9. Mechanical oscillations and resonances

9.1 Introduction

Oscillations play a major role in all domains of physics. Examples include the pendulum like the spring pendulum in mechanics, resonator circuits in electronics, oscillating strings or membranes in acoustics, or oscillating electric and magnetic fields in optics. The mathematical treatment always yields to similar generic equations.

Using the rotary oscillation of a disk as an example, we will learn about the general properties of (harmonic) oscillators in this experiment. In particular, we will focus on the dependence of these properties on damping, where we assume that the damping force is proportional to the velocity of the oscillator. Furthermore, an harmonic external force to drive the oscillation and the behaviour close to resonance will be investigated. The findings can easily be used to describe analogous oscillating system.

9.2 Theory

Treating oscillations we distinguish undamped, damped, and forced oscillations. An undamped oscillation is said to be harmonic if the restoring force is proportional to the instantaneous displacement from the equilibrium position. In most cases, this condition is fulfilled for small displacements. Harmonic oscillations can be described by means of sine or cosine functions. Undamped oscillations, *i.e.* oscillations without loss of energy, are an idealized model, which does not exist in nature owing to omnipresent friction. As a consequence, the amplitude of the oscillation decreases with time.

If, in addition, an external force drives the oscillation, the oscillation is said to be a forced oscillation. If the driving force is a periodic (harmonic) function of time, the oscillation amplitude will depend on the frequency of the driving force and resonance phenomena occur.
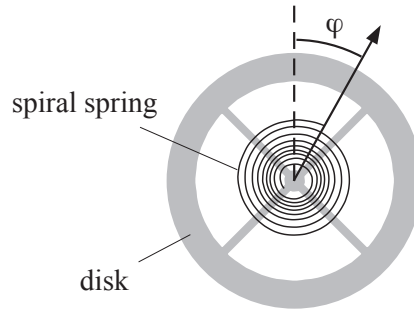
Figure 9.1: The rotating pendulum

a) Undamped rotary oscillation of a disk

A metallic disk is fixed such that it may rotate around its center axis. A spiral spring ensures that the disk, once rotated out of its equilibrium position, experiences a restoring torque. A sketch of this pendulum is shown in Fig. 9.1. If the pendulum is deflected by a small angle φ out of equilibrium, the restoring torque is approximately proportional to the angle and becomes:

$$M_D = -k_D \cdot \varphi. \tag{9.1}$$

Neglecting damping forces we can deduce from the principle of angular momentum the equation of motion:

$$\theta_s \cdot \frac{d^2\varphi}{dt^2} = -k_D \cdot \varphi, \tag{9.2}$$

where

$$\theta_s \quad = \quad \text{moment of inertia and} \tag{9.3}$$

$$k_D \quad = \quad \text{spring constant of the spiral spring.} \tag{9.4}$$

Eqn. 9.2 represents the differential equation describing an harmonic oscillator with the solution

$$\varphi(t) = \varphi_0 \cdot \cos\left(\omega_0 \cdot t - \delta\right), \tag{9.5}$$

where (see Fig. 9.2)

$$\varphi_0 \quad = \quad \text{oscillation amplitude,} \tag{9.6}$$

$$\omega_0 = \sqrt{\frac{k_D}{\theta_s}} \quad = \quad \textit{eigenfrequency of the undamped oscillator, and} \tag{9.7}$$

$$\delta \quad = \quad \text{phase offset.} \tag{9.8}$$

The eigenfrequency (or natural frequency) ω_0 is given by the properties of the oscillator. It determines the time of period of the oscillation

$$T = \frac{2\pi}{\omega_0}. \tag{9.9}$$

The amplitude φ_0 and the phase δ are determined by the starting conditions.

As shown in Fig. 9.2 the deflection is passing through extremal positions at times $t_n = \delta/\omega_0 = 0, \pi, 2\pi, \ldots, n\pi$, the amplitude of this oscillations remains constant.
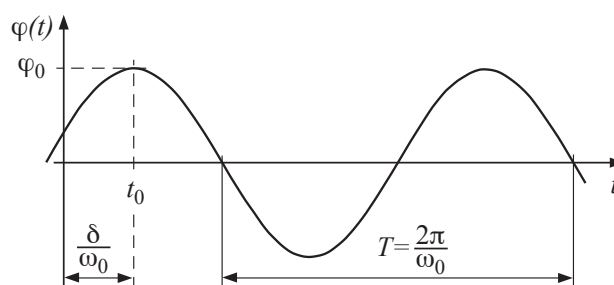
Figure 9.2: Harmonic oscillation.

b) Damped oscillation

The oscillation of the pendulum is said to be damped if frictional forces act on the pendulum in the direction opposite to its movement. If the torque M_R produced by the frictional forces is proportional to the angular speed, we may write

$$M_R = -\beta \cdot \frac{d\varphi}{dt}. \tag{9.10}$$

The principle of angular momentum yields

$$\theta_s \cdot \frac{d^2\varphi}{dt^2} = -k_D \cdot \varphi - \beta \cdot \frac{d\varphi}{dt} \tag{9.11}$$

or

$$\frac{d^2\varphi}{dt^2} + \frac{\beta}{\theta_s} \cdot \frac{d\varphi}{dt} + \frac{k_D}{\theta_s} \cdot \varphi = 0. \tag{9.12}$$

This is the differential equation of a damped oscillation. The solution depends on the strength of the damping.

Weak damping

For weak damping the solution becomes (see Fig.. 9.3):

$$\varphi(t) = \varphi_0 \cdot e^{-\alpha \cdot t} \cdot \cos\left(\omega' \cdot t - \delta'\right), \tag{9.13}$$

where

$$\alpha = \frac{\beta}{2\,\theta_s} \quad = \quad \text{damping coefficient,} \tag{9.14}$$

$$\varphi_0 \quad = \quad \text{maximum amplitude,} \tag{9.15}$$

$$\omega' = \sqrt{\omega_0^2 - \alpha^2} \quad = \quad \text{angular frequency of the damped oscillation, and} \tag{9.16}$$

$$\delta' \quad = \quad \text{phase offset.}$$

The (angular) frequency ω' is smaller than the natural frequency ω_0 of the undamped oscillator. This means that the oscillation is slowed down by the damping. The phase offset δ' and the maximum amplitude φ_0 are given by the starting conditions.
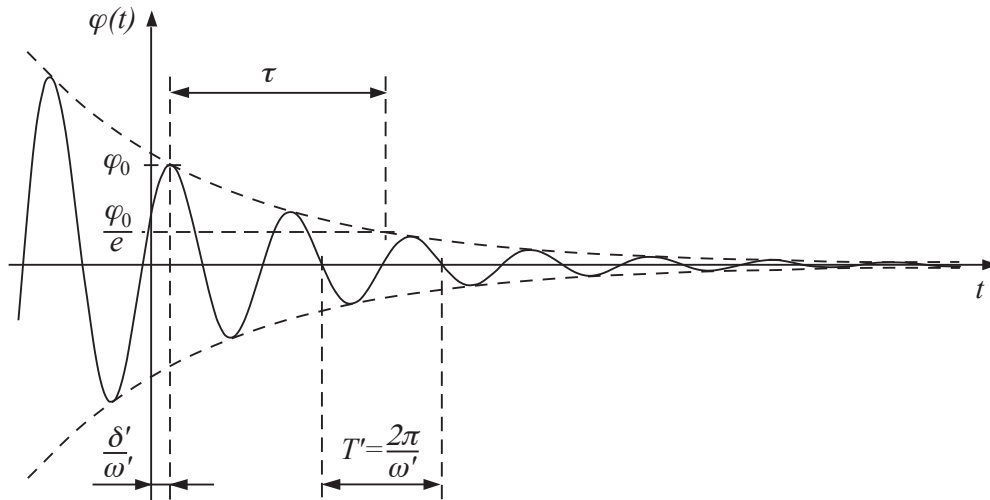
Figure 9.3: Oscillation with weak damping.

The deflection passes through extrema at times $t_n = \delta'/\omega' = 0, \pi, 2\pi, \ldots, n\pi$. The amplitude

$$\varphi(t_n) = \varphi_0 \cdot e^{-\alpha \cdot t} \tag{9.17}$$

decreases exponentially with time, the stronger the damping α the faster the amplitude decays. The decay time $\tau = \alpha^{-1}$ equals the time interval after which the amplitude decays to $1/e$ of its initial value.

Critical damping

As can be seen from Eqn. 9.16 ω' decreases with increasing damping α. In the case of strong damping $\alpha > \omega_0$ the motion is no longer periodic but the pendulum returns to its initial equilibrium position without overshooting. The limiting case of $\alpha = \omega_0$ or $\omega' = 0$, which marks the transition between damped oscillation and aperiodic motion, is called critical damping. Typical trajectories for these three cases of weak, critical and strong damping are schemtically shown in Fig. 9.4.
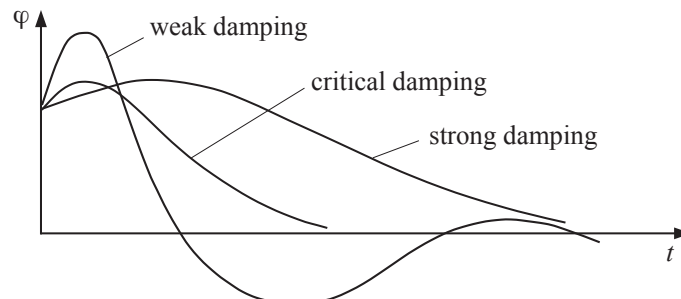

Figure 9.4: Weak, critical and strong damping.

The case of critical damping is important for technological applications like e.g. buffers, vibration dampers, or in weighing scales or galvanometers.

c) Forced oscillation

If an additional torque, produced by an external force of frequency ω, acts on the disk

$$M_A = M_0 \cdot \cos{(\omega \cdot t)} \tag{9.18}$$

the principle of angular momentum leads to the following differential equation of a forced oscillation:

$$\frac{d^2\varphi}{dt^2} + \frac{\beta}{\theta_s} \cdot \frac{d\varphi}{dt} + \frac{k_D}{\theta_s} \cdot \varphi = \frac{M_0}{\theta_s} \cdot \cos{(\omega \cdot t)}. \tag{9.19}$$

In the beginning, the damped oscillation at frequency ω' and the forced oscillation at frequency ω are superimposed. The damped oscillation decays with time, but the forced oscillation remains for times $t \gg \alpha^{-1}$. As a result the pendulum will oscillate with certain phase lag with respect to the driving force and at driving frequency ω. Using the *ansatz*

$$\varphi(t) = A \cdot \cos{(\omega t - \delta'')} \tag{9.20}$$

together with Eqn. 9.19, we can show that the amplitude of the forced oscillation behaves like

$$A(\omega) = \frac{M_0}{\theta_s \cdot \sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \beta^2\omega^2/\theta_s^2}}. \tag{9.21}$$

A typical example is plotted in Fig. 9.5).

The peak maximum is located at

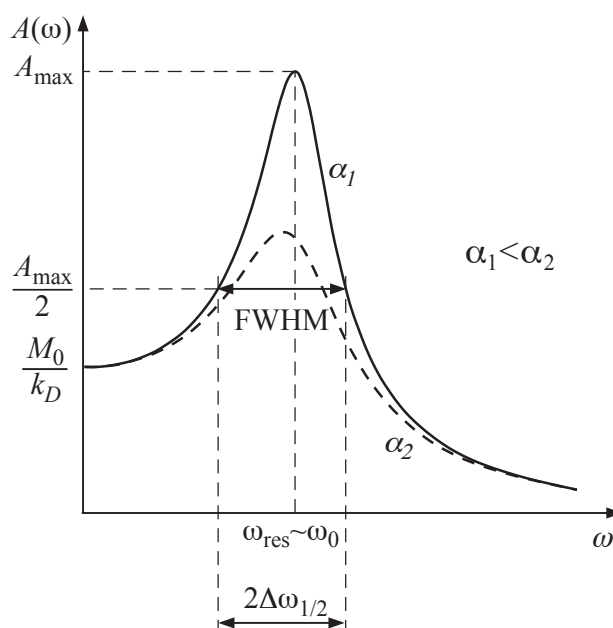$$\omega_{\text{res}} = \omega' = \sqrt{\omega_0^2 - 2\,\alpha^2} \tag{9.22}$$

Figure 9.5: Resonance curve with full width at half maximum (FWHM).

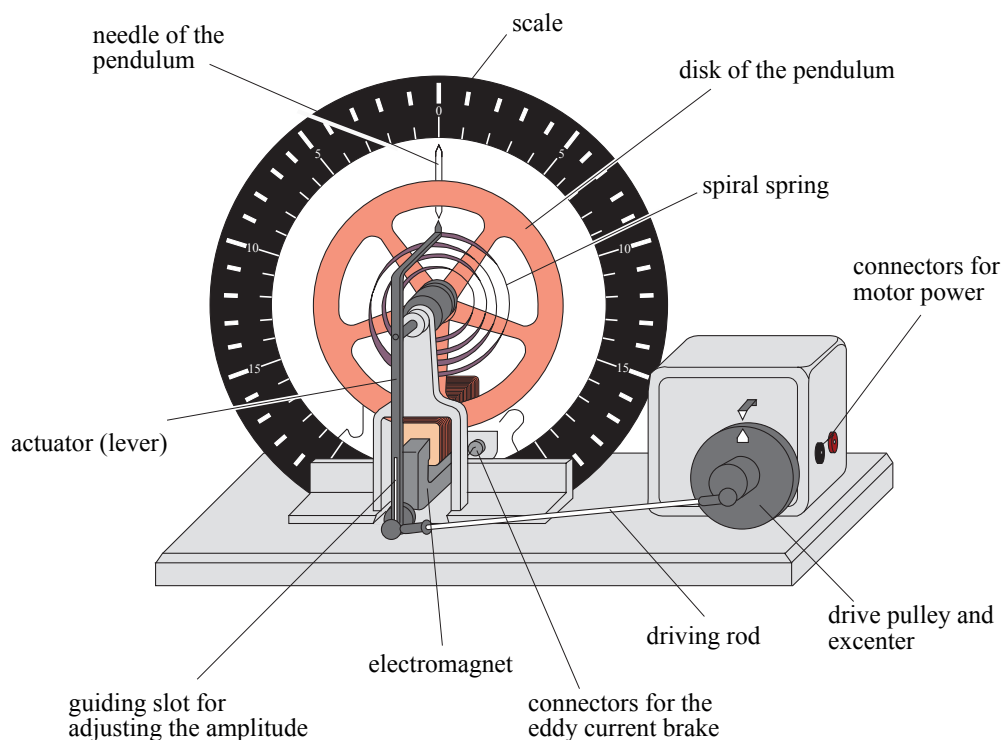and the corresponding amplitude amounts to

$$A_{\max} = \frac{M_0}{\beta \cdot \omega_0} = \frac{M_0}{2\,\alpha \cdot \omega_0 \cdot \theta_s}. \tag{9.23}$$

Often, the so-called full-width-at-half-maximum (FWHM) $2\,\Delta\omega_{1/2}$ is used for characterizing the width of the peak. It depends solely on the strength of the damping

$$\Delta\omega_{1/2} = \alpha \cdot \sqrt{3}. \tag{9.24}$$

Therefore, the damping coefficient α can be determined by the width of the resonance curve for any oscillator.

9.3   Experimental

Specifications:

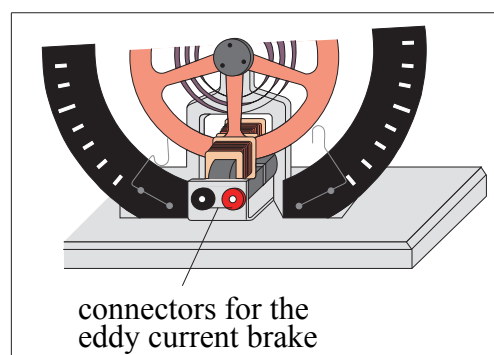|	
eigen frequency:	ca. 0, 5 Hz
voltage of the motor:	2...16 V
– close to resonance:	8 V
eddy current damping:	0...10 V
	maximum 1.0 A
	(short-term (!!!) up to 1.5 A)

Figure 9.6: Rotary pendulum according to Pohl with eddy current damping.

The setup is shown in Fig. 9.6: It consists of a rotary pendulum driven by an electric motor and damped by an eddy-current brake. The motor and the pendulum are connected mechanically by an extender wheel and a driving rod. The damping is controlled by the current passing through the coils which produce the fields and, thereby, the eddy currents in the disk. The motor produces a periodic torque, whose frequency is controlled via the voltage of the motor. The power supplies and the principle of the brake will be explained by the assistant.

This experiment consists of three parts:

1. A qualitative observation ad description of the pendulum for weak, critical and strong damping. Then, the current setting for critical damping shall be determined.

2. For weak damping and two different values, the exponential decay of the amplitude is to be recorded. From the decay, the damping coefficient α can be calculated.

3. For the same (weak) damping strengths and external torque, full resonance curves are recorded. The damping coefficients α can be calculated from the FWHM of the two curves and compared to the value obtained in the second part.

a) Critical damping

In order to observe the critical damping, we use a special power supply capable to supply high enough currents to the eddy-current brake.

- Observe the oscillations of the pendulum at weak damping. Increase the current passing through the brake until the pendulum does no longer overshoot. Note this current as corresponding to critical damping.

- *Attention: the current passing through the brake is limited to 1.5 A maximum for a short period! Execute this experiment speedily!*

b) Determination of the damping coefficient from exponential decay

For a given damping strength the damping coefficient shall be determined from the exponential decay of the amplitude according to Eqn. 9.17.

Logarithmising Eqn. 9.17 yields:

$$\ln \varphi(t_n) = \ln \varphi_0 - \alpha \cdot t_n \tag{9.25}$$

or

$$\log \varphi(t_n) = \log \varphi_0 - \alpha \cdot t_n \cdot \log e. \tag{9.26}$$

Plotting $\ln \varphi(t_n)$ or $\log \varphi(t_n)$ as function of time on graduated paper or else $\varphi(t_n)$ on semi-logarithmic paper yields a straight line with slope $-\alpha$. [1] From the slope the damping coefficient can be calculated (see Fig.. 9.7).

- Adjust the damping current to the first of the two values given in the laboratory.

- Use a stop watch to determine the period of time T' out of three measurements of four oscillations each.

- Measure the decaying amplitude as function of time.

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[1]It does not matter which way is used to represent the data and to obtain the slope: Note, however, that the numerical value depends on the kind of logarithm used, the natural log or to the base 10. The latter is of advantage for graphical purposes, the first for calculations.
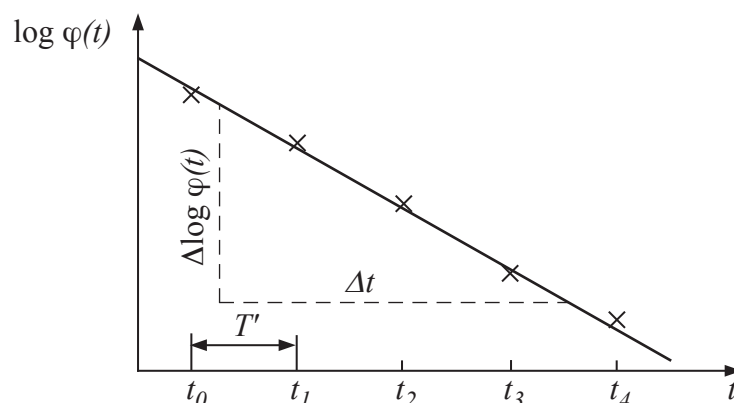
Figure 9.7: Determination of the damping coefficient α.

- Plot the amplitudes on semi-log paper versus time.

- Draw the best-fitting straight line passing through the data and determine α from the slope (pay attention to the y-scale when working with semi-log paper!) No error calculus is required.

- Calculate the half-value period of the damped oscillator.

- Repeat the measurements for the second current value given.

c) Determination of the damping coefficient from the resonance curve

- Set the damping current to the same value than in the previous part b).

- Connect the motor and power it up. The pendulum will start forced oscillations.

- Determine the driving frequency ω by measuring the time for 10 revolutions of the motor axis.

- Wait until the eigen oscillation decay and measure the amplitude of the forced oscillation.

- Change the frequency of the motor and repeat the measurements. Take the full resonance curve. Choose large frequency steps when far from resonance, but small steps around the resonance (should be at least 6 data points on the resonance peak).

- Plot the amplitude as function of driving frequency ω. Determine the FWHM and calculate the damping coefficient α according to Eqn. 9.24 (see Fig. 9.8). There is no error calculus required.

- Repeat this experiment for the second damping current value.
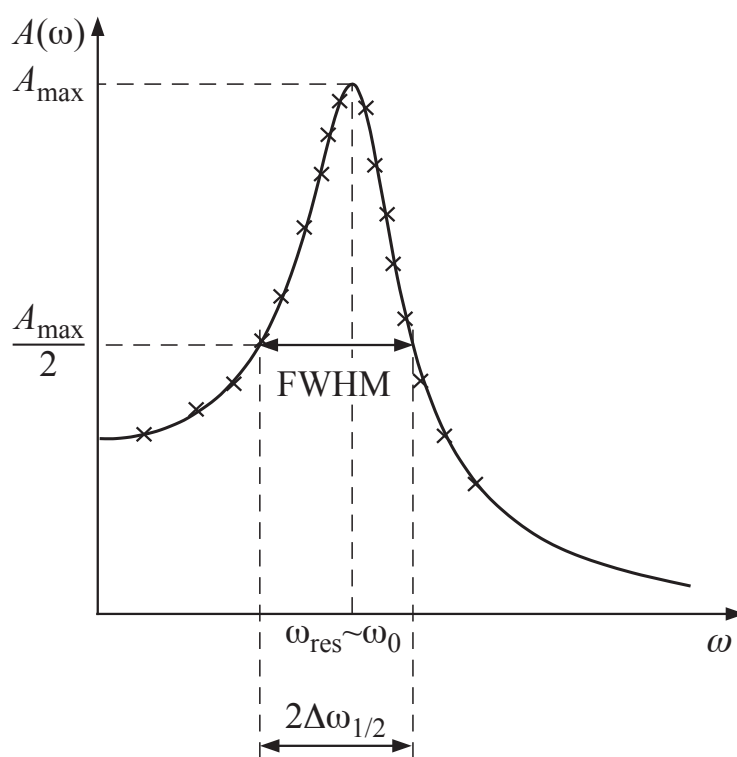
- Compare these results with values obtained in part b).

Figure 9.8: Full width at half maximum of the resonance peak.