

Part I

Student Laboratory Manuals

13. Torsion and bending of beams

13.1 Introduction

In this laboratory we approach the question of how to design a mechanical structure – a bridge, a crane – such that it will not collapse. Specifically we ask: what do we need to know about the construction material in order to predict stability?

We will not actually perform experiments to evaluate the limits of a design, but rather investigate deformations of structural elements, such as beams and shafts (which we somewhat indifferently just call *rods* below), as we bend and twist them. We shall see, that this allows us to determine parameters that may be used to calculate the response to any load.

In a *mechanics* course, as taken by physics students, structures of elements and bodies are assumed to be perfectly rigid; they are non-deformable. A general motion may thus be reduced to translations of the centre of mass and rotations about the centre of mass.

A rigid body is an idealisation, though, since in real world, macroscopic bodies are deformed under the action of a force or a torque. This would be the object of an engineering course in *solid mechanics*. Examples of different modes of deformation are *tension* and *compression*, *torsion*, *bending*, and *buckling*. The latter mode is characterised by a sudden failure of a structural member subjected to high compressive stresses. Examples of use of these terms are bending, or tension, or compression of a beam, buckling of a cylindrical steel column, torsion of a shaft, deflection of a thin shell. Within the science of applied mechanics, *mechanics of structures* is a field of study where the behavior of structures under mechanical load is investigated.

The required knowledge for this laboratory is Hooke's law¹ in its most basic form (the spring equation):

$$F = k x \tag{13.1}$$

This relation, here in its scalar form, applies to a spring extended to length x from its relaxed position under the load of a force F . The relation in this simple form requires that the spring is *linear elastic*, which is another way of saying that the force F is proportional to x through the scalar k , the *spring constant*. The spring is linear elastic for not too large extensions. We shall see that we can apply an equivalent reasoning to the deformation of material bodies.

¹So named after the 17th century British physicist and polymath Robert Hooke. First formulated in 1676 as a Latin anagram (ceiinossssttuu), whose solution he published in 1678 as *Ut tensio, sic vis* (As the extension, so the force).

In this laboratory we work solely in the *elastic* region of structural elements subjected to various loads: Torsional load of a cylindrical shaft, and bending of a beam with square cross section. We investigate the dependence of deformation on known external loads, and in addition explore the importance of different cross-sectional shapes and choice of different materials.

We study only cases where applied loads and shape of the beam show nice symmetries. General cases of orientations of forces and cross sections would only become much more difficult to analyse.

We further assume that the elements investigated are always isotropic. This is not generally the case, since the properties of a material may depend on molecular properties, as may be the case for polymers, or on manufacturing procedures that favour preferential orientation, as for metals subjected to surface treatment such as milling and grinding.

13.2 Theory

In order to discuss the fundamentals of deformation, we introduce here two basic concepts for the analysis of mechanical stability in solid mechanics: *stress* and *strain*.

A structure element is said to be mechanically stable if it remains in equilibrium under load. This requires a balance between externally applied forces and internal reactions. The element may deform under the external load, but it will not break if in equilibrium.

Stress is the internal reaction to an external load.

Strain is the *relative deformation* of an object.

13.2.1 Strain and stress

Starting from quite general principles we might think of a continuum body as an assembly of points through which we imagine reference lines in a three dimensional grid. This allows us to consider deformations and the corresponding internal reactions in any direction: changes in length of lines (i.e. normal strain) and changes in angles between pairs of lines initially perpendicular to each other (i.e. shear strain).

For such general cases it is sufficient to know the normal and shear components of strain on a set of three mutually perpendicular directions, but we would have to treat material properties as tensors. This approach requires a somewhat more involved analysis than we need in this laboratory, and we chose here to demonstrate the principles by using one-dimensional cases, where the material properties are scalars.

Normal strain – a measure of deformation along a line of force

If a one-dimensional body (e.g. a rod) of length l is stretched or compressed by forces along to the axis, attaining the new length l' , then the deformation is $\Delta l \equiv l' - l$ and the *strain* ε is a measure of the deformation, representing the relative displacement along a line of the material body, i.e. normal strain,

$$\varepsilon \equiv \frac{l' - l}{l} \equiv \frac{\Delta l}{l} \quad (13.2)$$

The terms *Cauchy strain* or *engineering strain* are also used synonymously for ε . Obviously strain is a dimensionless quantity, since it is the ratio of a difference in length and a length; strain can be expressed as a decimal fraction, a percentage or in parts-per million.

In particular, in the case of an increase in length, the normal strain is called *tensile strain*; if there is reduction in length, (compression) it is called *compressive strain* (sometimes tensile strain is used for both).

We mention here in passing, that the *stretch ratio* or *extension ratio* is sometimes used as a measure for normal strain. It is defined as the ratio between the final length and the initial length of the material line.

$$\lambda = \frac{l'}{l} \quad (13.3)$$

Shear strain – deformation when a body is twisted or cut

If a body is subjected to a pair of transverse forces (not acting along the same line) or a torque, the body is deformed, not along lines, but over sliding parallel planes. The amount of distortion associated with the attempted sliding of plane layers over each other, is called *shear strain*. It may be described as change of angles between lines within a deforming body.

The *engineering shear strain* is defined as the change in the angle between two material line elements initially perpendicular to each other in the undeformed or initial configuration.

Stress – a measure of internal forces

The external force, divided by the area to which the force is applied, is equal to the internal reaction that we call stress, if stability is maintained. The unit is therefore the same as for pressure: force divided by area, or pascal (or Nm^{-2}). Common practical units are megapascals (MPa or N/mm^2) or gigapascals (GPa or kN/mm^2).

If the deformations are recovered, i.e. if the material returns to its initial shape after the external forces have been removed, we speak of *elastic deformations*. This is the range of stress up to a certain threshold value known as the elastic limit or *yield stress*, sometimes *yield strength*. Keep in mind, that elastic deformations are not necessarily linear (as prescribed by Hooke's law). In fact, linear elastic deformation, exist for most materials only in a small part of the region of elastic deformation.

Irreversible deformations, i.e. deformations remaining after the external load has been removed, are called *plastic deformations*. On a microscopic level these result from slip between grains, or dislocations at the atomic level. For many engineering applications plastic deformation is unacceptable, and the yield strength is used as the design limitation. In other cases, for instance when bolts are tightened for certain critical applications, as e.g. bearing caps in internal combustion

engines, bolts may be specified to be tightened with a torque that exceeds the yield strength in order to obtain maximum stability for a given size of bolt. Such bolts cannot be reused, but has to be discarded and replaced if removed.

If the external loads increase beyond the limit where the internal stress cannot respond, the structure will ultimately experience failure.

a) Normal stress and shear stress; elastic modulus and shear modulus – Hooke’s law

Already in the introduction we stated the most simple form of Hooke’s law, the spring equation:

$$F = k x \quad (13.4)$$

We shall see below that this is a generic form for equations describing normal deformation as well as shear deformation.

Our object here is to describe the internal reactions set up in a structural element because of external loads. In particular we want to be able to answer the question: How much deformation can we expect, knowing the load applied?

In order to do so, we apply the method of making an imaginary plane cut through the structural element – the beam, the shaft, or whatever element we are working with – at some material point where we would like to analyse the internal loads (we may want to make certain that the yield stress is not exceeded).

The principle is to study any of the two parts that was separated by the cut, but choose the one that is most suited because of shape and external forces given. At each point of the cut we have to compensate for the missing part with a force that acts to represent that part, such that there is no change in the balance of forces on the part of the element chosen for study. At each point of the cut, which now appears at both of the separated parts, mutually opposite forces and torques, formally cancels, which is necessary since the cut is only imaginative, and should not add anything to any of the parts separated by the cut.

Since an external force may vary over a given surface, both in magnitude and direction, a general analysis requires a force $d\vec{F}$ acting on the differential surface element dA . The force, being arbitrary in direction, may be decomposed in a normal component dF_n perpendicular to the surface and a tangential component dF_t parallel to it, cf Fig. 13.1. The normal strain σ and the shear strain τ are now defined as:

$$\sigma = \frac{dF_n}{dA} \quad \text{and} \quad \tau = \frac{dF_t}{dA} \quad (13.5)$$

In the following we leave the generality aside, and resort to simple one-dimensional cases, as the one shown in Fig. 13.2: A rod of length l with constant circular cross section A is extended by a force F (we may think of the rod as an actual test sample used in typical yield strength test, where the test rod is pulled in a machine with a controlled, gradually increasing force until the sample changes shape or breaks). In any cross section perpendicular to the axis of the rod there is a normal stress $\sigma = F/A$, which is approximately constant over the cross section, as long as deformation is

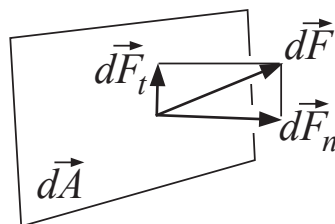


Figure 13.1: Definition of normal strain and shear strain.

not too big. The shear stress is approximately zero in the perpendicular cross section, again until plastic deformation starts become appreciable.

The external load brings about a change in length Δl of the rod. If the force is not too large, the material behaves *linearly elastic*, and Hooke's law applies, meaning that the relative change in length $\epsilon = \Delta l/l$ is proportional to the magnitude of the force applied and thus to the normal stress σ .

For the case of normal stress, the relation we require is not between an external force and the actual elongation of a spring. We state here the equivalent form for the material parameters introduced above.

$$\sigma = E \epsilon \quad (13.6)$$

Recall from above Hooke's law, the spring equation, a comparison tells us, at glance, that σ corresponds to the force F , and that ϵ corresponds to elongation x .

The elastic properties of a structural material under load may be conveniently assessed in a diagram of stress versus strain. The constant of proportionality in Hooke's law is then defined as the slope of the stress-strain curve in the elastic deformation region:

$$E = \frac{\text{stress}}{\text{strain}} \quad (13.7)$$

where E (sometimes lambda) is the elastic modulus. For the case of an isotropic medium, (or one dimensional sample), as is the case in this laboratory, E is a scalar, sometimes called the *Young's modulus*², also designated Y . It describes uniaxial, or isotropic tensile elasticity, or the tendency of an object to deform along an axis on which opposing forces are applied. Young's modulus is a measure of the stiffness of an isotropic elastic material. In a general case the modulus of elasticity is a tensor.

²Thomas Young, 19th century British scientist. The concept was developed in 1727 by Leonhard Euler, and the first experiments were performed by the Italian scientist Giordano Riccati in 1782, 25 years before the work of Young.

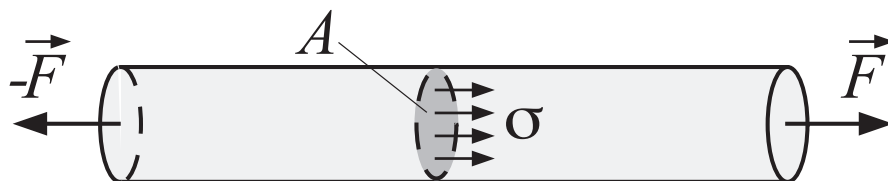


Figure 13.2: Extension of a rod.

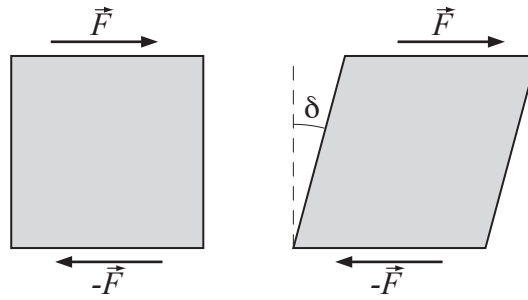


Figure 13.3: Schematic diagram of shearing of a cube with surface area A .

Using the result above we may write:

$$E = \frac{\sigma}{\varepsilon} \quad (13.8)$$

The modulus of elasticity is a materials constant of great importance in solid mechanics and mechanical design.

We may define other elastic moduli, each describing in different ways the tendency of an object to deform elastically (i.e., non-permanently) when a load is applied.

If a cube is subjected to forces parallel to one of its faces, as shown in Fig. 13.3, the amount of shearing may be defined using a certain angle δ as in the figure. Any surface cut parallel to the faces where the forces act (horizontal cuts in the figure) experiences a shear stress $\tau = F/A$, whereas the normal stress is zero. The shear stress is actually a tensor entity, but remaining with our one-dimensional examples, we abstain from developing this further. If the external forces F are not too large, the shearing angle δ is proportional to the magnitude of the force, and therefore to the shear stress τ . The constant of proportionality is the *shear modulus* or *modulus of rigidity*, commonly designated with symbols G or μ .

For shear stress the equivalent Hooke's law expression becomes

$$\tau = G \delta \quad (13.9)$$

We may take as a formal definition of shear modulus:

$$G \equiv \frac{\tau}{\delta} \quad (13.10)$$

The shear modulus (or modulus of rigidity) G describes an object's tendency to *shear* (the deformation of shape at constant volume) when acted upon by opposing forces that are not in-line; it is defined as shear stress over shear strain. The shear modulus is part of the derivation of viscosity.

Just as the modulus of elasticity, G is an important materials constant, either tabulated or made directly available in CAD software. The generic definition above again gives this parameter as shear stress over shear strain. The shear modulus has important implications in the derivation of a theory of viscosity of fluids.

For the sake of completeness, we finally mention the *bulk modulus* K , which describes volumetric elasticity, or the tendency of an object's volume to deform under pressure; it is defined as volumetric

stress over volumetric strain, and is the inverse of the compressibility, another important material's property. As an example, the bulk modulus is of use for comparison of actual data with computer simulations of molecular processes (molecular dynamic, MD). It is a generalisation of Young's modulus to three dimensions.

b) Torsion of a cylindrical rod

A cylindrical rod with length l and radius R is fixed and rigidly supported at one end, and may be loaded at the other end with an axial torque, as shown in Fig. 13.4. If the rod under load is in equilibrium, the external torque is balanced internal by the shear stress. The shear stress may be seen as acting in each imaginary perpendicular cut with a torque equal but opposite to the external torque.

Twisting a rod by applying an external torque makes it respond as in Fig. 13.5. A volume element is shown, that in the relaxed (unloaded) equilibrium state would be a strip cut from an annular cylinder (cylindrical shell).

As the external axial torque M_0 twists the rod, a section perpendicular to the axis at a distance x from the end rotates through the angle $\phi(x)$. With the chosen geometry the angle increases along the rod with position x . If we would draw a straight line on the surface of the relaxed rod, parallel to the axis, then this line would be a helix (screw shaped line), making everywhere an angle δ to the undisturbed straight line.

From a simple geometric consideration, we may write, using the approximation $\tan \delta \approx \delta$, valid for small angles:

$$x \cdot \delta = r \cdot \phi \quad (13.11)$$

The volume element shown in Fig. 13.5 is sheared through the angle δ , and the corresponding shear stress τ acts at the end surface. The situation is equivalent to that for the cube in Fig. 13.3.

For not too large a torque Hooke's law is valid in its shear form

$$\tau = G \cdot \delta = G \cdot \frac{r \cdot \phi}{x} \quad (13.12)$$

For the volume element in Fig. 13.5 to be in equilibrium, the external torque M_0 and the internal reaction τ must be equal. For simplicity we calculate the internal reaction relative to a point P :

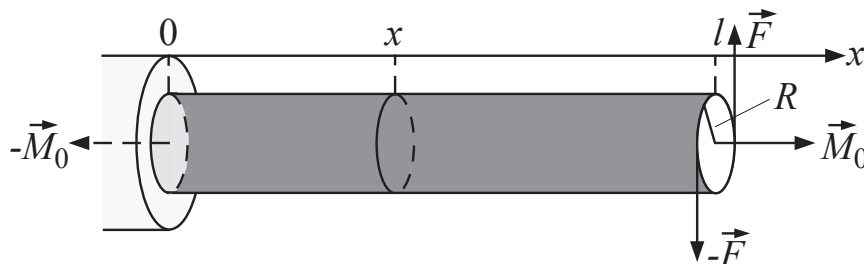


Figure 13.4: Torsion of a cylindrical rod – schematic overview.

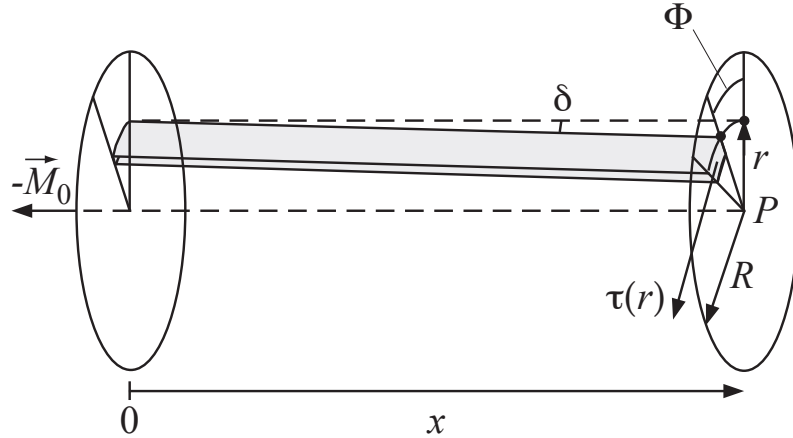


Figure 13.5: Torsion of a cylindrical rod under axial load – analysis shown for a twisted volume element.

$$M_0 = \int r \cdot \tau dA = \frac{G \cdot \phi(x)}{x} \cdot \int r^2 dA = \frac{G \cdot \phi(x)}{x} \cdot \int_0^R r^2 \cdot 2\pi \cdot r dr = \frac{G \cdot \phi(x)}{x} \cdot \frac{\pi \cdot R^4}{2} \quad (13.13)$$

Note, that the variable x is a constant in this integration, which is performed with the surface element dA .

This allows us in particular to extract an expression for the angle of deformation $\phi(x)$, and evaluate at the end of the rod, putting $x = l$:

$$\phi(l) = \frac{2l \cdot M_0}{\pi \cdot G \cdot R^4} \quad (13.14)$$

If the rod is not a solid cylinder, but an annular (hollow) cylinder, then the integration in Eq. 13.13 is performed over a range from the inner radius R_i to the outer R_a , yielding:

$$\phi(l) = \frac{2l \cdot M_0}{\pi \cdot G \cdot (R_a^4 - R_i^4)} \quad (13.15)$$

d) Bending of a rectangular section rod

A rod of length l with a rectangular cross-sectional area A , as in Fig. 13.6, is rigidly fixed at one end, while the other is unsupported. At the free end an external force \vec{F}_0 may be applied perpendicular to the symmetry line of the rod. Equilibrium requires again that external forces are balanced by internal reactions.

In Fig. 13.7 a situation is shown when the rod is being acted on by the external force \vec{F}_0 .

This is called a *bending moment*, by physicists also *torque*³. In the rod there will be both tensile stresses and compressive stresses; the upper side the material will be stretched, the lower side will be

³For mechanical engineers, a torque is created by two equal but opposite forces not in-line, also called a *couple*. There is no resulting translational force in this case, as there is for the rod discussed here.

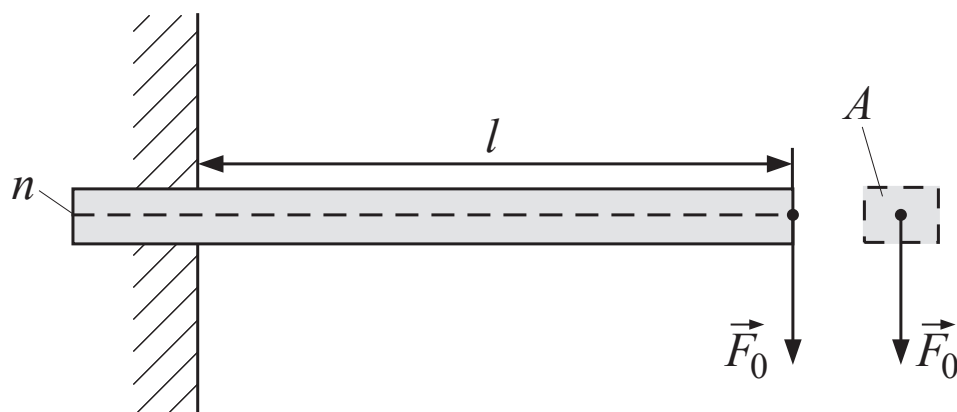


Figure 13.6: Rod with rectangular cross section before force is applied.

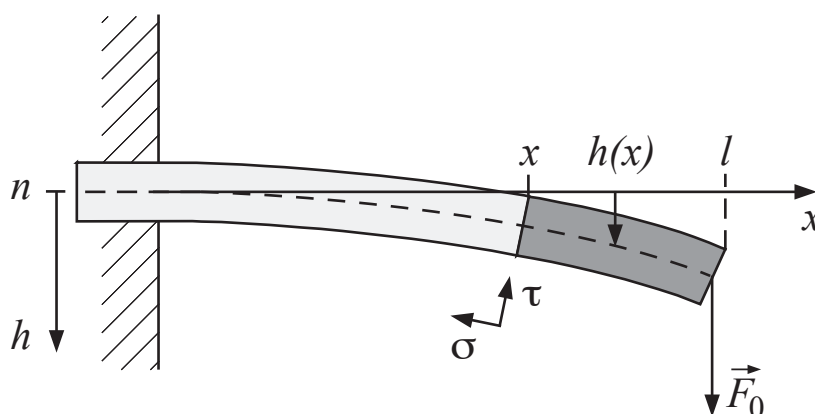


Figure 13.7: Bending of a rod with rectangular cross section under the action of a perpendicular force.

compressed. This means that in the rod there must be an un-deformed, 'neutral' line (or surface), often called the *neutral axis*, n . A cross section perpendicular to n will remain perpendicular independent of the bending, as long as the material remains in the elastic region.

Failure in bending will occur when the bending moment is sufficient to induce tensile stresses greater than the yield stress of the material, either in extension or compression. Failure because of shear may occur before failure in bending.

A cross-sectional surface perpendicular to the neutral axis is subject to a constant shear stress τ , (whereas the normal stress varies because of the bending).

$$\tau = \frac{F_0}{A} \quad (13.16)$$

The external force F_0 is thus compensated by stress as long as the rod remains in equilibrium. cf Fig. 13.7.

A formula for the deformation

Next our purpose is to develop expressions that connect the measurable deformation with the parameters of stress introduced above. Macroscopic equilibrium in particular means that external forces and moments must be globally balanced. The equation for equilibrium of the moments for the rod would look like this:

$$M_0 = F \cdot l \quad (13.17)$$

where M_0 is the moment at the point of fixture in the wall.

Since we are not dealing with a rigid mechanics problem, this relation is of little use. We therefore apply the same reasoning to the part of the rod shaded grey in Fig. 13.7, and separated from the rod by an imaginary perpendicular cut at co-ordinate x . By doing so, we are able to introduce internal stress into the equilibrium equations.

First, we write the global equilibrium for the piece of the rod cut at x :

$$M(x) = F_0 \cdot (l - x) \quad (13.18)$$

This relation is valid for any cut at a position x . The moment $M(x)$ varies with the co-ordinate x and assumes its maximum at $x = 0$, i.e. at the position where the rod is rigidly fixed.

If the cross section of the rod is constant and equal in form everywhere, we the bent rod assumes the form of a circular shape with constant radius R . We introduce the radial co-ordinate z , with origin at the neutral axis of Fig. 13.8.

The strain ε , the relative change in length, may be written:

$$\varepsilon(z) = \frac{(R + z) \cdot \phi - R \cdot \phi}{R \cdot \phi} = \frac{z}{R} \quad (13.19)$$

The normal strain $\varepsilon(z)$ is accompanied by a corresponding normal stress $\sigma(z)$

In the elastic range Hooke's law is valid, and we write, using that the relative change of length (the normal stress) is proportional to the normal strain, with the elastic modulus E as factor of proportionality:

$$\sigma(z) = E \cdot \varepsilon(z) = E \cdot \frac{z}{R} \quad (13.20)$$

In an infinitesimal area element dA , cf Fig. 13.8, the normal strain is $\sigma(z)$ at distance z from the neutral axis (the origin). Calculating the moment relative to the neutral axis, the moment arm is likewise z , so that the infinitesimal moment contributed by area element dA is $z \cdot \sigma(z)$. Integrating over the cross section A , and inserting $\sigma(z)$ from Eq. 13.20, we get the total moment for the neutral axis:

$$M'_0 = \int_A z \cdot \sigma(z) dA = \frac{E}{R} \cdot \int_A z^2 dA = \frac{E}{R} \cdot I_z \quad (13.21)$$

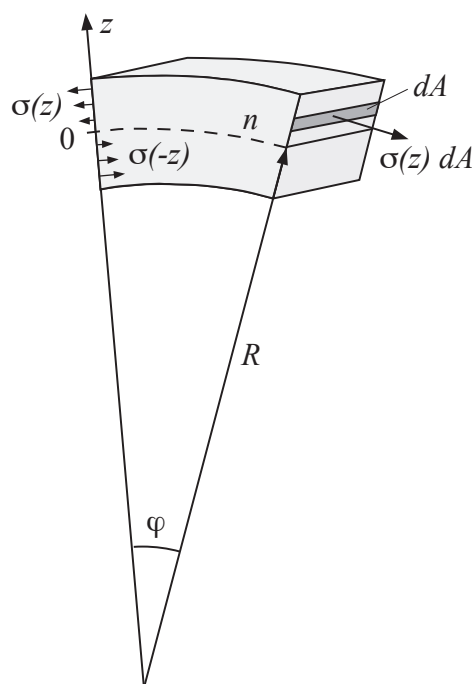


Figure 13.8: Bending a rectangular rod – detail.

Appearing in this expression is the so called *area moment of inertia* or *second moment of inertia* $I_z = \int_A z^2 dA$, (sometimes the second moment of area), making a short digression due here.

The resistance of a beam to bending and deflection depends not only on the load but also on the geometry of the beam's cross-section, or more precisely, how area is distributed in the cross section.

A rectangular beam of width b and thickness d in the plane of bending has the area moment:

$$I_z = \frac{b \cdot d^3}{12} \quad (13.22)$$

A beam with I-shaped cross section will have a slightly smaller area moment. We might think of it as cut out of the rectangular beam, its profile fitting inside. The much smaller mass, however, in the end gives a stiffer and lighter structure, which is a consequence of the relatively large area far from the neutral line. For this reason, beams with higher area moments of inertia, such as I-beams, are often preferred in building construction, as opposed to other beams with the same area⁴.

Despite the apparent likeness, the second moment of area is not at all the same thing as the moment of inertia, as used, i.e. in Newton's second law for calculating angular acceleration. As a word of warning, It is not uncommon to refer to the second moment of area as the moment of inertia and use the same symbol I for both. It is usually clear from the context, though, if accelerational or bending moment of inertia is meant, and it becomes obvious from the units: second moment of area has unit length to the fourth power, whereas a moment of inertia has the unit mass times length squared.

⁴For the case of torsional load, the *polar moment of inertia*, characterises an object's ability to resist torsion.

In order that the structure element be in equilibrium, we require that

$$M_0'(x) = M_0(x) \quad \text{equivalent to} \quad \frac{E}{R(x)} \cdot I_z = F_0 \cdot (l - x) \quad (13.23)$$

For large bending radii R , i.e. a small deformation, we approximately have:

$$\frac{1}{R} = \frac{d^2h}{dx^2} \quad (13.24)$$

with $h(l)$ the deflection from the centre line at the free end, cf Fig. 13.7, and $h(x)$ the deflection at co-ordinate x . Inserting in Eq. 13.23 yields a differential equation

$$\frac{d^2h}{dx^2} = \frac{F_0}{E \cdot I_z} \cdot (l - x) \quad (13.25)$$

which is twice integrated to give:

$$h(x) = \frac{F_0}{E \cdot I_z} \cdot \left(\frac{l \cdot x^2}{2} - \frac{x^3}{6} \right) + C_1 \cdot x + C_2 \quad (13.26)$$

The two integration constants may be determined from the boundary conditions at $x = 0$ where the beam is fixed. Here we have $h(0) = 0$ and $\frac{dh}{dx} = 0$, from which follows $C_1 = C_2 = 0$. The maximum deflection at the end ($x = l$) is then

$$h(l) = \frac{F_0}{E \cdot I_z} \cdot \left(\frac{l^3}{2} - \frac{l^3}{6} \right) = \frac{F_0 \cdot l^3}{3E \cdot I_z} \quad (13.27)$$

13.3 Experimental

In this laboratory we use the relations 13.14 for an external torque and the corresponding torsion angle, and 13.27 for an external force and the deflection. Our purpose is to determine the elastic constants for the materials used.

a) Torsion of a cylindrical rod

In the first part of the experiments, different cylindrical rods are loaded with external torque and the torsion angle $\phi = \phi(l)$ measured at the end $x = l$ of the rod. The torsion angle may be read directly from an angular scale. In addition, provision is made for the use of a laser beam to determine the deflection to calculate the angle. The assistant explains the use of the second method. Try both methods and compare the results. Discuss with the assistant pros and cons of the two methods.

The external moment M_0 is produced with the aid of masses m attached to the periphery of a disk with radius R_s , fastened to the end of the rod. Then $M_0 = m \cdot g \cdot R_s$ and Gl. 13.14 gives us:

$$\phi = \frac{2l \cdot g \cdot R_s}{\pi \cdot G \cdot R^4} \cdot m \quad (13.28)$$

The masses are chosen for each rod so that the torsion angle ϕ never increases beyond 15° and the deformation remains linear elastic.

Relevant parameters for the available rods are tabulated in table 13.1. Using these rods

- determine the shear modulus G of aluminium, brass and iron (Rods #1, #2 and #3),
- verify the R^4 -dependence of the torsion angle (Rods #3 and #4), and
- compare a solid cylinder and an annular cylinder of approximately the same cross section (Rods #4 and #5).

While mounting the rods, some pre-stress should be attempted in order that a reproducible zero angle is attained each time the rod is relaxed (the torque unloaded).

- Use a micrometer to verify the diameters of all rods with best possible precision. Use these measured values in the calculations, not the approximate values in table 13.1.

Table 13.1: Cylindrical rods for torsion experiments

Rod	material	length [mm]	inner radius [mm]	external radius [mm]
1	Aluminium	750	-	2.5
2	Brass	750	-	2.5
3	Iron	750	-	2.5
4	Iron	750	-	2.75
5	Iron	750	1	3.0

Determination of the shear modulus of aluminium, brass and iron

- Mount aluminium rod #1, allowing for the required pre-stress.
- Load the rod with a torque consecutively using weights of mass 0.25 kg, 0.5 kg and 0.75 kg, and measure the corresponding torsion angle ϕ at the end of the rod.
- Draw a diagram of angle ϕ as function of mass m . Draw the best fitting straight line for the points representing the measurements. Note, that the line not necessarily have to pass through the origin (why?). Determine G from the slope. Estimate the error in G .
- Repeat the measurement with rod #2, using masses 0.25 kg, 0.5 kg, 0.75 kg and 1.0 kg.
- Repeat the measurement with rod #3, using masses 0.5 kg, 1.0 kg, 1.5 kg, and 2.0 kg.

Verifying the R^4 -dependence of the torsion angle

- Mount the thin iron rod #3, allowing for pre-stress.
- Load the rod with a torque using masses 0.25 kg and 2.0 kg and measure the torsion angle ϕ at the end of the rod.
- Repeat the experiment for the thicker iron rod #4. Use the same masses as for rod #3.
- Compare the torsion angle for both rods for equal torques. Verify that

$$\frac{\phi_4}{\phi_3} = \frac{R_3^4}{R_4^4} \quad (13.29)$$

Comparison of a solid cylinder shape and annular cylinder shape

- Mount the solid cylinder shaped rod #4 with pre-stress applied as before.
- Load the rod with a torque, using masses 0.25 kg and 2.5 kg. Measure the torsion angle ϕ at the end of the rod.
- Repeat the measurement for the annular cylinder shaped rod #5. Use the same masses as for the solid cylinder shaped rod #4.
- Compare the torsion angle of both rods at equal torsional load. Verify that:

$$\frac{\phi_4}{\phi_5} = \frac{R_{5,a}^4 - R_{5,i}^4}{R_4^4} \quad (13.30)$$

- Calculate the cross sectional area of both rods from the measurements and discuss the ratio of mass to stiffness.

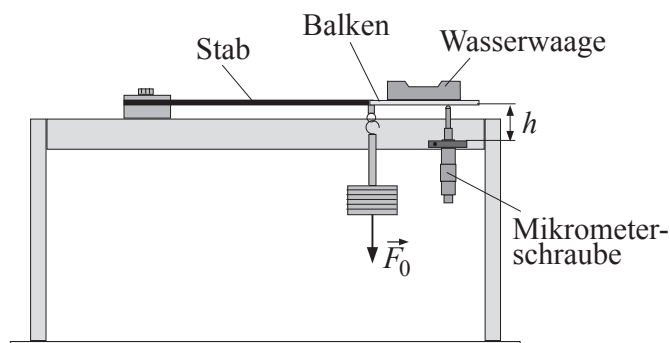


Figure 13.9: Set-up for the measurement of the deflection of a rectangular rod.

b) Bending of a rod with rectangular cross section

In the second experiment rectangular rods are loaded with known forces and the deflection $h = h(l)$ at the end of the rod measured. The set-up is shown in Fig. 13.9. The deflection of the rod is measured with the help of a beam that is connected to the rod at one end, and a micrometer at the other. A water level is used with the micrometer to position the beam horizontally. The vertical position read from the micrometer represents the position of the rod end, and any change in the reading for a different load is the corresponding deflection of the rod. The external force F_0 is applied by using weights of known masses m that are attached to the end of the rod. Therefore $F_0 = m \cdot g$, and with Eq. 13.27 and Gl. 13.22 we have:

$$h = \frac{12g \cdot l^3}{3E \cdot b \cdot d^3} \cdot m \quad (13.31)$$

The weights are chosen for each rod such that the deflection does not exceed the linear elastic range.

Relevant parameters for the rods used in the experiments may be found in Table 13.2. Using these rods:

- determine the elasticity modulus E for steel, aluminium, and brass (rods #3, #6 and #7),

Table 13.2: Rectangular rods for deflection experiments

Rod	material	length [mm]	width [mm]	height [mm]	wall thickness [mm]
1	Steel	250	8	6	-
2	Steel	350	8	6	-
3	Steel	350	6	8	-
4	Steel	300	16	4	-
5	Steel	300	15	10	1.5
6	Aluminium	350	6	8	-
7	Brass	350	6	8	-

- verify the l^3 -dependence of the deflection (rods #1 and #2).
- compare deflections for rods with different cross-sectional shapes but approximately equal area. (Rods #2, #3, #4 and #5).

For all measurements pay attention to the following:

- Using a micrometer, measure the dimensions of all rods
- With the rods mounted, measure the length l of all rods from the point of fixture to the free end.
- For all calculations, use measured values of the parameters, not the approximate values in Table 13.1.

Determination of the elastic modulus of steel, aluminium and brass

- Mount rod #3 (steel) in the measurement set-up without a load. Adjust the micrometer using the water level and determine the reference position.
- Load the rod consecutively with weights of masses 0.5 kg, 1.0 kg, 1.5 kg und 2.0 kg. For each load read the position indicated by the beam adjusted with the micrometer to horizontal position with the water level. Calculate the deflection from the reference position.
- Draw a diagram of the deflection h as function of mass m . Fit a best straight line to the experimental points, and determine the elasticity modulus E from the slope. Estimate the error of E .
- Repeat the measurement with rod #6 (aluminium), using masses 0.25 kg, 0.5 kg, 0.75 kg, 1.0 kg and 1.25 kg.
- Repeat the measurement with rod #7 (aluminium), using masses 0.25 kg, 0.5 kg, 0.75 kg, 1.0 kg, 1.25 kg and 1.5 kg.

Verification of the l^3 -dependence of the deflection

- Mount the short rod #1 (steel) and read the position without external load.
- Load the mounted rod with a weight of mass 1.0 kg. Measure the position and calculate the deflection h of the end.
- Repeat the measurement for the longer rod #2 (steel). For the calculation, use the same mass as for rod #1.
- Compare the deflections of the two rods and verify that for equal load

$$\frac{h_2}{h_1} = \frac{l_2^3}{l_1^3} \quad (13.32)$$

Dependence of the deflection on the area moment of inertia

- Mount the flat rod #2 (steel) and measure the reference position without external load as before.
- Load the rod with a weight of mass 1.5 kg and measure the position and calculate the deflection h at the end of the rod.
- Repeat the measurement for rods #3, #4 and #5. Use the mass of rod #2 in the calculations.
- Calculate the area moment of inertia of the four rods according to Eq. 13.22 (consider the area moment of inertia for a hollow rectangular cross section, and devise a method in analogy with Eq. 13.22 for its determination. Verify that for equal load the ratio of the deflections for the two possible orientations i and j ('portrait' and 'landscape'), that the following relation holds:

$$\frac{h_i}{h_j} = \frac{I_{z,j}}{I_{z,i}} \quad (13.33)$$

- Calculate the cross section of the four rods and discuss the influence of the cross-sectional shape and the ratio of mass of the rod to the stiffness.