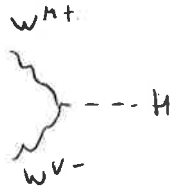
FEYNMAN RULES FOR HIGGS COUPLINGS

We have seen that the mass term for the vector bosons is

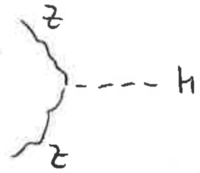
$$\mathcal{L}_M = \frac{1}{2} v^2 \left[\frac{g^2}{4} (W W^+ + h.c) + \frac{1}{4} \frac{g^2}{\cos^2\theta} Z^2 \right]$$

Since v comes from $\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+H \end{pmatrix}$ we can obtain the Higgs couplings with the replacement $v \to v + H$

$$\mathcal{L}_{WWH} = vH \frac{g^2}{4} (W W^+ + h.c) \quad \Rightarrow \quad -i g m_W g^{\mu\nu}$$

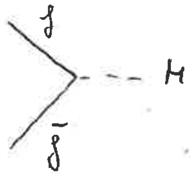


$$\mathcal{L}_{ZZH} = vH \frac{g^2}{4} \frac{Z^2}{\cos^2\theta} \quad \Rightarrow \quad -ig \frac{m_Z}{\cos\theta} g^{\mu\nu}$$



Analogously, for the fermions we can write

$$\mathcal{L}_{ffH} = -\lambda_f \bar{F}_L H f_R + h.c \quad \Rightarrow \quad +i \frac{m_f}{v}$$



(assuming a down type fermion)

$\Rightarrow$ From the Feynman rules we clearly see that the Higgs couplings to the vector bosons and fermions <u>ARE PROPORTIONAL TO THEIR MASSES</u>

$\Rightarrow$ The Higgs boson will decay to the heaviest particles allowed by kinematics

THE GOLDSTONE BOSONS AND THE EQUIVALENCE THEOREM

In the unitary gauge the physical spectrum of the EW theory is exposed: we have the fermions, the massless photon and the massive vector bosons (with three polarizations) plus the Higgs boson. In this gauge the propagator of the W and Z boson is

$$\frac{i}{p^2 - m_V^2 + i\epsilon}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m_V^2}\right)$$

This gauge choice, although clear as far as the physical content of the theory is concerned, makes the renormalization program of the theory rather complicated. Indeed the propagator has a bad behavior as $p \to \infty$, and complicated cancellations in the scattering amplitudes are required. To this purpose, it is more convenient to work in the so called R_ξ gauges, where one adds to the Lagrangian a gauge fixing term of the form

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}\Big[2\left(\partial^\mu W_\mu^+ - i\xi m_W \eta^+\right)\left(\partial_\mu W - i\xi m_W \eta^-\right) + \left(\partial^\mu Z_\mu - i\xi m_Z \eta_3\right)^2 + \left(\partial_\mu A_\mu\right)^2\Big].$$

In this case the propagator becomes

$$\frac{i}{p^2 - m_V^2 + i\epsilon}\left[g_{\mu\nu} + (\xi - 1)\frac{p^\mu p^\nu}{p^2 - m_V^2 \xi}\right]$$

for $\xi \to \infty$ we recover the unitary gauge

for $\xi = 1$ we have the 't Hooft-Feynman gauge, in which the $p^\mu p^\nu$ term is absent.

The physical matrix elements should of course be independent on ξ

The propagators for the Goldstone bosons are

$$\underset{\eta^\pm \quad \eta^\pm}{--\rightarrow--} \qquad \frac{i}{p^2-\xi m_W^2+i\epsilon} \qquad\qquad \underset{\eta_3}{--\rightarrow-} \qquad \frac{i}{q^2-\xi m_Z^2+i\epsilon}$$

and they decouple in the $\xi\to\infty$ limit.

We now consider a W (or Z) gauge boson at rest and choose a basis for its polarizations

$$\varepsilon_1^\mu = (0,1,0,0) \qquad \varepsilon_2^\mu = (0,0,1,0) \qquad \varepsilon_L^\mu = (0,0,0,1)$$

We go in a frame in which the boson momentum is $p^\mu = (E,0,0,p)$ with $E^2-p^2=m_V^2$. By doing the Lorentz transformation we get

$$(\varepsilon_L^0)' = \gamma v\, \varepsilon_L^3 = \gamma v$$

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{E}{m_V}$$

$$(\varepsilon_L^1)' = (\varepsilon_L^2)' = 0$$

$$(\varepsilon_L^3)' = \gamma\, \varepsilon_L^3 = \gamma \qquad\qquad \Rightarrow \quad \varepsilon_L^{\mu\,\prime} = \left(\frac{p}{m_V}, 0, 0, \frac{E}{m_V}\right)$$

In the high energy limit we have $\varepsilon_L \to \dfrac{p_\mu}{m_V}$

Since this polarization is proportional to the boson momentum, at very high energies the contribution of longitudinal polarizations will dominate in the scattering of vector bosons.

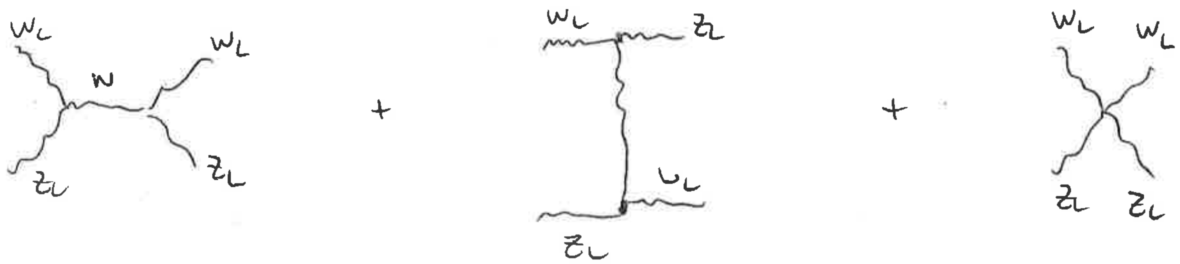
Actually, there is a theorem, called EQUIVALENCE THEOREM which states that at very high energies the amplitude for the scattering of longitudinal massive vector bosons can be computed by replacing them with the corresponding Goldstone bosons

$$\mathcal{M}(V_L^1 \ldots V_L^n \to V_L^1 \ldots V_L^{n'}) \approx \mathcal{M}(\eta^1 \ldots \eta^n \to \eta^1 \ldots \eta^{n'})_R + O\left(\frac{m_V}{E_i}\right)$$

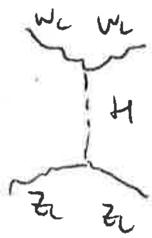
where the subscript R denotes a covariant renormalizable gauge choice.
The theorem is valid to all orders in gauge and symmetry breaking interactions, and makes explicit the fact that the longitudinal gauge bosons modes behave as quanta of the symmetry breaking Lagrangian at high energy.

UNITARITY AND THE HIGGS BOSON

If the contribution of the Higgs boson is not considered the scattering amplitudes of weak vector bosons violate unitarity. Considering for example $W_L Z_L \to W_L Z_L$ we have the following diagrams



The sum of these diagrams would produce an amplitude $\mathcal{M}(W_L Z_L \to W_L Z_L) \approx \frac{t}{v^2} + \dots$ which violates unitarity at very high energies. The additional diagram with the Higgs exchange



provides a contribution $\sim -\frac{t}{v^2}$ which exactly cancels the bad high energy behavior of the other diagrams.

Before the Higgs discovery, it was precisely this kind of arguments that lead people to state the so called <u>NO-LOSE THEOREM</u>. If the Higgs boson does not exist, then the gauge boson scattering become strong, and something has to happen to cure the bad high energy behavior, so some new physical mechanism must necessarily show up.

THEORETICAL CONSTRAINTS ON THE HIGGS BOSON MASS

The only parameter which is completely undetermined in the Higgs sector is the Higgs mass m_H. Although the theory does not provide the value of m_H, interesting theoretical constraints can be derived from assumptions of consistency of the theory and on the energy range up to which it should be valid.
These include constraints from unitarity in the scattering amplitudes of weak bosons, stability of the electroweak vacuum, perturbativity of the Higgs self coupling and fine tuning.

UNITARITY AND WW SCATTERING

The main limit of the old Fermi theory of weak interactions was that it violates unitarity at energies of the order of the Fermi scale. The problem is solved by the introduction of the W boson as mediator of charged current weak interactions. However the unitarity problem is still a potential problem in the SM at very high energies. In this regime the strength of the interactions of the longitudinal components of the vector bosons grow with the energy. The possibility of a strongly interacting WW sector and of new particles in the TeV region is an interesting alternative to the weakly coupled Higgs sector. At the same time, this regime can be used to derive important constraints on the theory. Let us discuss this issue by focusing on WW scattering.
In the limit $s, m_H^2 \gg m_W^2$ the elastic scattering amplitude can be approximated as

$$\mathcal{M}(W^+W^- \to W^+W^-) = -\sqrt{2}\, G_F\, m_H^2 \left(\frac{s}{s-m_H^2} + \frac{t}{t-m_H^2} \right)$$

This result can violate unitarity at high energy. To see this explicitly one can perform a partial wave expansion of the amplitude

$$\mathcal{M} = 16\pi \sum^{\infty} (2\ell+1)\, P_\ell(\cos\theta)\, a_\ell$$

and compute the a_0 amplitude, which is

$$a_0 = -\frac{m_H^2}{16\pi v^2}\left(2 + \frac{m_H^2}{s-m_H^2} - \frac{m_H^2}{s}\log\left(1+\frac{s}{m_H^2}\right)\right)$$

and in the large s limit we obtain

$$a_0 \to -\frac{m_H^2}{8\pi v^2}$$

The partial amplitudes a_ℓ, however, are bound from unitarity. In particular, we must have

$$|\mathrm{Re}(a_\ell)| \leq \frac{1}{2} \qquad \text{(see ex. sheet 2)}$$

From this condition we have (Lee, Quigg, Thacker, 1977)

$$\frac{m_H^2}{8\pi v^2} < \frac{1}{2} \quad \Rightarrow \quad m_H^2 < 4\pi v^2 = \frac{4\pi}{\sqrt{2}G_F} \approx (870\,\mathrm{GeV})^2$$

In practice, the limit can be made more restrictive by considering the other scattering channels $Z_L Z_L$, HH and HZ_L, and one obtains

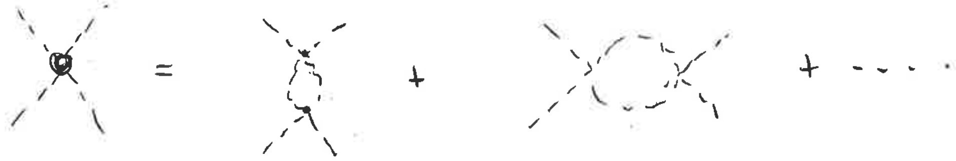
$$m_H \lesssim 710\ \mathrm{GeV}$$

Thus we can conclude that if the Higgs mass exceeds values of $O(700\,\mathrm{GeV})$, unitarity will be violated unless there are new phenomena to restore it.

TRIVIALITY BOUND

The behavior of the couplings in the SM Lagrangian, because of quantum corrections, is driven by the renormalization group, and in general, the couplings depend on the energy. This in particular applies to the Higgs self coupling λ. In the limit in which the Higgs boson is heavy, and the gauge and Yukawa couplings can be neglected, the effective Higgs coupling is controlled by the renormalization group equation

$$\frac{d\lambda}{d\log Q^2} = \frac{3}{4\pi^2}\lambda^2$$



TYPICAL DIAGRAMS

The equation can be solved to give

$$\frac{d\lambda}{\lambda^2} = \frac{3}{4\pi^2}\, d\log Q^2$$

$$\Rightarrow \quad \lambda(Q^2) = \frac{\lambda(Q_0^2)}{1-\frac{3}{4\pi^2}\lambda(Q_0^2)\log\frac{Q^2}{Q_0^2}} \qquad Q_0 \simeq v$$

The quartic coupling depends logarithmically on the energy. If Q^2 is much smaller than the electroweak scale we have $\lambda(Q^2)\to 0$ and the theory is non interacting. On the contrary, when $Q^2\to\infty$ the coupling grows and eventually diverges, due to the Landau pole $Q_L = v\exp\left(\frac{4\pi^2}{3\lambda_0}\right)$. This singularity, which is analogous to what we have in QED, is just a signal of the fact that the Higgs sector is not asymptotically free. The triviality argument simply says that, in order for the theory to be valid at all scales, the coupling has to vanish exactly ($\lambda=0$). However, one can use this to obtain an upper bound on the Higgs mass.

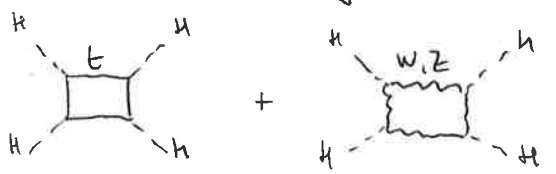
Let us assume that the SM is valid up to a scale Λ $\Rightarrow$ since the Higgs sector is not asymptotically free, if M_H is too heavy, we can hit the Landau pole at energies $Q < \Lambda_L$. Imposing that there is no Landau pole below Λ, we get an upper limit on M_H. In particular, if Λ is large, M_H must be small, to keep the logarithmic term smaller than one. If on the contrary, the cutoff is small, and new physics is behind the corner, then the Higgs mass can be higher.

VACUUM STABILITY BOUND

The renormalization group equation we have discussed before becomes more involved when the gauge and Yukawa couplings are also considered.

$$\frac{d\lambda}{d\log Q^2} = \frac{1}{16\pi^2}\left\{12\lambda^2 + 6\lambda\lambda_t^2 - 3\lambda_t^4 - \frac{3}{2}\lambda(3g^2+g'^2) + \frac{3}{16}\left(2g^4 + (g^2+g'^2)^2\right)\right\}$$

where, for simplicity, only the top Yukawa coupling has been considered.
The additional contributing diagrams are shown below



If the Higgs is light and the top is heavy, the quartic term λ_t^4 can dominate, and turn the vacuum to be unstable.
In the limit in which the Higgs boson is light we can neglect the λ terms in the evolution and write

$$\frac{d\lambda}{d\log Q^2} = \frac{1}{16\pi^2}\left\{-3\lambda_t^4 + \frac{3}{16}\left(2g^4+(g^2+g'^2)^2\right)\right\}$$

which gives

$$\lambda(Q^2) - \lambda(v^2) = \frac{1}{16\pi^2}\left\{-3\lambda_t^4 + \frac{3}{16}\left(2g^4+(g^2+g'^2)^2\right)\right\}\log\frac{Q^2}{v^2}$$

where the dependence on Q^2 in λ_t, g and g' has been neglected.

Imposing $\lambda(Q^2) > 0$ implies

$$\lambda(v^2) > \frac{1}{16\pi^2}\left[3\lambda_t^4 - \frac{3}{16}\left(2g^4 + (g^2+g'^2)^2\right)\right] \log\frac{Q^2}{v^2}$$

or

$$m_H^2 > \frac{v^2}{8\pi^2}\left[3\lambda_t^4 - \frac{3}{16}\left(2g^4 + (g^2+g'^2)^2\right)\right] \log\frac{Q^2}{v^2}$$

This equation sets a strong constraint on the Higgs mass, which depends on the cut-off Λ (the larger is the cut-off and the stronger is the constraint!)

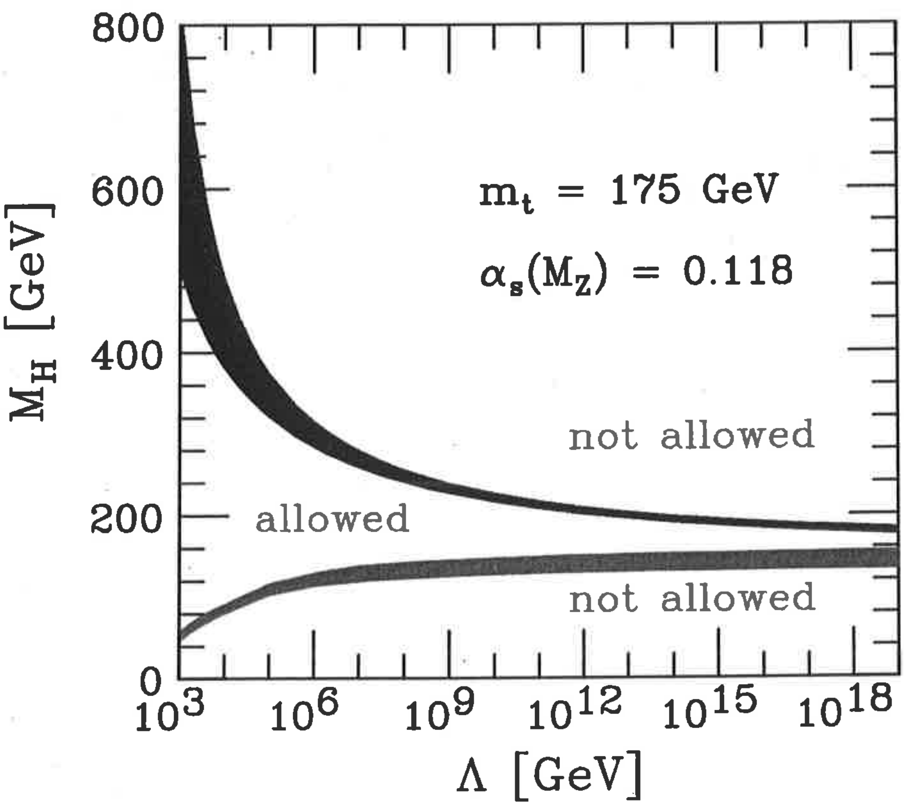
$$\Lambda \sim 10^3\ \text{GeV} \quad \Rightarrow \quad m_H \gtrsim 70\ \text{GeV}$$

$$\Lambda \sim 10^{16}\ \text{GeV} \quad \Rightarrow \quad m_H \gtrsim 130\ \text{GeV}$$

However, the stability bound can be relaxed if we assume the vacuum is METASTABLE

The SM effective potential can have a minimum that is DEEPER than our EW minimum, provided that the tunneling probability to this minimum is small enough that the lifetime of our EW vacuum is LARGER than the LIFETIME of our universe!

The fact that $m_H \approx 125$ GeV indeed suggests that we live in a METASTABLE VACUUM



T. HAMBYE, K. RIESSELMANN (1997)

Custodial symmetry

We consider the EW theory spontaneously broken without specifying the Higgs sector.

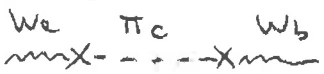
We want to show that the relation $m_W^2 = m_Z^2 \cos^2\theta$ is a consequence of a symmetry.

Let us consider the gauge currents and the corresponding Goldstone bosons.

The most general parametrization of the matrix element $\langle 0 | J_\mu^a | \pi^b \rangle$ is

$$\langle 0 | J_\mu^a | \pi^b \rangle = i f^{ab} p_\mu$$

The Lagrangian will contain a term of the form $g J_\mu^a W_a$ so the mass matrix is generated by the diagram



$$g_a g_b f^{ac} f^{bc}$$

We now assume that there is an unbroken $SU(2)_V$ global symmetry under which the three Goldstone bosons transform as a TRIPLET $\Rightarrow$ one can show that in this case $f^{ab} = f \delta^{ab}$ (Schur's lemma) (the same argument is used for the pions in QCD)

We now impose that the $U(1)_{EM}$ is unbroken

$$\langle 0 | J_\mu^{L3} + \frac{J_\mu^Y}{2} | \pi^3 \rangle = 0 \quad \Rightarrow \quad \langle 0 | J_\mu^Y | \pi^3 \rangle = -i 2 f p_\mu$$

The mass matrix can be written as

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & Y \end{array} \\ f^2 & \begin{pmatrix} g^2 & & & \\ & g^2 & & \\ & & g^2 & -gg' \\ & & -gg' & g'^2 \end{pmatrix} \end{array}$$

$$(g^2-\lambda)(g'^2-\lambda) - g^2 g'^2 = 0$$

$$\lambda^2 - \lambda(g^2+g'^2) = 0 \quad \Rightarrow \quad \begin{cases} \lambda = 0 & \text{photon} \\ \lambda = g^2+g'^2 \end{cases}$$

$$\Rightarrow \quad m_W^2 = m_Z^2 \frac{g^2}{g^2+g'^2} \qquad \boxed{\rho = 1}$$

We see that the result $\rho=1$ comes out quite generally, without making other assumptions on the structure of the Higgs sector

Let us now assume that the Higgs sector is the one of the SM

Instead of introducing the Higgs as a complex doublet $\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_3 + i\phi_4 \\ \phi_1 + i\phi_2 \end{pmatrix}$

we can define

$$M = \begin{pmatrix} \phi_1 + i\phi_2 & -(\phi_3 - i\phi_4) \\ \phi_3 + i\phi_4 & \phi_1 - i\phi_2 \end{pmatrix} \qquad \det M = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 \equiv \phi^2$$

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}\left(\partial_\mu M^\dagger \partial^\mu M\right) - \frac{\lambda}{4}\left(\det M - v^2\right)^2$$ neglect gauge interactions

$SU(2)_L \otimes SU(2)_R$ invariance $\quad M \to g_L^\dagger M g_R \quad \Rightarrow$ spontaneously broken to $SU(2)_V$

Note also that Yukawa interaction could be written

as

$$\mathcal{L}_Y = K\,(u_L^+ \; d_L)\, M \begin{pmatrix} u_R \\ d_R \end{pmatrix} + h.c.$$ ONLY IF $m_u = m_d$!

$\Rightarrow$ $SU(2)_L \otimes SU(2)_R$ symmetric

The symmetry of the Yukawa interaction is broken already at tree level by $m_u - m_d$ diff.

So, if we neglect gauge interactions the lagrangian can be written in this $SU(2)_L \otimes SU(2)_R$ symmetric form and the SCREENING THEOREM HOLDS : " Up to one loop the only observables radiative corrections that grow with the Higgs mass are logarithmic".

Gauge interactions break the symmetry

$\Rightarrow$ we expect the $\rho=1$ relation to be broken at loop level

Since the custodial symmetry is also broken by Yukawa couplings we expect corrections to $\rho=1$ quadratic in the top mass (the heaviest quark)