}\sum_m e^{-i(m+\nu)\vartheta}|m+\nu\rangle=e^{i\nu\vartheta}|\vartheta\rangle$$

<u>Vacuum to vacuum amplitude</u>

$$\langle\vartheta|e^{-iHt}|\vartheta\rangle=\sum_{m,n}e^{i(n-m)\vartheta}\int[dA]_{m-n}\,e^{-i\int(\mathcal{L}+JA)\,d^4x}$$

↳ summation over paths connecting m to n

$$=\sum_\nu e^{-i\nu\vartheta}\int(dA)_\nu\,e^{-i\int(\mathcal{L}+JA)\,d^4x}$$

$$=\sum_\nu\int[dA]_\nu\,e^{-i\int\mathcal{L}_{eff}+JA}$$

$$\boxed{\mathcal{L}_{eff}=\mathcal{L}+\frac{\vartheta g^2}{32\pi^2}\,\mathrm{Tr}\,F_{\mu\nu}\widetilde{F}_{\mu\nu}}$$

new parameter of the theory