

## 7. The gyroscope

### 7.1 Introduction

This experiment concerns a special type of motion of a gyroscope, called precession. From the angular frequency of the precession, the moment of inertia of the spinning wheel of the gyroscope can be determined. In a second part, the moment of inertia will be measured using a simple method based on the law of energy conservation.

### 7.2 Theory

#### a) The gyroscope

A body moving in such a way that only one single point of the body remains fixed in space, is called a *gyroscope* or short gyro. The point is the center of rotation O. The motion has three degrees of freedom and obeys the principle of angular momentum:

$$\frac{d\vec{L}}{dt} = \vec{M}. \quad (7.1)$$

Here,  $\vec{L}$  denotes the angular momentum of the gyro and  $\vec{M}$  the external torque acting onto the gyro. We describe the motion of the gyro as a rotation around a transient axis passing through the pivot point O. The axis of rotation changes orientation with respect to space and with respect to the body of the gyro. This general treatment of the gyro motion is exact but quite complex. We will consider only the following simple case (see Fig. 7.1):

1. The gyro has cylindrical symmetry with respect to the body axis  $z'$ . The pivot point, which is fixed in space, is situated on the axis  $z'$ .
2. The gyro is revolving fast around  $z'$  compared to the motion of  $z'$  in space. The moment of inertia, i.e. the rotational analogon to the inertial mass for translations, be  $I_{z'}$ .
3. The gyro is solely subject to the gravitational force.

Briefly, we consider a fast revolving gyroscope in the gravitational field of the Earth. If, in particular, the pivot point coincides with the center of gravity of the gyro, no net force or torque is

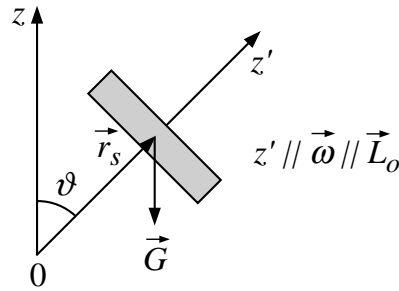


Figure 7.1: Geometry of the gyroscope.

acting on the gyro. According to the principle Eqn. 7.1 cited above the angular momentum must stay constant:

$$\frac{d\vec{L}}{dt} = \vec{M} = 0, \text{ i.e. } \vec{L} = \text{constant.}$$

If the axis of rotation of the gyro  $z'$  coincides with the space fixed axis of the angular momentum  $\vec{L}$ , the axis of rotation  $z'$  will be fixed in space as well. Otherwise, the  $z'$ -axis moves on a cone around the space fixed angular momentum axis. This kind of motion is called *nutation*.

### b) Precession

Now, we assume that the axis of angular momentum  $\vec{L}$  and the body axis and axis of rotation  $z'$  coincide. The absolute value  $L$  can be evaluated according to condition (2) above:

$$L \approx L_{z'} = I_{z'} \cdot \omega_{z'}. \quad (7.2)$$

The center of gravity be displaced by a distance  $r_s$  off the pivot point O along the axis. Therefore, gravity produces a torque  $\vec{M}_0$  with

$$\vec{M}_0 = \vec{r}_s \times \vec{G}.$$

It follows for the absolute value of  $\vec{M}_0$ :

$$|\vec{M}_0| = M_0 = r_s M g \sin \vartheta, \quad (7.3)$$

where  $M$  denotes the mass of the gyro. The torque  $\vec{M}_0$  is perpendicular to the plane spanned by  $\vec{r}_s$  and  $\vec{G}$ . Moreover, the vectors  $\vec{r}_s$  and  $\vec{G}$  form a right-handed trihedron with  $\vec{M}_0$ .

Let us assume that the angular momentum has a direction  $\vec{L}$  at time  $t$ . The *change* in angular momentum  $d\vec{L}$  during the infinitesimal time interval  $dt$  must have the same direction than the torque  $\vec{M}_0$  according to the principle of angular momentum Eqn. 7.1. At time  $t + dt$  the vector of angular momentum  $\vec{L}$  and, thereby, the axis of rotation  $z'$  will have a new direction  $\vec{L} + d\vec{L}$ . Since at the same time, the vector of the torque  $\vec{M}_0$  rotates by the same angle  $d\varphi$  (see Fig. 7.2) the situation is the same then before and the whole movement continues: The end of the vector of angular momentum moves on a horizontal circle around the space fixed vertical axis  $z$ . The center of gravity of the gyro moves on a horizontal circle, too, perpendicular to the field lines of gravity. This motion is called *precession*.

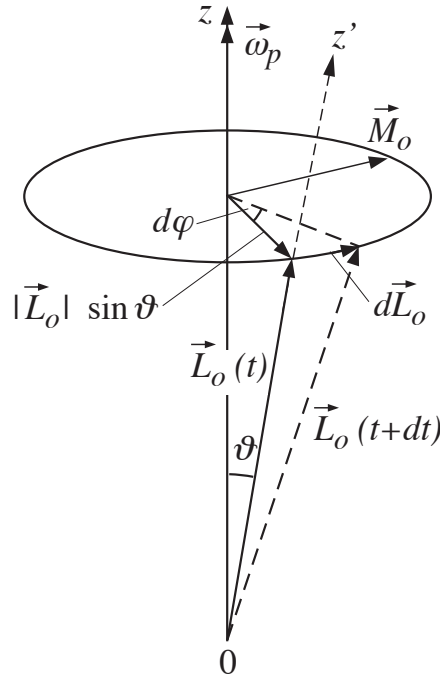


Figure 7.2: Precession: angular momentum, torque and resulting change in angular momentum.

The angular velocity of the precession is  $\omega_p = d\varphi/dt$ , where  $d\varphi$  denotes the angle by which the vectors  $\vec{L}$  and  $z'$  rotate around the space fixed axis  $z$  in the time interval  $dt$ . From Fig. 7.2 we can deduce that

$$d\varphi = \frac{\text{length of arc}}{\text{radius}} = \frac{|d\vec{L}_0|}{|\vec{L}_0| \cdot \sin \vartheta}.$$

Since according to the principle of angular momentum Eqn. 7.1  $d\vec{L}_0 = \vec{M}_0 dt$  we obtain

$$d\varphi = \frac{|\vec{M}_0| dt}{|\vec{L}_0| \cdot \sin \vartheta}.$$

Inserting Eqns. 7.2 and 7.3 allows one to eliminate the angle  $\vartheta$ . Finally we obtain for the angular velocity of the precession

$$\omega_p = \frac{d\varphi}{dt} = \frac{r_s G \sin \vartheta}{I_{z'} \omega_{z'} \sin \vartheta} = \frac{r_s M g}{I_{z'} \omega_{z'}}, \quad (7.4)$$

where we used the following definitions:

$\omega_p$  = angular velocity of the precession around  $z$  and

$\omega_{z'}$  = angular speed of the revolution around the  $z'$ -axis with  $\omega_{z'} \gg \omega_p$ .

## 7.3 Experimental

### a) Experimental setup

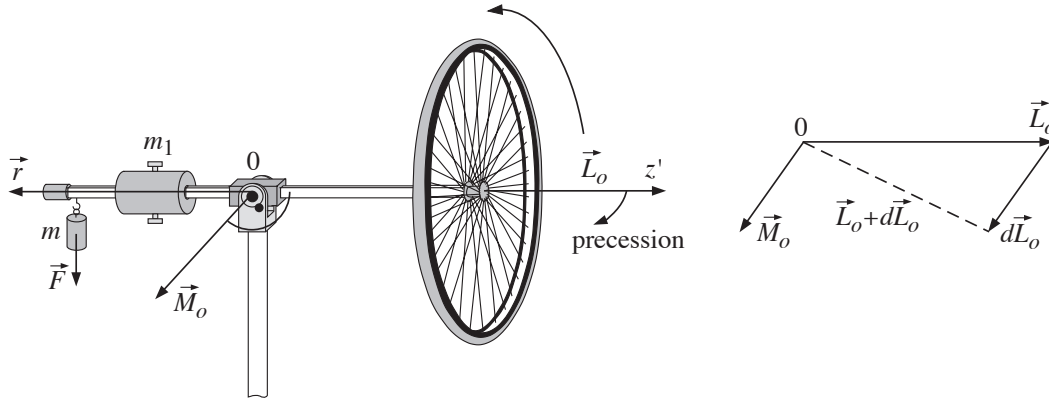


Figure 7.3: Setup of the gyroscope.

A gyroscope with a wheel be mounted like shown in Fig. 7.3. The mass of the wheel be compensated by a mass  $m_1$  such that no net force or torque is acting on the gyro. An additional mass  $m$ , fixed at a distance  $r$  from the pivot point produces a torque  $\vec{M}_0$ , which is given by the additional mass  $\vec{F}_G = m \vec{g}$  and which yields:

$$\vec{M}_0 = \vec{r} \times \vec{F} = m \cdot \vec{r} \times \vec{g}.$$

This leads to precession with angular frequency

$$\omega_p = \frac{m g r}{I_{z'} \omega_{z'}}. \quad (7.5)$$

In order to determine the angular velocity, the time of period of the rotation shall be measured. Using Eqn. 7.5 the moment of inertia  $I_{z'}$  can be calculated from the angular velocity.

### b) Realization and data analysis

By adjusting the mass  $m_1$  and without additional mass  $m$ , the gyro is put into an indifferent equilibrium. Before each measurement the wheel is set into rapid revolution ( $\omega_{z'} \gg \omega_p$ ).

1. Qualitative observation: a little kick makes the axis of the gyro  $z'$  deviate from the direction of the angular momentum  $\vec{L}_0$ . The gyro starts to nutate. Repeat this experiment with the wheel revolving in the opposite sense.
2. By adding a mass  $m$  the horizontal gyroscope starts to precess. In order to avoid nutation, the gyro should be gently guided in the beginning. This is justified by the fact that Eqn. 7.4 was derived under the assumption of a stationary angular velocity of the precession. It is necessary to measure both time  $T_{z'}$  and  $T_p$  simultaneously because the wheel slows down owing to friction.

The experiment is to be carried out 5 times. In order to observe the vector relation between  $\vec{M}_0$  and  $\vec{dL}_0$ , the wheel shall revolve in the opposite sense of rotation at least once. The result for the moment of inertia and the corresponding measurement error are obtained by means of Eqn. 7.5 from averaging over the five measurements.

### c) Acceleration of free fall; energy conservation

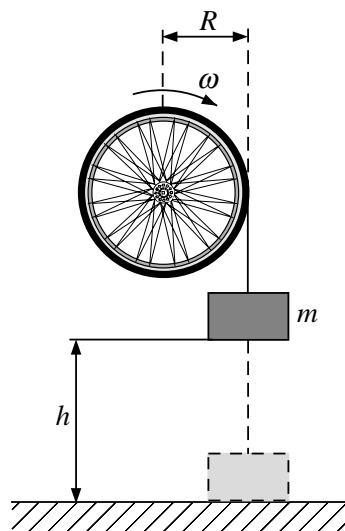


Figure 7.4: Prinziple of the second method.

For the second experiment the gyroscope is blocked. A mass  $m$  is suspended at the end of a cord, which is wound around and fixed at the wheel. The mass falling from a height  $h$  above ground produces a constant torque on the wheel. The angular velocity of the wheel increases until the mass touches ground.

Energy balance: the sum of potential and kinetic energies of wheel and mass  $m$  is a constant. Comparing the situation before and directly after the free fall we obtain

$$mgh + 0 + 0 = 0 + \frac{mv_{end}^2}{2} + \frac{I_s \omega_{end}^2}{2}. \quad (7.6)$$

This means that the potential energy of mass  $m$  at beginning equals the sum of kinetic energies of mass *and* wheel at the end. The final translational speed  $v_{end}$  of mass  $m$  equals the speed of a point on the circumference of the wheel

$$v_{end} = v_{Umfang} = \omega_{end} \cdot R,$$

where  $R$  denotes the radius of the wheel and  $v_{Umfang}$  the speed on the circumference of the wheel..

### d) Realization and calculation

It is important to check that the cord is always tangential to the circumference of the wheel. Repeat this experiment 5 times and vary the mass  $m$  and the height of the free fall  $h$ .

- Determine the angular velocity by measuring the time of revolution  $T_{end} = 2\pi/\omega_{end}$  during the first three revolutions after the mass has touched the ground.
- The moment of inertia  $I_s$  is to be calculated from Eqn. 7.6 for the second method,  $I_{s, \text{Fall}}$  as well as the measurement error. Compare these values with the results from the first experiment.