Homework 3 Solution

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Exercise 1. Potential and field lines

Draw the equipotential lines, the field lines and L_1 for the system depicted in Figure 1 (M_1=10M_2).

The dotted lines can be used as a guide for drawing the field and equipotential lines

The necessary steps to solve this exercise are shown in Figure 2.
The first step is to draw the equipotential lines of the field. To draw the lines one can follow the three following prescriptions:

· the gravitational potential of a spherical body has spherical symmetry and scale as

$$U(r) = -GM/r$$

· the gravitational potential of two or more bodies is the superposition of the gravitational potential of each body

· equipotential lines at different potentials can never cross

As a first consideration, on the surface of each body, the contribution of the gravitational potential of the other one can be neglected. This means that the potential near the surface is the spherical body unperturbed potential, see Figure 2a.

Far from the surfaces, the effective gravitational potential is the superposition of the two potentials, which can still be considered unperturbed far away from one of the body, but is heavily modified in the region between them and merge in the center, see Figure 2b.

As one move away from the system, the equipotential lines smooth out and envelope both the bodies, see Figure 2c.

Now that the equipotential lines are drawn, is possible easy to draw the field lines. By definition, field lines are perpendicular to the equipotential lines, and the density of field lines is proportional to the strength of the field. In Figure 2d the "trivial" lines are drawn.

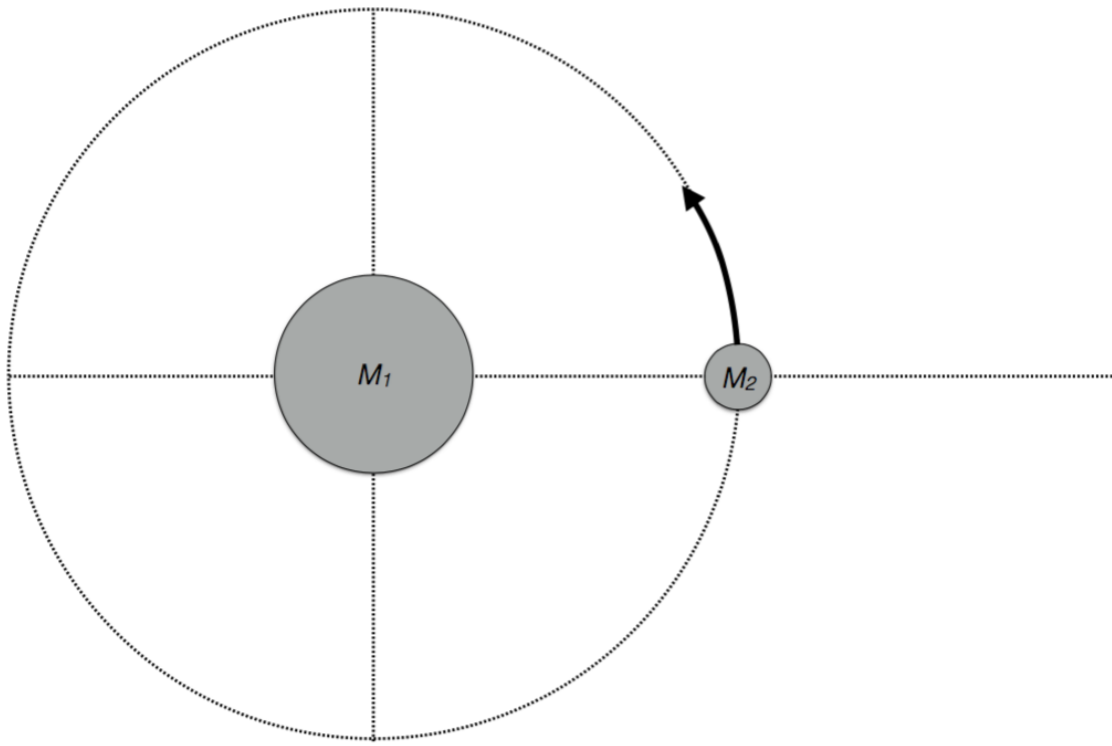

Figure 1: Two body system, M_2 gravitate around the central massive body M_1

Given the two body system, there is a point on the axis between the bodies where the sum of the gravitational pull of the two object is equal to zero. It can be shown that the distance between M_1 and the this point is:

$$x = \frac{d}{1 + \sqrt{\frac{M_2}{M_1}}} \approx 0.76d \tag{1}$$

Where d is the distance between M_1 and M_2. Using simple considerations one can deduce that a body close to this point with a speed *v=0* will not be attracted to it but will "fall" toward one of the two bodies, see Figure 2e. This is an unstable equilibrium point.

Following the prescription that field lines are always perpendicular to the equipotential lines, one add field lines in the region in between the two bodies, see Figures 2f and 2g.

The last task is to draw L_1. The Lagrangian point L_1 is close to the point on the axis between the bodies where the sum of the gravitational pull of the two object is equal to zero, but a little bit closer to the heavier object. This is because L_1 is *always* in between the two bodies, and is moving on a circular trajectory. To compensate for the centrifugal force, L_1 have to be closer to the heviest object.

Exercise 2. Dark matter and rotational velocity of galaxies

The first hint of the existence of dark matter was found looking at the rotational velocity of galaxies. Assuming galaxies are cylinders with a mass density profile that depends only on radius:

$$\rho(h, r, \theta) = \rho(r) \tag{2}$$

and hence the mass contained in a concentric cylinder with radius r is $M(r) = \int_0^r \rho(R)dR$. Assuming that the rotational velocity of the galaxy is a known function $v(R)$.

a) Calculate the function $M(r)$;

The centripetal force of a star in the galaxy is given by the gravitational force. Hence, we can say:

$$m\frac{v(R)^2}{R} = G\frac{mM(R)}{R^2} \tag{3}$$

Simplifying and rearranging the terms one obtain the following expression for *M(R)* as a function of R and *v(R)*:

$$M(R) = \frac{v(R)^2 R}{G} \tag{4}$$

b) In the approximation $v(r) = v_0$, find density mass profile $\rho(r)$.

In this approximation, we have:

$$M(R) = \frac{v_0^2 R}{G} \tag{5}$$

$$M(R) = \int_0^R \rho(r) \cdot 2\pi h r dr = \frac{v_0^2 R}{G} \tag{6}$$

derivating both members of the equation, we get:

$$\rho(R) \cdot 2\pi h R = \frac{v_0^2}{G} \Rightarrow \rho(R) = \frac{v_0^2}{2\pi RhG}. \tag{7}$$

Points (10 pt)

Exercise 1. 5 points divided as follow:
2 points for the correct drawing of equipotential lines (Figure 2c).
2 points for the correct drawing of the field lines (Figure 2f or 2g).
1 point for using equation (1) or doing analogous considerations before drawing L_1 (Figure 2h).
Exercise 2. 5 points divided as follow:
a) 2 points for using equation (3) and 1 point for consideration on the Gauss's law
b) 1 point for equation (6) and 1 point for equation (7)

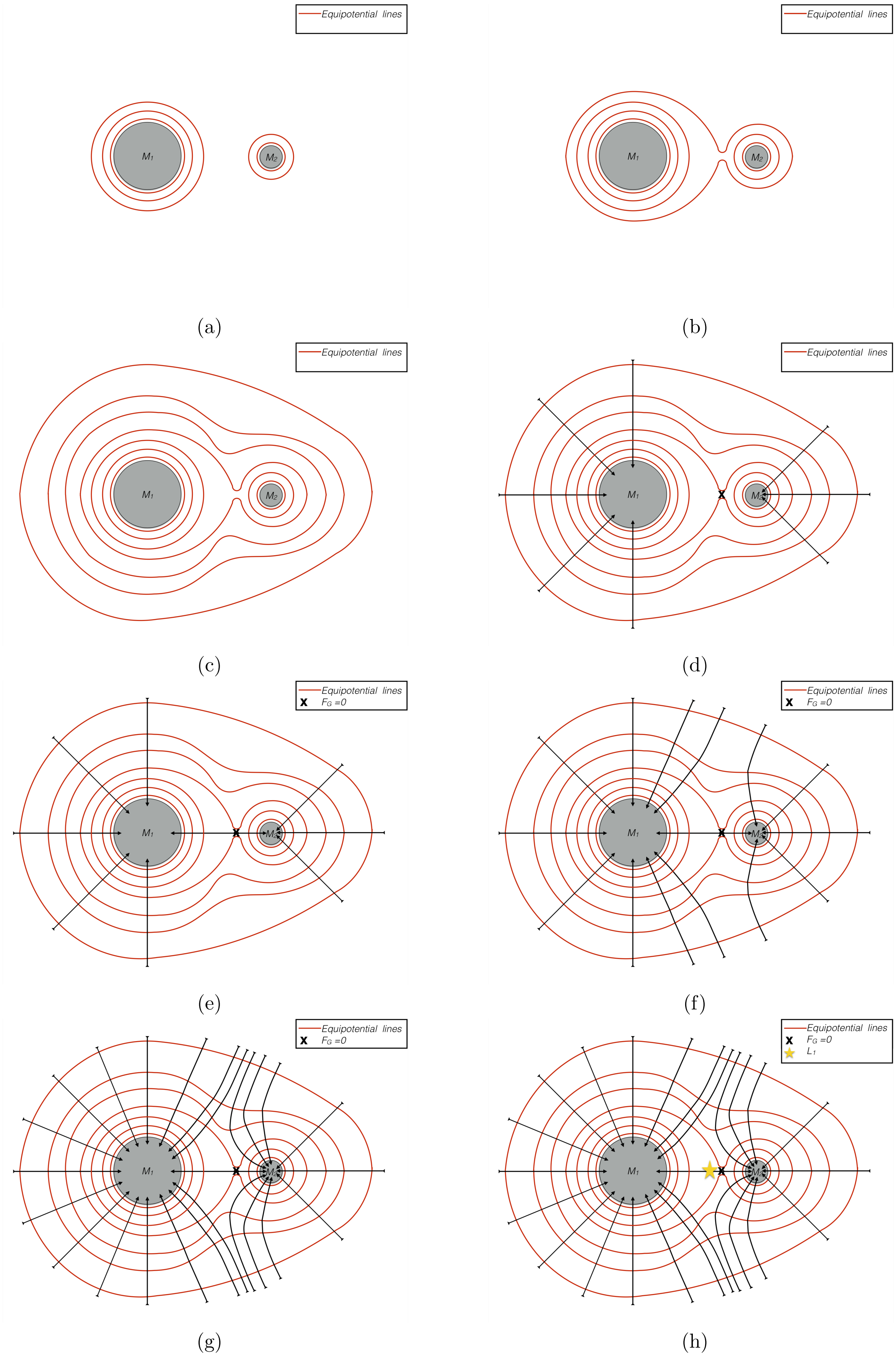

Figure 2: Equipotential lines and field lines for a two body system with $M_1 = 10\ M_2$

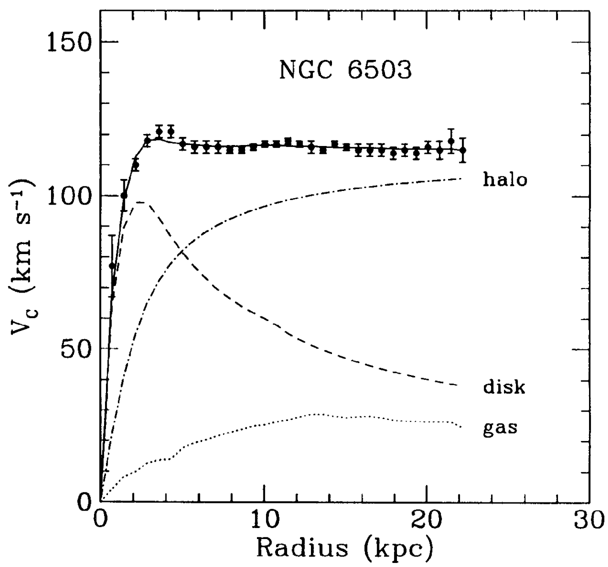

Figure 3: arXiv:0812.4005 [astro-ph].