11. Coupled Oscillations

11.1 Introduction

Oscillations of multiple interlinked coupled systems play an important part in almost all areas of physics (see Experiment R). The major phenomena appear already in the coupling of only two systems. In this experiment, mechanical linear oscillators are used as oscillatory systems. The goal of the experiment is to characterise the coupled system with the aid of so-called normal modes and how these are related to the resonance frequencies of the individual system as well as the coupling between the systems.

11.2 Theory

a) The Linear Oscillator

The linear oscillator consists of a mass m, that is attached to a spring with spring constant k and that can move in one dimension, e.g. along the x-axis (see fig. 11.1a).

Assuming that the spring is subject to tension in Hooke's range, meaning that the restoring force F acting on m is given by $F = -k \cdot x$, the equation of motion (ignoring friction forces) is given by

$$m \cdot \ddot{x} = -k \cdot x \tag{11.1}$$

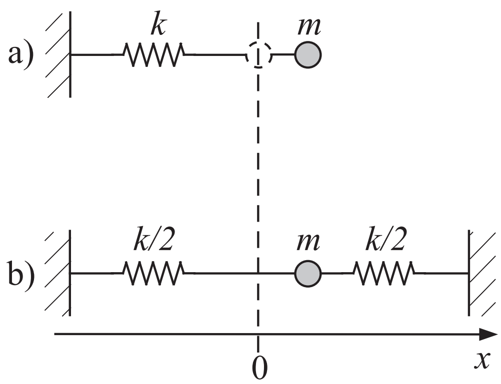

Figure 11.1: Linear oscillator

with the solution being

$$x = A \cdot \cos\left(\omega_0 \cdot t + \delta\right) \quad \text{with} \quad \omega_0 = \sqrt{\frac{k}{m}} \tag{11.2}$$

The oscillator oscillates purely harmonically with a characteristic frequency ω_0, the resonance frequency of the system. The two integration constants, namely the amplitude A and phase δ are set by the initial conditions. Out of practicality, instead of a single spring, there are two symmetricallly aligned springs in use for each oscillator in the experiment. The above relations are still valid if the two springs have the spring constant $k/2$ (see fig. 11.1ab).

b) Two Coupled Oscillators

Looking at the oscillations of two oscillators, which are coupled by a spring (see fig. 11.2), the two equations of motions for the masses m_1 and m_2 are

$$m_1 \cdot \ddot{x}_1 = -k_1 \cdot x_1 - k' \cdot (x_1 - x_2) \tag{11.3}$$

$$m_2 \cdot \ddot{x}_2 = -k_2 \cdot x_2 - k' \cdot (x_2 - x_1) \tag{11.4}$$

Assuming that both oscillators are alike, meaning $m_1 = m_2 = m$ and $k_1 = k_2 = k$, then

$$m \cdot \ddot{x}_1 = -k \cdot x_1 - k' \cdot (x_1 - x_2) \tag{11.5}$$

$$m \cdot \ddot{x}_2 = -k \cdot x_2 - k' \cdot (x_2 - x_1) \tag{11.6}$$

This coupled system of equations can be solved in different ways. In the following, a descriptive ansatz is described.

Experimental observation shows, that harmonic solutions exist in which both masses oscillate at the same frequency. This allows for the following ansatz:

$$x_1 = A \cdot e^{i\,\omega \cdot t} \tag{11.7}$$

$$x_2 = B \cdot e^{i\,\omega \cdot t} \tag{11.8}$$

where A and B are complex amplitudes (they contain the phase constant).

By inserting eq. 11.8 in eq. 11.5 and 11.6, it follows

$$(\,-m \cdot \omega^2 + k + k'\,) \cdot A - k' \cdot B = 0 \tag{11.9}$$

$$-k' \cdot A + (\,-m \cdot \omega^2 + k + k'\,) \cdot B = 0 \tag{11.10}$$

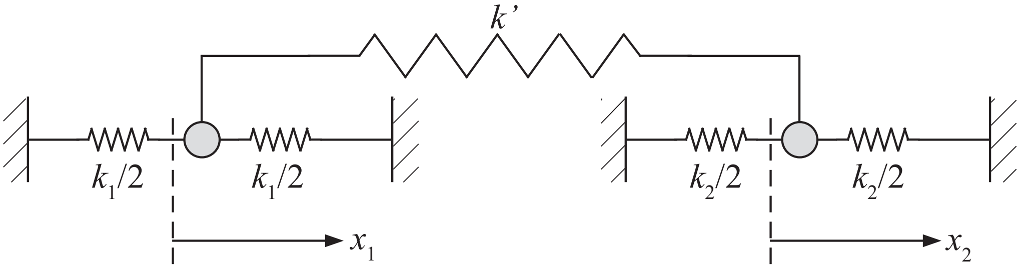

Figure 11.2: Two coupled linear oscillators

This system of two linear homogeneous equations for A and B has only a non-trivial solution for a vanishing determinant of the coefficient matrix, i.e.

$$\begin{vmatrix} -m \cdot \omega^2 + k + k' & -k' \\ -k' & -m \cdot \omega^2 + k + k' \end{vmatrix} = (\,-m \cdot \omega^2 + k + k'\,)^2 - k'^2 = 0 \tag{11.11}$$

There are thus two frequencies that satisfy the ansatz, namely the two positive solutions of the quadratic eq. 11.11

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{k + 2\,k'}{m}} \tag{11.12}$$

From eq. 11.9 follows

$$B = \frac{-m \cdot \omega^2 + k + k'}{k'} \cdot A \tag{11.13}$$

and by inserting the two solutions ω_1 and ω_2 for the frequency, the solutions for the amplitudes become $A_1 = B_1$ and $A_2 = -B_2$. To sum up, the two pairs of solutions that solve the ansatz are

$$x_{1,\omega_1} = A_1 \cdot e^{i\omega_1 \cdot t} \quad \text{and} \quad x_{2,\omega_1} = A_1 \cdot e^{i\omega_1 \cdot t} \tag{11.14}$$

for the resonance frequency ω_1 and

$$x_{1,\omega_2} = A_2 \cdot e^{i\omega_2 \cdot t} \quad \text{and} \quad x_{2,\omega_2} = -A_2 \cdot e^{i\omega_2 \cdot t} \tag{11.15}$$

for the resonance frequency ω_2. In real form, the solutions are

$$x_{1,\omega_1} = A_1 \cdot \cos\,(\omega_1 \cdot t + \delta_1) \quad \text{and} \quad x_{2,\omega_1} = A_1 \cdot \cos\,(\omega_1 \cdot t + \delta_1) \tag{11.16}$$

for the resonance frequency ω_1 and

$$x_{1,\omega_2} = A_2 \cdot \cos\,(\omega_2 \cdot t + \delta_2) \quad \text{and} \quad x_{2,\omega_2} = -A_2 \cdot \cos\,(\omega_2 \cdot t + \delta_2) \tag{11.17}$$

for the resonance frequency ω_2.

These two purely harmonic solutions are called the normal modes of the system of two coupled oscillators.

The most general solution of the equation of motion is attained by the superposition of the normal modes

$$x_1 = A_1 \cdot \cos\,(\omega_1 \cdot t + \delta_1) + A_2 \cdot \cos\,(\omega_2 \cdot t + \delta_2) \tag{11.18}$$

$$x_2 = A_1 \cdot \cos\,(\omega_1 \cdot t + \delta_1) - A_2 \cdot \cos\,(\omega_2 \cdot t + \delta_2) \tag{11.19}$$

As expected, there are four constants A_1, A_2, δ_1, and δ_2 to be set by initial conditions, two for each eq. 11.5 and 11.6.

c) Excitation of Specific Oscillation States

An oscillation state is completely determined by the four intial conditions. In the following, three important special cases are looked at.

Case 1: Both Oscillators are Released from Rest with Same Displacement

The initial conditions at $t = 0$ are:

$$x_1(0) = x_2(0) = A \quad , \quad \dot{x}_1(0) = \dot{x}_2(0) = 0 \tag{11.20}$$

Inserting into eq. 11.18 and 11.19 gives the solution $A_1 = A$, $A_2 = 0$, $\delta_1 = 0$ with arbitrary δ_2. It follows

$$x_1 = x_2 = A \cdot \cos\,(\,\omega_1 \cdot t\,) \quad \text{with} \quad \omega_1 = \sqrt{\frac{k}{m}} \tag{11.21}$$

The two oscillators oscillate together. The solution is a normal mode with the lowest frequency. The spring constant k' is not present in the solution, the spring is thus not participating in the oscillation and its tension state does not change (try to observe it!)

Case 2: The two Oscillators are Released from Rest with Opposite Displacement

The intial conditions at $t = 0$ are:

$$x_1(0) = -x_2(0) = B \quad , \quad \dot{x}_1(0) = \dot{x}_2(0) = 0 \tag{11.22}$$

Inserting into eq. 11.18 and 11.19 gives the solution $A_1 = 0$, $A_2 = B$, arbitrary δ_1, and $\delta_2 = 0$. It follows

$$x_1 = -x_2 = B \cdot \cos\,(\,\omega_2 \cdot t\,) \quad \text{mit} \quad \omega_2 = \sqrt{\frac{k + 2\,k'}{m}} \tag{11.23}$$

This is the second normal mode. The two oscillators oscillate opposingly.

Case 3: One Oscillator is Released from Rest, the Other from Rest with Displacement

The initial conditions at $t = 0$ are:

$$x_1(0) = C \quad , \quad x_2(0) = 0 \quad , \quad \dot{x}_1(0) = \dot{x}_2(0) = 0 \tag{11.24}$$

Inserting into eq. 11.18 and 11.19 gives the solution $A_1 = A_2 = C/2$ and $\delta_1 = \delta_2 = 0$. It follows

$$x_1 = \frac{C}{2} \cdot (\,\cos\,(\,\omega_1 \cdot t\,) + \cos\,(\,\omega_2 \cdot t\,)\,) = C \cdot \cos\,(\,\frac{\omega_2 - \omega_1}{2} \cdot t\,) \cdot \cos\,(\,\frac{\omega_2 + \omega_1}{2} \cdot t\,) \tag{11.25}$$

$$x_2 = \frac{C}{2} \cdot (\,\cos\,(\,\omega_1 \cdot t\,) - \cos\,(\,\omega_2 \cdot t\,)\,) = C \cdot \sin\,(\,\frac{\omega_2 - \omega_1}{2} \cdot t\,) \cdot \sin\,(\,\frac{\omega_2 + \omega_1}{2} \cdot t\,) \tag{11.26}$$

For weak coupling ($k' << k$), $\omega_2 \approx \omega_1$ and thus $\omega_2 - \omega_1 << \omega_1 + \omega_2$. Illustrated in fig. 11.3, this solution can be seen as an oscillation of the frequency $\frac{\omega_1+\omega_2}{2}$ with time varying amplitude.

This is called a *beat*, that always appears in the superposition of oscillations of almost identical frequencies. Because the amplitude of an individual oscillator depends here on the time, its energy is in contrast to the cases 1 and 2 not constant. The energy of the system travels with frequency $\Omega = \omega_2 - \omega_1$ from one oscillator to the other. The total energy of the two oscillators however remains constant throughout.

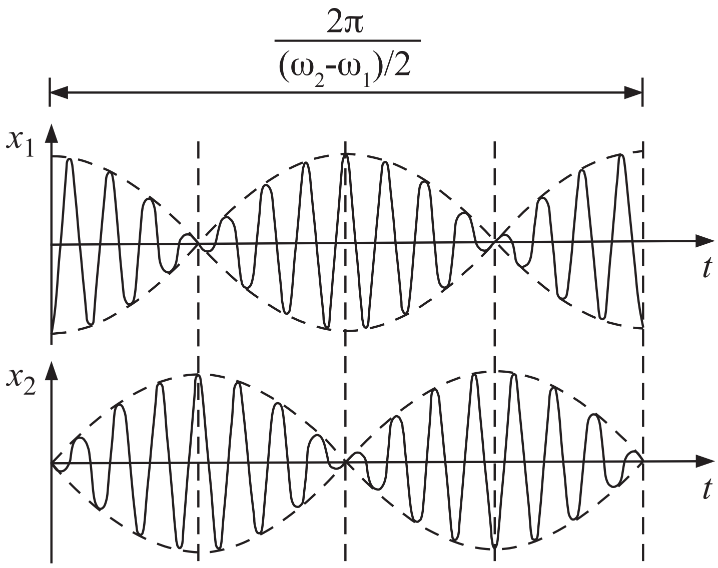

Figure 11.3: Beat: In the middle antinode, the oscillator initially has opposite phase; the beat thus extends in this case over two antinodes.

d) Generalisation for a System with N Coupled Oscillators

In general, the number of normal modes is equal to the number of degrees of freedom of the whole system. The generalisation for N coupled oscillators would yield a system of N coupled equations of motion. Equation 11.11 would be of degree $2\,N$ with generally N different positive solutions for the normal or resonance frequencies ω_N.

11.3 Experimental Part

In the experiment, the derived relations of the theoretical part are to be studied qualitatively and quantitatively. As friction forces are completely ignored in the derivations, the experiment is built to minimise friction forces. The oscillators consist of two riders, that glide on an air cushion and that can be coupled with each other using springs.

The springs must not be overstretched in the experiment, as otherwise the underlying assumption of a linear force law ($F = -k \cdot x$) would not be justified.

a) Qualitative Observations

- Open the compressed air valve until the riders are lifted off the underlay
- Excite the above mentioned special oscillation cases by chosing the right initial conditions and compare the behaviour of the oscillators with the expectations.

b) Measurement of the Oscillation Frequencies

The following frequencies are to be experimentally determined:

$$\begin{aligned}
\omega_1 &= \sqrt{\frac{k}{m}} && \text{(1. Normal mode)} \\
\omega_2 &= \sqrt{\frac{k+2\,k'}{m}} && \text{(2. Normal mode)} \\
\Omega &= \omega_2 - \omega_1 && \text{(Beat frequency)} \\
\omega' &= \sqrt{\frac{k+k'}{m}} && \text{(Frequency of an oscillator, if the other is retained)}
\end{aligned}$$

- Measure ω_1, ω_2, and ω' by determining the period T of each oscillation through five measurements each over 20 oscillation cycles and calculation of angular frequency from $\omega = 2\,\pi/T$. In each case, calculate the mean of the measurements and estimate the error on ω_1, ω_2, and ω'.
- Determine the beat frequency Ω and the error on Ω in the same way. Because of the inevitable damping, the period T has to be identified over 10 oscillation cycles in this case, e.g. over 10×2 nodes of one pendulum (see Fig. 11.3).
- Confirm the expected relations $\Omega = \omega_2 - \omega_1$ und $\omega_1^2 + \omega_2^2 = 2\,\omega'^2$ using the measured frequencies.
- Think about the reasons why the calculated and the experimentally determined values of Ω differ so much.

c) Measurement of the Spring Constant and Calculation of the Oscillation Frequency

The spring constant k_0 of the employed springs have to be measured separately. For this, the two methods illustrated in fig. 11.4 are used. Because the preceding experiments use two springs per oscillator, use $k_0 = k/2$.

- Put tension on the springs one after another by attaching a known mass m and measure the difference in length h. Calculate the spring constant using $k_0 = m \cdot g/h$.
- Using the mass m, oscillate the stressed spring vertically and measure the oscillation period T. Calculate the spring constant using $k_0 = m \cdot \omega^2 = m \cdot 4\pi^2/T^2$.
- Calculate the expected values of the resonance frequency ω_1 and ω_2 using the measured spring constant $k_0 = k/2$ and the known masses of the riders used in the first parts of the experiment. Compare the expected values of the resonance frequencies ω_1 and ω_2 with the measured ones.

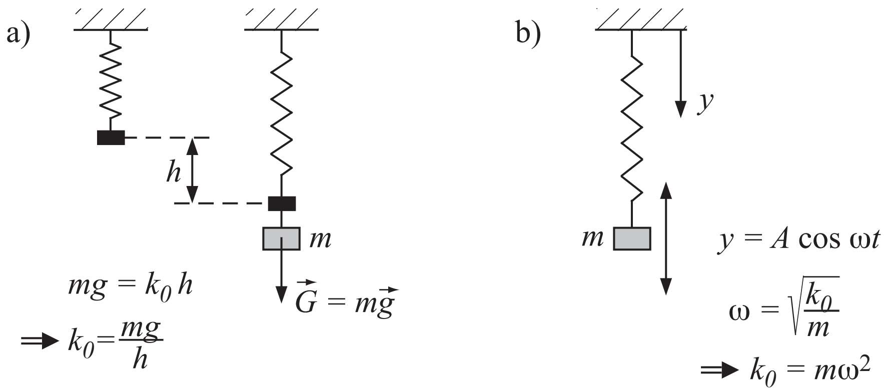

Figure 11.4: Measuring the spring constant k.