Part I

Student Laboratory Manuals

4. The Gravitational Constant G

4.1 Introduction

Gravitation, or gravity, is one of the four fundamental interactions of nature[1], giving rise to attractive forces between objects with mass. Gravitation is the agent that determines the weight of objects with mass and causes them to fall when dropped. Gravitation causes dispersed matter to coalesce, thus accounting for the existence of planets, stars and galaxies. Gravitation is responsible for keeping the Earth and the other planets in their orbits around the Sun; for keeping the Moon phase locked in its orbit around the Earth; for the formation of tides in the oceans; and is one cause of heating of the interiors during the formation of stars and planets. In a different context, gravitation is responsible for natural convection in fluids and gases, causing flow to occur under the influence of a density gradient.

In 1684, Isaac Newton (4 January 1643 - 31 March 1727) formulated the law of *universal gravitation*[2], according to which the mutual attractive force between two masses, m_1 and m_2, is proportional to the product of the masses and inversely proportional to the square of the distance r between their respective centre of mass (CM).

$$F = G \cdot \frac{m_1 \cdot m_2}{r^2} \tag{4.1}$$

The factor of proportionality G, the *universal gravitational constant*[3], is an empirical physical constant that appears as in Eq.4.1. It is universal in the sense that it is valid in the whole of universe. The gravitational constant is also known as *Newton's constant*, and colloquially 'Big G'. It must not be confused with 'little g', g, which, which is not a constant, and has the dimension acceleration; it includes other effects, not only the strength of the *local gravitational field*[4]. 'Little g' is equivalent to the free-fall acceleration, especially that at Earth's surface.

Although Newton's theory has been superseded, most modern non-relativistic gravitational calculations are still using some variety of Eq.4.1, because of its simplicity, and sufficient accuracy for most Earth bound applications.

A general treatment of gravitation requires Einstein's theory of general relativity, which describes

[1] Along with the *strong force*, *electromagnetism* and the *weak force*)

[2] *Philosophiae Naturalis Principia Mathematica* ("the Principia"), first published on 5 July 1687.

[3] Occasionally Γ may be encountered.

[4] See appendix.

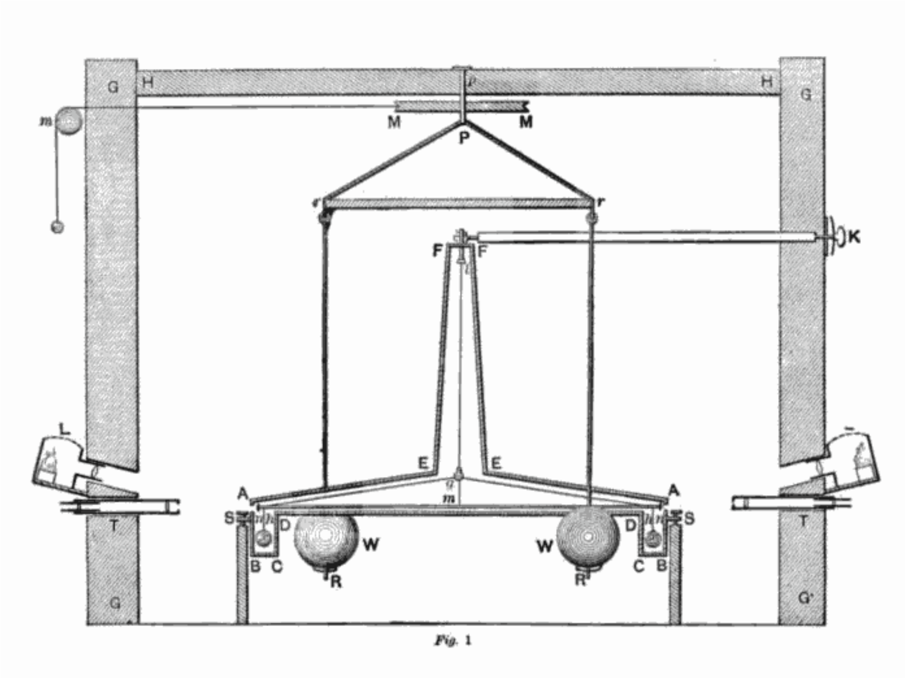

Figure 4.1: Torsion balance according to Henry Cavendish. Note that the fixed large balls (W) are placed inside the smaller dumbbell masses. Dumbbell beam 1.8 m; dumbbell masses (lead) 0.73 kg; fixed masses (lead) 158 kg; torsion wire 51 mm diameter.

the motion of inertial objects, again using the constant G, but now with gravitation is a consequence of the curvature of spacetime,[5].

The gravitational constant is one of the five *Planck units*, or five *universal physical constants*[6] These five units have the inherent quality that any one of them assumes the value exactly 1 if expressed in the other units. The five units are also known as *natural units*, since they are properties of nature and does not depend on any human design. In particular this means that they have to be determined by measurement; they cannot be calculated from any other experiments.

In this laboratory, we determine G by means of a *torsion balance*[7] of the kind (cf. Fig.4.1) that in 1797-98 was used by British scientist Henry Cavendish (10 October 1731 - 24 February 1810), who was the first to measure the force of gravity between masses in the laboratory. The data were accurate enough for the gravitational constant and the mass of the Earth[8] to be determined.

[5]The long standing issue with the orbit of Mercury was only resolved in 1915, using Albert Einstein's new theory of general relativity, which could account for the small discrepancy observed already in the 19th century.

[6]After Max Planck, who conceived the idea in 1899; the other four units are the reduced Planck constant, $\hbar$, the speed of light in vacuum, c, the Coulomb constant k_e and Bolzmann's constant k_{B}.

[7]The experiment was devised sometime before 1783 by geologist John Michell, who did not complete the work, After his death in 1793, the apparatus eventually passed on to Henry Cavendish, who rebuilt it but kept close to Michell's original plan.

[8]Cavendish did not derive a value for G, and neither was his intention to do so, but rather to derive a value for Earth's density.

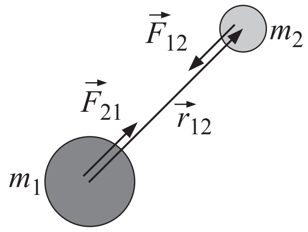

Figure 4.2: Diagram explaining the law of gravitation Eq.4.3

4.2 Theory

Newton's law of universal gravitation

The gravitational constant, G, is arguably the physical entity most difficult to measure, which to some extent might become obvious during the laboratory. In SI units, the 2006 CODATA-recommended value of the gravitational constant is:

$$G = 6.67428 \cdot 10^{-11}\,\mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-1} \quad [\equiv \mathrm{N\,m}^2 \cdot \mathrm{kg}^{-2}]. \tag{4.2}$$

with a relative standard uncertainty of 1 part in 10^4. At this instance, it must be pointed out that reported values from different experiments may differ widely, and even be mutually exclusive, reflecting the difficulties because of the small magnitude of the gravitational interaction.

The law of gravitation is generally put on vector form:

$$\vec{F}_{12} = -G \cdot \frac{m_1 \cdot m_2}{r_{12}^3} \cdot \vec{r}_{12} = -G \cdot \frac{m_1 \cdot m_2}{r_{12}^2} \cdot \hat{r}_{12} \tag{4.3}$$

where (also refer to Fig. 4.2):

$\vec{F}_{12}$	:	The force acting on mass 2 because of 1, from 1 to 2.
m_1, m_2	:	Masses of two material bodies 1 und 2
G	:	The gravitational constant
$\vec{r}_{12}$	:	Vector directed from the CM of body 1 to the CM of body 2.
$\hat{r}_{12}$	:	Unit vector from the CM of body 1 to the CM of body 2.

Determination of the gravitational constant G with a torsion balance

In this laboratory we determine the constant of gravitation G with a torsion balance as shown in Fig. 4.3

In essence, the torsion balance consists of a horizontal dumbbell with two smaller lead spheres, each of mass m, at the ends of a thin beam, suspended vertically from its midpoint with a thin metal

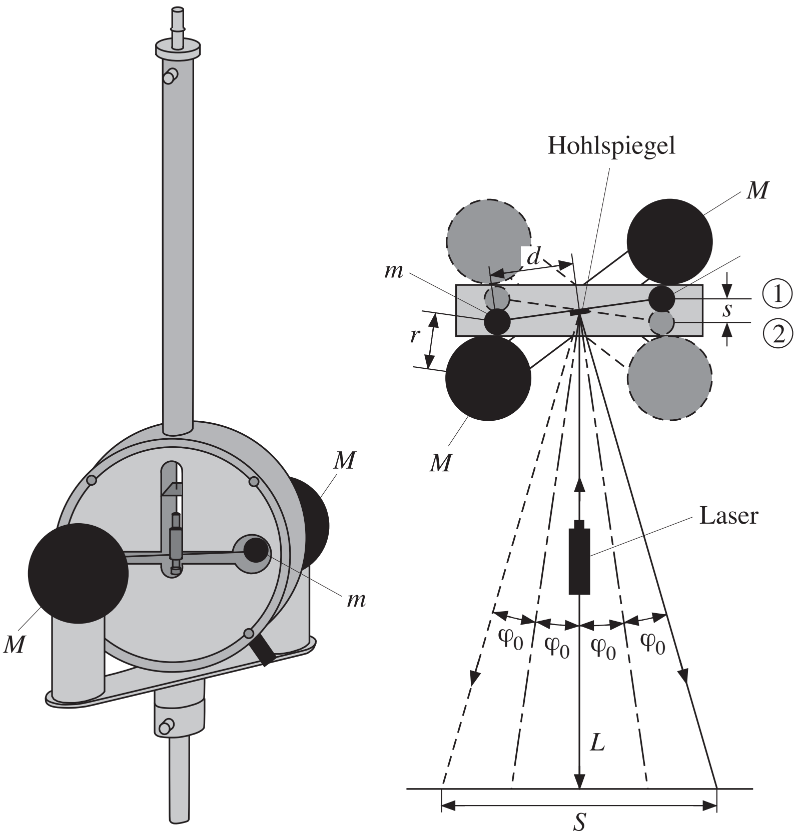

Figure 4.3: Torsion balance and principle of measurement

wire or band. The arrangement is enclosed in a casing that allows free movement of the dumbbell, but protects the sensitive balance from air convection. Two larger spheres of mass M each are mounted outside the dumbbell casing on an horizontal arm, pivoted at the midpoint, below the casing with the midpoint such that a line along the torsion band would extend through it. The arrangement is such that the centers of mass of both the external masses and the dumbbell end spheres move in the same horizontal plane. The influence of Earth's gravitational attraction is assumed to be perpendicular to the horizontal plane, which is where the mutual attraction of the spheres takes place, and may thus, for good reason, be disregarded.

The principle of operation is that the to smaller spheres are pairwise gravitationally attracted to the larger, external spheres in either one of two equivalent positions[9], thus creating a torque on the dumbbell that is balanced by a counter-torque from the supporting wire. In the initial position, here (1), the attraction between the dumbbell end masses and the external spheres will be assumed to have settled in mechanical equilibrium. A note of warning: This takes approximately one day, so please do not disturb the set-up prior to the actual experiment.

The measurement starts when the external spheres are moved to position (2). In this position the external spheres and the other dumbbell sphere come closest and attracts each other such that the

[9]The gravitational attraction between any one of the external spheres and the sphere at the farther end of the dumbbell is neglected

torque of the wire is not in equilibrium, and a net torque acts to turn the dumbbell. Because of the distributed mass of the dumbbell, its inertia together with the gravitational attraction, and the torsion of the suspending wire, results in a slow oscillation (period more than ten minutes) in the horizontal plane about a new equilibrium.

This angular oscillation is reproduced by a laser beam reflected from a mirror a the centre of the dumbbell, and projected onto a screen several metres distant, where it should be recorded with pen marks as function of time. The evaluation requires that the differential equation describing the dynamics of the dumbbell is solved, which is done as follows.

In equilibrium, i.e. no motion, the torque from the gravitational forces D, and that of the wire, τ_w, are opposite but equal in magnitude[10]:

$$D = -\tau_w \tag{4.4}$$

Using the law of gravitation Eq.4.1, we have $D = d \cdot F_{12} + d \cdot F_{21}$, which is the total torque because of the gravitational forces for the two sets of spheres *We do not require the vector form Eq.4.3, since the two forces are parallel.*.

In equilibrium, we characterise the resistance of the wire to torsion with the torque $\tau_w = k \cdot \varphi_0$, which is an angular form of Hooke's law, valid below the elastic limit of the material[11]. The constant k is known as the *torsion coefficient* (or constant), *torsion elastic modulus*, or just *spring constant.*

$$2\,d \cdot \Gamma \cdot \frac{m \cdot M}{r^2} = -k \cdot \varphi_0 \tag{4.5}$$

The factor 2 on the left is because of the pair of gravitational forces acting on the dumbbell.

d	:	Distance from CM of a dumbbell sphere to the centre of oscillation (the wire)
m	:	Mass of one dumbbell sphere
M	:	Mass of one external sphere
r	:	Distance between a dumbbell sphere and an external sphere in equilibrium
k	:	Torsion coefficient of the wire
φ_0	:	Angle of twist from centre position to equilibrium position in radians

The equation of motion for the angle of oscillation $\varphi(t)$ for the torsion balance takes the general form of a damped oscillation:

$$\Theta_s \cdot \frac{d^2\varphi}{dt^2} + \beta \cdot \frac{d\varphi}{dt} + k \cdot \varphi = D \tag{4.6}$$

where we used the following notations:

[10] The minus sign enters because we are required to define a direction for the torque, which in general should written as a vector, in order to get the resulting differential equation on the correct form.

[11] This is the equation for a mousetrap in equilibrium; the principle was used for the chassis springs in the original VW Beetle.

Θ_s : Moment of inertia of the dumbbell
β : Damping[12] of the oscillation
k : Torsion coefficient of the wire
D : Torque of the gravitational forces.

The homogeneous solution of this ordinary second order differential equation ($D = 0$) is an exponentially damped harmonic oscillation (cf. laboratory R), which is easily verified through direct substitution.

$$\varphi(t) = \varphi_0 \cdot e^{-t/\tau} \cdot \cos(\omega \cdot t - \delta) \tag{4.7}$$

with

$$\tau = \frac{2\,\Theta_s}{\beta} \quad \text{and} \quad \omega = \frac{2\,\pi}{T} = \sqrt{\frac{k}{\Theta_s} - \frac{1}{\tau^2}} \tag{4.8}$$

The moment of inertia Θ_s for the dumbbell is approximately:

$$\Theta_s = 2\,m \cdot d^2 \tag{4.9}$$

where we assumed that the dumbbell spheres are point objects, and the mass of the dumbbell beam may be neglected, i.e. no contribution to the moment of inertia.

Using Gl. 4.7 and 4.9 we may solve for the torsion coefficient:

$$k = 2\,m \cdot d^2 \cdot \left(\frac{4\,\pi^2}{T^2} + \frac{1}{\tau^2}\right) \tag{4.10}$$

Assuming a perfectly adjusted torque wire, i.e. central position $\varphi_0 = 0$ without the external spheres in place, the two equilibrium positions, labelled (1) and (2), in the presence of the external spheres, will be symmetrically deflected to the left and to the right of the central position. As may be seen in Fig.4.2, the reflected beam makes an angle $2\varphi_0$ with the axis of symmetry. With a simple trigonometric argument we calculate the distance S between the marks on the screen for the two equilibrium positions:

$$S = 4\,L \cdot \tan\varphi_0 \tag{4.11}$$

Since φ_0 is small, we may use the approximation $\tan\varphi_0 \approx \varphi_0$ and therefore

$$\varphi_0 = \frac{S}{4\,L} \tag{4.12}$$

From Eq. 4.5, 4.10 and 4.12 we finally have for the gravitational constant:

$$G = \frac{\left(\frac{4\pi^2}{T^2} + \frac{1}{\tau^2}\right) \cdot r^2 \cdot d \cdot S}{4\,M \cdot L} \tag{4.13}$$

4.3 The experiment

A note of warning: The torsion balance is a very delicate instrument and must be handled with utmost care. Avoid any commotion, in particular do not lean on the table, and do not use the marble slab supporting the balance for writing. Already a minor mistake may ruin your experiment and put it on hold for another day. Do not touch the adjusting knob at the top of the column where the torque wire is suspended.

a) Instructions for the measurements

The purpose of the laboratory is to determine the gravitational constant G, using Eq. 4.13. Variables d, r, and M with estimated errors are given in Table 4.1.

The measurements must yield values for the following variables:

- Period of oscillation T and the exponential damping time constant τ for the torsion balance.
- Distance S between position marks P_{initial} and P_{final} for the equilibrium positions (1) and (2).
- distance L between the mirror and the torsion wire and the screen.

The experiment has been prepared for you in such a way that the arm carrying the two external spheres have been positioned in either position (1) oder (2) with the balance in static equilibrium.

- On the sheet metal projection screen, fasten the custom-made strip of millimetre Cartesian graph paper provided by the assistant,
- Turn on the laser and mark the position on the graph paper of the reflected beam spot during 20 minutes for every 2 minutes, in order to compensate for building vibrations or any small oscillation that may still prevail. Determine the initial equilibrium position P_{initial} though some suitable averaging procedure, and estimate the error from the distribution of the marks.
- Start of experiment: Plan the work to be done; prepare notes; then with greatest of care, turn the arm with the external spheres into the alternative position. The spheres should touch the glass casing in order that the distance r is well defined, but avoid by all means to transfer vibrations to the dumbbell. Make a note of the absolute time when the external spheres were moved into the new position.

Table 4.1: Data for the apparatus

Mass of one external sphere	$M = (1500 \pm 2)\,\text{g}$
Diameter of external mass	$R = (63.7 \pm 0.1)\,\text{mm}$
Distance between the CM for a dumbbell sphere to centre of oscillation	$d = (50.0 \pm 0.2)\,\text{mm}$
Distance between the CM of a pair of spheres in equlibrium (external sphere touches the casing)	$r = (47.0 \pm 0.4)\,\text{mm}$

Because of the unbalanced gravitational forces the dumbbell immediately starts to turn towards a new equilibrium position. The given dynamical properties of the system makes it perform a damped oscillation about the new equilibrium position. Because of the small forces involved and the properties of the balance, the time required is very long, and we must be satisfied to record a few cycles of the oscillations, in order to determine the new equilibrium P_{Final}. For this purpose the laser mark on the graph paper screen is marked off in a suitable manner so that different cycles may be distinguished after the experiment.

- Record the position of the laser reflection mark on the graph paper screen during five full cycles of oscillations once a minute, using a suitable system of pen marks that allows the sequence of marks to be tabulated as a function of time. Make a note of the absolute time for the last mark made.
- Measure the distance L between the graph paper screen and the mirror at the dumbbell. Estimate the error of measurement.

b) Evaluation

- Prepare on a large sheet of millimetre graph paper a chart of the deflection of the laser marks as function of time (cf. Fig. 4.4).
- Think of a suitable method to find the best line representing null deflection, thereby taking the exponential decay of the deflection into account. Determine P_{final}, and the distance S between P_{final} and P_{initial}.
- Determine the period of oscillation T, from the points where the oscillation crosses the null. Collect separately the data for deflections to left and right. If there appears to be a systematic difference between the two data sets, the null was probably not correctly determined. If so, repeat the determination of the null.
- Determine the damping time constant τ. Read the amplitudes A_i of oscillations from the chart for both left and right deflections. Find the time where the maxima are expected, i.e.

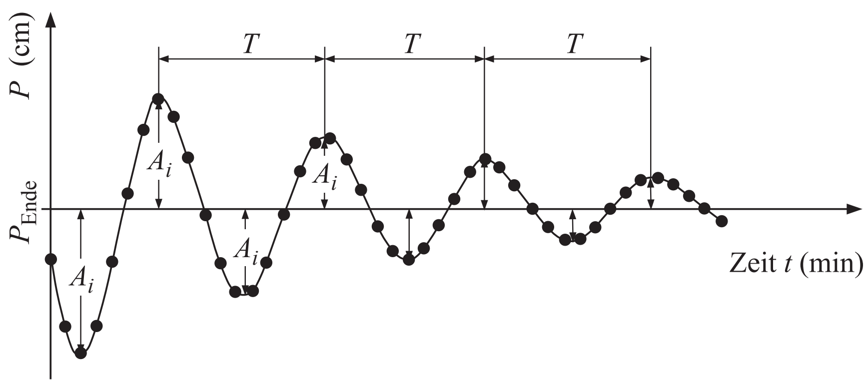

Figure 4.4: Recorded marks of laser reflection position.

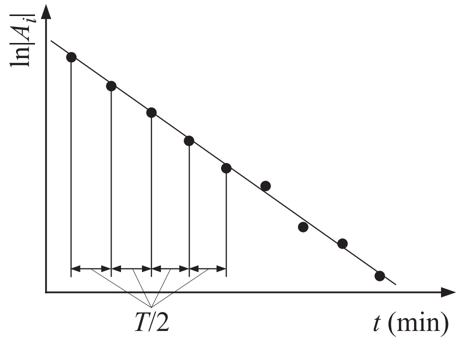

Figure 4.5: Deflection as function of time.

in middle between two consecutive null crossings, and read the deflection amplitude at these times[13]. Prepare a table of A_i, their logarithms and the corresponding time. Finally draw a diagram of the logarithm against time, and determine the slope of the expected straight line. From this, calculate τ. Estimate the error.

- Prepare a table listing the values of all measured variables and those given. Enter the corresponding errors, and the method by which they were determined.
- Calculate the gravitational constant G according to Eq. 4.13. Using error propagation to calculate the absolute error. Determine if any of the errors my be neglected. If so, the choice must be carefully motivated in your report.

[13]Taking the apparent extrema directly from the chart may not make much of a practical difference, but since the oscillation is damped by an exponential, strictly, the true maximum is not the apparent.