ADVANCED FIELD THEORY

The course covers various (sometimes disconnected) topics in (quantum) field theory

Anomalies → Anomaly cancellation in the SM
Anomalies → $U_A(1)$ problem
Anomalies → $\pi^0 \to 2\gamma$
Anomalies → In the Schwinger model

Topological Objects → Sphalerons
Topological Objects → Axions
Topological Objects → Instantons
Topological Objects → Monopoles
Topological Objects → Solitons

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BIBLIOGRAFY

- S. Weinberg "The quantum theory of fields" vol 1, 2
- S. Pokorski "Gauge field theories"
- Peskin–Schroeder "An introduction to Quantum Field Theory"
- Shifman "Anomalies in gauge theories", PR 209 (1991) 341

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The energy momentum tensor and anomalous breaking of scale invariance

Recall the Noether theorem:

given an action invariant under a general set of coordinates and field transformations

$$S = \int d^4x\, \mathcal{L}(\phi_i, \partial_\mu \phi_i) \qquad \phi_i \text{ arbitrary local fields}$$

$$\begin{cases} x_\mu \to x_\mu + \delta x_\mu \\ \phi_i(x) \to \phi_i'(x') = \phi_i(x) + \delta\phi_i(x) \end{cases}$$

$\Rightarrow$ there is a conserved current

$$\delta J^\mu = \mathcal{L}\,\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\,\bar{\delta}\phi_r$$

<u>Noether current</u>

where $\bar{\delta}\phi_r$ is the LOCAL variation of the field

$$\bar{\delta}\phi_r = \phi'_r(x') - \phi_r(x') = \phi'_r(x') - \phi_r(x) + \phi_r(x) - \phi_r(x') = \delta\phi_r - (\partial_\mu \phi_r)\,\delta x^\mu$$

Let us consider the simple example of a space-time translation

$$x_\mu \to x_\mu + \delta x_\mu = x_\mu + \varepsilon_\mu$$

$$\bar{\delta}\phi_r = \underbrace{\delta\phi_r}_{=0} - (\partial_\mu \phi_r)\,\delta x^\mu = -(\partial_\mu \phi_r)\,\delta x^\mu$$

$$\Rightarrow \quad \delta J^\mu = \mathcal{L}\,\delta x^\mu - (\partial_\nu \phi_r)\,\delta x^\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} = \left(\mathcal{L}\, g^\mu{}_\nu - (\partial_\nu \phi_r)\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\right)\delta x^\nu$$

We can define the <u>CANONICAL ENERGY MOMENTUM TENSOR</u>

$$T^{\text{can}}_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_r)}\,\partial_\nu \phi_r - \mathcal{L}\, g_{\mu\nu} \qquad\qquad \partial_\mu T^{\mu\nu}_{\text{can}} = 0$$

Note that this tensor is in general NOT SYMMETRIC AND GAUGE INVARIANT

- In the case of a scalar theory with lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$

we get $T^{\text{can}}_{\mu\nu} = (\partial_\mu \phi)(\partial_\nu \phi) - \mathcal{L}\, g_{\mu\nu}$ $\Rightarrow$ IT IS SYMMETRIC

<u>EXERCISE</u> Try with the EM case (pure $U(1)$ gauge theory)

and show that it is NOT symmetric and gauge invariant

The canonical energy-momentum tensor can be made symmetric and gauge invariant by using the <u>BELIFANTE</u> procedure

F.J. Belifante (1940)

③

We first observe that we can always add to the energy-momentum tensor a term of the form $\partial^\rho \Sigma_{\rho\mu\nu}$ where $\Sigma_{\rho\mu\nu}$ is ANTISYMMETRIC in the first two indices $\Sigma_{\rho\mu\nu} = -\Sigma_{\mu\rho\nu}$

$$T_{\mu\nu} \to T_{\mu\nu} + \partial^\rho \Sigma_{\rho\mu\nu} \equiv T'_{\mu\nu}$$

$\partial^\mu \partial^\rho \Sigma_{\rho\mu\nu} = 0 \Rightarrow$ the new tensor is still conserved

$$P'_\nu = \int d^3x\, T'_{0\nu} = \int d^3x\, T_{0\nu} + \int d^3x\, \partial^\rho \Sigma_{\rho 0 \nu}$$

$$= P_\nu + \int d^3x \left(\cancel{\partial^0 \Sigma_{00\nu}} + \cancel{\partial^i \Sigma_{i0\nu}} \right)$$

↳ vanishes if Σ vanishes sufficiently fast at infinity

↳ vanishes for the antisymmetry

$\Rightarrow$ the conserved quantities are the same

- Let us now consider Lorentz transformation

Under a general coordinate transformation the FIELDS are NOT INVARIANT but transform under the corresponding representation of the symmetry group

An infinitesimal Lorentz transformation can be written as

$$x^\mu \to x'^\mu = x^\mu + \delta\omega^{\mu\nu} x_\nu \qquad \text{where (to preserve } x'^2 = x^2) \qquad \delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu}$$

$$\phi'_r(x') = \phi_r(x) + \frac{1}{2}\delta\omega_{\mu\nu} \left(I^{\mu\nu}\right)_{rs} \phi_s(x)$$

↳ generators of the Lorentz group in the representation of the fields

(example: if we have a single scalar field ϕ $\quad (I^{\mu\nu})_{rs} = 0$ and $\phi'(x') = \phi(x)$)

Let us evaluate the conserved current in the case of a Lorentz transformation

$$\delta J^\mu = \mathcal{L}\, \delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \bar{\delta} \phi_r$$

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$$\delta\phi_1 = \phi'_1(x') - \phi_1(x) = \frac{1}{2}\,\delta w_{\mu\nu}\,(I^{\mu\nu})_{12}\,\phi_2$$

$$\bar{\delta}\phi_1 = \delta\phi_1 - (\partial_\mu\phi_1)\,\delta x^\mu = \frac{1}{2}\,\delta w_{\mu\nu}\,(I^{\mu\nu})_{12}\,\phi_2 - (\partial_\mu\phi_1)\,\delta x^\mu$$

$$\delta J^\mu = \mathcal{L}\,\delta w^{\mu\nu}\,x_\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\left(\frac{1}{2}\,\delta w_{\rho\sigma}\,(I^{\rho\sigma})_{12}\,\phi_2 - (\partial_\rho\phi_1)\,\delta w^{\rho\sigma}x_\sigma\right)$$

$$= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\,\frac{1}{2}\,\delta w_{\rho\sigma}\,(I^{\rho\sigma})_{12}\,\phi_2 - T^{con}_{\mu\rho}\,\delta w^{\rho\sigma}\,x_\sigma$$

canonical energy momentum tensor

Since $\delta w^{\rho\sigma}$ is antisymmetric we can also write

$$\delta J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\,\frac{1}{2}\,\delta w_{\rho\sigma}(I^{\rho\sigma})_{12}\phi_2 - \frac{1}{2}\left(T^{con}_{\mu\rho}\,x_\sigma - T^{con}_{\mu\sigma}\,x_\rho\right)\delta w^{\rho\sigma}$$

$$\Rightarrow \quad 0 = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\,\frac{1}{2}(I^{\rho\sigma})_{12}\,\phi_2\right] - \frac{1}{2}\left(T^{\mu\rho}_{con}\,g_\mu{}^\sigma - T^{\mu\sigma}_{con}\,g_\mu{}^\rho\right)$$

$$= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\,\frac{1}{2}\,(I^{\rho\sigma})_{12}\,\phi_2\right] - \frac{1}{2}\left(T^{\sigma\rho}_{con} - T^{\rho\sigma}_{con}\right) \qquad (*)$$

This equation tell us that the ANTISYMMETRIC PART of the canonical energy momentum tensor is related to the way in which the field transforms under the Lorentz group

If we define

$$T_{\mu\nu} = T^{con}_{\mu\nu} + \partial^\rho\xi_{\rho\mu\nu}$$

where

$$\xi_{\rho\mu\nu} = -\frac{1}{2}\left[-\frac{\partial\mathcal{L}}{\partial(\partial^\rho\phi_1)}(I_{\mu\nu})_{12} + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_1)}(I_{\rho\nu})_{12} + \frac{\partial\mathcal{L}}{\partial(\partial^\nu\phi_1)}(I_{\rho\mu})_{12}\right]\phi_2$$

5 The tensor $\Sigma_{\rho\mu\nu}$ is antisymmetric in the first two indices, so the new tensor is also conserved, and the conserved quantities are the same. Moreover, thanks to eq. (*), the new tensor is symmetric!

$$T_{\mu\nu} - T_{\nu\mu} = T^{can}_{\mu\nu} - T^{can}_{\nu\mu} + \partial^\rho\left(\Sigma_{\rho\mu\nu} - \Sigma_{\rho\nu\mu}\right)$$

$$= T^{can}_{\mu\nu} - T^{can}_{\nu\mu} - \frac{1}{2}\partial^\rho\left[-\frac{\partial\mathcal{L}}{\partial(\partial^\rho\phi_r)}(I_{\mu\nu})_{rs} + \cancel{\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_r)}(I_{\rho\nu})_{rs}} + \cancel{\frac{\partial\mathcal{L}}{\partial(\partial^\nu\phi_r)}(I_{\rho\mu})_{rs}}\right]$$

ϕ_s

$$+\frac{1}{2}\partial^\rho\left[-\frac{\partial\mathcal{L}}{\partial(\partial^\rho\phi_r)}(I_{\nu\mu})_{rs} + \cancel{\frac{\partial\mathcal{L}}{\partial(\partial^\nu\phi_r)}(I_{\rho\mu})_{rs}} + \cancel{\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_r)}(I_{\rho\nu})_{rs}}\right]$$

ϕ_s

$$= T^{can}_{\mu\nu} - T^{can}_{\nu\mu} + \partial^\rho \frac{\partial\mathcal{L}}{\partial(\partial^\rho\phi_r)}(I_{\mu\nu})_{rs} = 0$$

The above procedure allows us to construct a symmetric tensor out of the canonical tensor. Note that in the case of the scalar field this procedure is trivial because $(I_{\mu\nu})_{rs} = 0$

The modified energy-momentum tensor allows us to define one more conserved tensor density

$$M^{\lambda\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}$$

$$\partial_\lambda M^{\lambda\mu\nu} = T^{\mu\nu} - T^{\nu\mu} = 0$$ due to the symmetry of T

In the case of a scalar theory the canonical energy momentum tensor is still not satisfactory since it does not have finite matrix elements between physical states. We define

$$\Theta_{\mu\nu} = T^{can}_{\mu\nu} - \frac{1}{6}\left(\partial_\mu\partial_\nu - \Box g_{\mu\nu}\right)\varphi^2$$ Callan - Coleman - Jackiw (1970)

This improved tensor can be written as: $\Theta_{\mu\nu} = T^{can}_{\mu\nu} + \partial^\rho\Sigma_{\rho\mu\nu}$

where $\Sigma_{\rho\mu\nu} = -\frac{1}{6}\left(g_{\nu\rho}\,\partial_\mu - g_{\mu\nu}\,\partial_\rho\right)\phi^2$ is ANTISYMMETRIC in ρ and μ

$\Rightarrow$ The new improved tensor is still conserved and the conserved quantities are the same

This procedure, applied after the Belifante procedure, provides the correct energy momentum tensor that acts as a source of the gravitational field

Scale invariance

The scale transformations are defined by

$$x \to x' = e^{-\varepsilon}x$$

and form an Abelian group. Scale invariance is the invariance under the scale transformations

Through a scale transformation the local fields transform as

$$\phi(x) \to \phi'(x') = T(\varepsilon)\,\phi(x)$$ the group is abelian $\Rightarrow$ irreducible rep. have dimension 1

The representation to which the field belongs is identified by a quantity d_ϕ $\quad T(\varepsilon) = e^{d_\phi \varepsilon}$

d_ϕ can be identified with the CANONICAL FIELD DIMENSION ($d_\phi = 1$ scalar field
$d_\phi = \frac{3}{2}$ fermion field

This is necessary to preserve ET commutation relations

$$\phi'(x) = T(\varepsilon)\,\phi(e^{\varepsilon}x)$$

$$\delta x = x' - x \approx -\varepsilon x$$

$$\phi'(x) \approx (1+\varepsilon d_\phi)\left(\phi(x) + (\partial_\mu\phi)\,\varepsilon x^\mu\right) \Rightarrow \delta\phi = \varepsilon\left(d_\phi\,\phi(x) + \partial_\mu\phi\, x^\mu\right)$$

The Noether current corresponding to the scale transformation is

$$\delta J_\mu = \varepsilon\,\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\left(d_\phi\,\phi + \partial_\nu\phi\, x^\nu\right) - \varepsilon x_\mu \mathcal{L}$$

$\Rightarrow$ the dilatation current is

$$D_\mu = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} d_\phi \phi + \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \partial_\nu \phi - \mathcal{L} g_{\mu\nu} \right)}_{\hookrightarrow T^{can}_{\mu\nu}} x^\nu$$

By using the improved tensor $\Theta_{\mu\nu}$ by Callan - Coleman - Jackiw

we can write for a scalar theory

$$D_\mu = (\partial_\mu \phi) \phi + x^\nu \left(\Theta_{\mu\nu} + \frac{1}{6} (\partial_\mu \partial_\nu - g_{\mu\nu} \Box) \phi^2 \right)$$

$$= \frac{1}{2} (\partial_\mu \phi^2) + x^\nu \left(\Theta_{\mu\nu} + \frac{1}{6} (\partial_\mu \partial_\nu - g_{\mu\nu} \Box) \phi^2 \right)$$

$$= x^\nu \Theta_{\mu\nu} - \frac{1}{6} \left(g_{\mu\nu} \partial^\nu + x_\mu \Box - 4 \partial_\mu - x^\nu \partial_\nu \partial_\mu \right) \phi^2$$

$$= x^\nu \Theta_{\mu\nu} - \frac{1}{6} \partial_\nu \left(x_\mu \partial^\nu - x^\nu \partial_\mu \right) \phi^2$$

The second term can be dropped because it is the derivative of an antisymmetric tensor and thus it does not contribute to the charge

$\Rightarrow$ we can define $D_\mu = x^\nu \Theta_{\mu\nu}$

$\partial^\mu D_\mu = \Theta_\mu{}^\mu$ — the divergence of the dilatation current is equal to the trace of the improved tensor

Let us compute $\Theta_\mu{}^\mu$ in the case of the scalar theory

$$\Theta_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \mathcal{L} g_{\mu\nu} - \frac{1}{6} (\partial_\mu \partial_\nu - g_{\mu\nu} \Box) \phi^2$$

$$\Theta_\mu{}^\mu = (\partial_\mu \phi)^2 - 4 \mathcal{L} - \frac{1}{6} (-3 \Box) \phi^2$$

$$= (\partial_\mu \phi)^2 - 4 \left(\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \right) + \frac{1}{2} \Box \phi^2$$

$$= (\partial_\mu \phi)^2 - 2 (\partial_\mu \phi)^2 + 2 m^2 \phi^2 + \partial_\mu (\phi \partial^\mu \phi)$$

$$= 2 m^2 \phi^2 + \phi \Box \phi \rightarrow m^2 \phi^2 \quad \text{using } (\Box + m^2) \phi = 0$$

$\Rightarrow$ The trace of the improved energy-momentum tensor vanishes in the massless limit

This symmetry of the classical theory is broken by quantum corrections. The necessity of regularizing the UV divergences forces us to introduce a mass scale either through a cut-off or by introducing the renormalization scale

$\Rightarrow$ $\partial_\mu D^\mu \neq 0$ at quantum level even if the theory has no mass scales at the classical level

(8)

If the quantum field theory under consideration is coupled to gravity, the energy momentum tensor is the source of the gravitational field

$\Rightarrow$ it can be obtained by varying the Lagrangian of matter fields with respect to $g_{\mu\nu}(x)$

This construction gives a manifestly symmetric and gauge invariant tensor

$$\vartheta^{\mu\nu} = 2 \frac{\delta}{\delta g_{\mu\nu}(x)} \int d^4x \, \mathcal{L}_m$$

When quantum corrections are included we know that a scale transformation is not a symmetry of the theory, since changing the scale we get a different renormalized coupling

$$g \to g + \delta g = g + \frac{\partial g}{\partial \mu} \delta\mu = g + \beta(g) \frac{\delta\mu}{\mu} = g + \varepsilon \beta(g)$$

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since $x \to x e^{-\varepsilon}$

$\mu \to \mu e^{\varepsilon}$ $\qquad \delta\mu = (1+\varepsilon)\mu - \mu \approx \varepsilon\mu$

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the corresponding change in the Lagrangian is $\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial g} \varepsilon \beta(g)$

$$\Rightarrow \boxed{\partial_\mu D^\mu = \Theta^\mu{}_\mu = \beta(g) \frac{\partial \mathcal{L}}{\partial g} \neq 0} \;!$$

In massless QED we can apply this formula more easily by rescaling the gauge field such that the electric charge is removed from the covariant derivative $eA \to A$

$\Rightarrow$ the charge appears only in $\mathcal{L} = -\frac{1}{4e^2}(F_{\mu\nu})^2$

$$\Rightarrow \quad \partial_\mu D^\mu = \frac{\beta(e)}{2e^3}(F_{\mu\nu})^2$$

<u>TRACE ANOMALY</u>