6. Torsion and bending of beams

6.1 Introduction

In this laboratory we approach the question of how to design a mechanical structure – a bridge, a crane – such that it will not collapse. Specifically we ask: what do we need to know about the construction material in order to predict stability?

We will not actually perform experiments to evaluate the limits of a design, but rather investigate deformations of structural elements, such as beams and shafts (which we somewhat indifferently just call *rods* below), as we bend and twist them. We shall see, that this allows us to determine parameters that may be used to calculate the response to any load.

In a *mechanics* course, as taken by physics students, structures of elements and bodies are assumed to be perfectly rigid; they are non-deformable. A general motion may thus be reduced to translations of the centre of mass and rotations about the centre of mass.

A rigid body is an idealisation, though, since in real world, macroscopic bodies are deformed under the action of a force or a torque. This would be the object of an engineering course in *solid mechanics*. Examples of different modes of deformation are *tension* and *compression*, *torsion*, *bending*, and *buckling*. The latter mode is characterised by a sudden failure of a structural member subjected to high compressive stresses. Examples of use of these terms are bending, or tension, or compression of a beam, buckling of a cylindrical steel column, torsion of a shaft, deflection of a thin shell. Within the science of applied mechanics, *mechanics of structures* is a field of study where the behavior of structures under mechanical load is investigated.

The required knowledge for this laboratory is Hooke's law[1] in its most basic form (the spring equation):

$$F = k\,x.$$

This relation, here in its scalar form, applies to a spring extended to length x from its relaxed position under the load of a force F. The relation in this simple form requires that the spring is *linear elastic*, which is another way of saying that the force F is proportional to x through the scalar k, the *spring constant*. The spring is linear elastic for not too large extensions. We shall see that we can apply an equivalent reasoning to the deformation of material bodies.

[1]So named after the 17th century British physicist and polymath Robert Hooke. First formulated in 1676 as a Latin anagram (ceiiinosssttuu), whose solution he published in 1678 as *Ut tensio, sic vis* (As the extension, so the force).

In this laboratory we work solely in the *elastic* region of structural elements subjected to various loads: Torsional load of a cylindrical shaft, and bending of a beam with square cross section. We investigate the dependence of deformation on known external loads, and in addition explore the importance of different cross-sectional shapes and choice of different materials.

We study only cases where applied loads and shape of the beam show nice symmetries. General cases of orientations of forces and cross sections would only become much more difficult to analyse.

We further assume that the elements investigated are always isotropic. This is not generally the case, since the properties of a material may depend on molecular properties, as may be the case for polymers, or on manufacturing procedures that favour preferential orientation, as for metals subjected to surface treatment such as milling and grinding.

6.2 Theory

In order to discuss the fundamentals of deformation, we introduce here two basic concepts for the analysis of mechanical stability in solid mechanics: *stress* and *strain*.

A structure element is said to be mechanically stable if it remains in equilibrium under load. This requires a balance between externally applied forces and internal reactions. The element may deform under the external load, but it will not break if in equilibrium.

Stress is the internal reaction to an external load.

Strain is the *relative deformation* of an object.

6.2.1 Strain and stress

Starting from quite general principles we might think of a continuum body as an assembly of points through which we imagine reference lines in a three dimensional grid. This allows us to consider deformations and the corresponding internal reactions in any direction: changes in length of lines (i.e. normal strain) and changes in angles between pairs of lines initially perpendicular to each other (i.e. shear strain).

For such general cases it is sufficient to know the normal and shear components of strain on a set of three mutually perpendicular directions, but we would have to treat material properties as tensors. This approach requires a somewhat more involved analysis than we need in this laboratory, and we chose here to demonstrate the principles by using one-dimensional cases, where the material properties are scalars.

Normal strain – a measure of deformation along a line of force

If a one-dimensional body (e.g. a rod) of length L is stretched or compressed by a couple of forces along the axis, attaining the new length L', then the deformation is $\Delta L \equiv L' - L$ and the *strain* ε is a measure of the deformation, representing the relative displacement along a line of the material body, i.e. normal strain,

$$\varepsilon \equiv \frac{L' - L}{L} \equiv \frac{\Delta L}{L}.$$

In particular, in the case of an increase in length, the normal strain is called *tensile strain*; if there is reduction in length, (compression) it is called *compressive strain* (sometimes tensile strain is used for both).

Shear strain – deformation when a body is twisted or cut

If a body is subjected to a pair of transverse forces (not acting along the same line) or a torque, the body is deformed, not along lines, but over sliding parallel planes. The amount of distortion associated with the attempted sliding of plane layers over each other, is called *shear strain.* It may be described as change of angles between lines within a deforming body.

The *engineering shear strain* is defined as the change in the angle between two material line elements initially perpendicular to each other in the undeformed or initial configuration.

Stress – a measure of internal forces

The external force, divided by the area to which the force is applied, is equal to the internal reaction that we call stress, if stability is maintained. The unit is therefore the same as for pressure: force divided by area, or pascal (or Nm^{-2}). Common practical units are megapascals (MPa or $\mathrm{N/mm}^2$) or gigapascals (GPa or $\mathrm{kN/mm}^2$).

If the deformations are recovered, i.e. if the material returns to its initial shape after the external forces have been removed, we speak of *elastic deformations.* This is the range of stress up to a certain threshold value known as the elastic limit or *yield stress*, sometimes *yield strength.* Keep in mind, that elastic deformations are not necessarily linear (as prescribed by Hooke's law). In fact, linear elastic deformation, exist for most materials only in a small part of the region of elastic deformation.

Irreversible deformations, i.e. deformations remaining after the external load has been removed, are called *plastic deformations.* On a microscopic level these result from slip between grains, or dislocations at the atomic level. For many engineering applications plastic deformation is unacceptable, and the yield strength is used as the design limitation. In other cases, for instance when bolts are tightened for certain critical applications, as e.g. bearing caps in internal combustion engines, bolts may be specified to be tightened with a torque that exceeds the yield strength in order to obtain maximum stability for a given size of bolt. Such bolts cannot be reused, but has to be discarded and replaced if removed.

If the external loads increase beyond the limit where the internal stress cannot respond, the structure will ultimately experience failure.

6.2.2 Normal stress and elastic modulus

A force $\vec{F}$ shall act on a rod of length L and cross-sectional area A as shown in Fig. 6.1. We assume the part, denoted by the grey shading, to be in equilibrium. This means that a second force $-\vec{F}$ must act on the surface A'. The normal stress is defined as:

$$\sigma = F/A$$

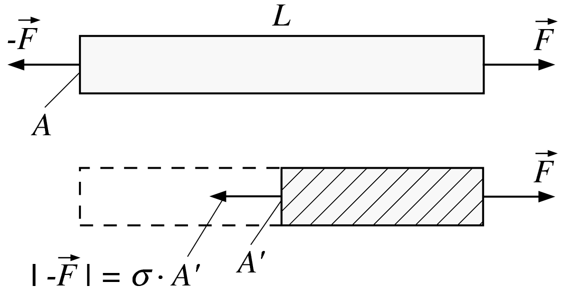


Figure 6.1: Normal strain..

Hooke's law states that the resulting strain, i.e. the relative elongation of the rod $\varepsilon = \Delta L/L$, is proportional to the stress σ for small strain:

$$\varepsilon = \frac{\Delta L}{L} = \frac{\sigma}{E} \tag{6.1}$$

The constant of proportionality depends on the material. It is called *elastic modulus* E.

6.2.3 Shear strain and shear modulus

We may define other elastic moduli, each describing in different ways the tendency of an object to deform elastically (i.e., non-permanently) when a load is applied.

If a cube is subjected to forces parallel to one of its faces, as shown in Fig. 6.2, the amount of shearing may be defined using a certain angle δ as in the figure. Any surface cut parallel to the faces where the forces act (horizontal cuts in the figure) experiences a shear stress $\tau = F/A$, whereas the normal stress is zero. The shear stress is actually a tensor entity, but remaining with our one-dimensional examples, we abstain from developing this further. If the external forces F are not too large, the shearing angle δ is proportional to the magnitude of the force, and therefore to the shear stress τ. The constant of proportionality is the *shear modulus* or *modulus of rigidity*, commonly designated with symbols G or μ.

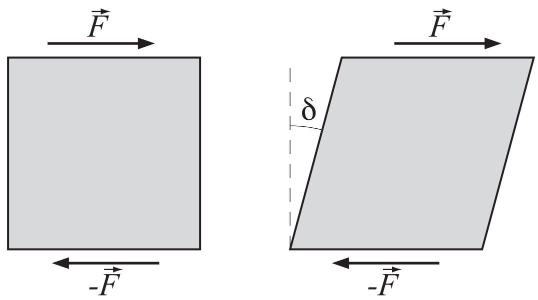


Figure 6.2: Schematic diagram of shearing of a cube with surface area A.

For shear stress the equivalent Hooke's law expression becomes

$$\tau = G\,\delta \tag{6.2}$$

We may take as a formal definition of shear modulus:

$$G \equiv \frac{\tau}{\delta}.$$

The shear modulus (or modulus of rigidity) G describes an object's tendency to *shear* (the deformation of shape at constant volume) when acted upon by opposing forces that are not in-line; it is defined as shear stress over shear strain. The shear modulus is part of the derivation of viscosity.

6.2.4 Torsion of a cylindrical rod

A cylindrical rod with length L and radius R is fixed and rigidly supported at one end, and may be loaded at the other end with an axial torque, as shown in Fig. 6.3. If the rod under load is in equilibrium, the external torque is balanced internal by the shear stress. The shear stress may be seen as acting in each imaginary perpendicular cut with a torque equal but opposite to the external torque (Fig. 6.4).

Referring to Fig. 6.4 the following equation holds:

$$\overline{AA'} \;=\; x\,\delta \;=\; r\varphi \tag{6.3}$$

According to Eq. 6.2 $\delta = \tau/G$, what yields

$$\tau(r,x) \;=\; G\,\frac{r\varphi}{x} \tag{6.4}$$

With respect to the center P of the cross-section, the total torque produces by the shear stress can be calculated by integrating over the cross-sectional area (see Fig. 6.4):

$$M_0 \;=\; \int_P r\tau\,df \;=\; \frac{G\,\varphi(x)}{x}\int_P r^2\,df \;=\; \frac{G\,\varphi(x)}{x}\int_0^R r^2\,2\pi r\,dr \;=\; \frac{G\,\varphi(x)}{x}\,\frac{\pi R^4}{2} \tag{6.5}$$

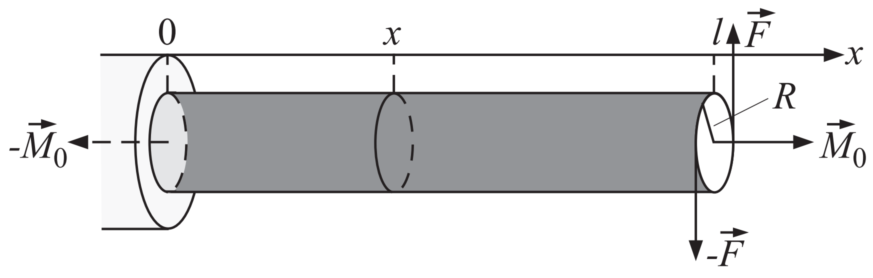


Figure 6.3: Torsion of a cylindrical rod – schematic overview.

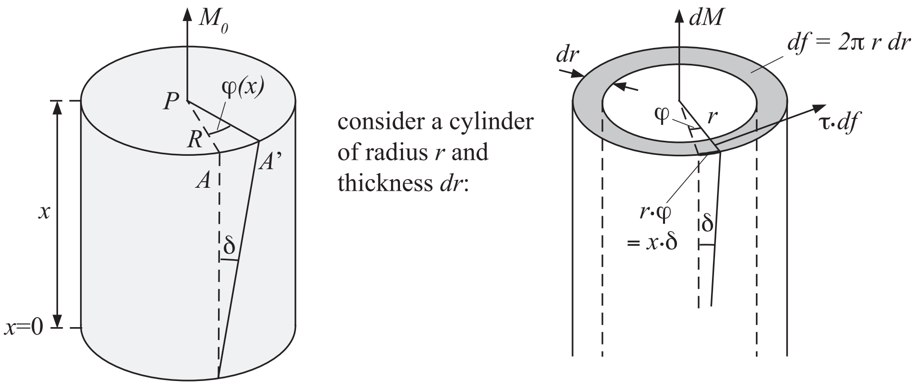


Figure 6.4: Torsion of a cylindrical rod under axial load and the torque produced by shear stress.

The angle of torsion at the end of the rod ($x = L$) then becomes:

$$\varphi(L) \;=\; \frac{2\,L\,M_0}{\pi\,G\,R^4} \tag{6.6}$$

In the case of a hollow cylindrical rod the integration extends from the inner radius R_i to the outer R_a:

$$M_0 \;=\; \frac{G\,\varphi(x)}{x}\,\frac{\pi\,(R_a^4 - R_i^4)}{2} \qquad \text{and} \qquad \varphi(L) \;=\; \frac{2\,L\,M_0}{\pi\,G\,(R_a^4 - R_i^4)} \tag{6.7}$$

6.2.5 Bending of a beam or rod of rectangular section

A beam or rod of rectangular section and length L be fixed at one end subject to an external load $\vec{F}_0$ at the other end, as shown in Fig. 6.5.

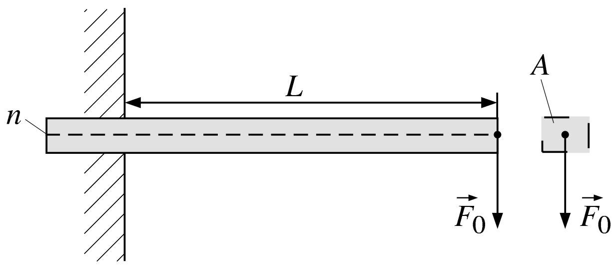


Figure 6.5: Load acting on a rod rigidly fixed at one end.

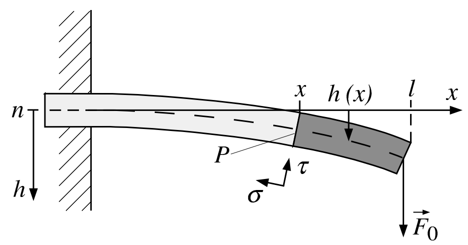


Figure 6.6: Bending of the rod under load.

The deformation of the rod consists of a compressive strain in the lower part of the rod and a tensile strain in the upper part. Both parts are separated by a line (actually better: surface), whose length remains unchanged under load. This line is called the *neutral line* (Fig. 6.6). Under load normal and shear forces act on any section of the rod. In equilibrium we obtain for the bending $h(x)$ of the piece extending from x to L and denoted by grey shading in Fig. 6.6:

$$h(x) \;=\; \frac{F_0}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right).$$

A detailed derivation of this equation can be found in the appendix.

In particular, for $x = L$:

$$h(L) = \frac{F_0 L^3}{3EI} \tag{6.8}$$

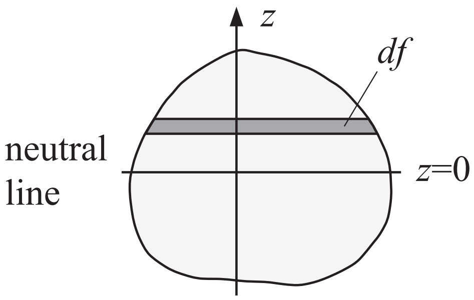


Figure 6.7: The area moment of inertia and the neutral line (surface).

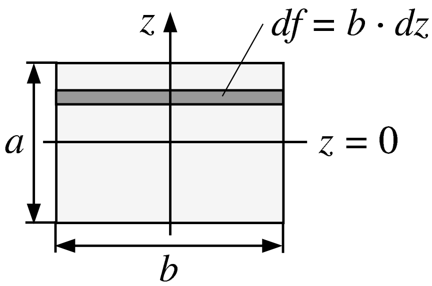
Here, the parameter I defines the *area moment of inertia*. It characterizes the distribution of material in a cross-section perpendicular to the neutral line. It is calculated by integration over the cross-sectional area with respect to the neutral line ($z = 0$):

$$I = \int_A z^2 \, df, \tag{6.9}$$

where z denotes the distance of the element *df* from the neutral line and along the direction of the external force. The bending of the beam is inversely proportional to the area moment of inertia.

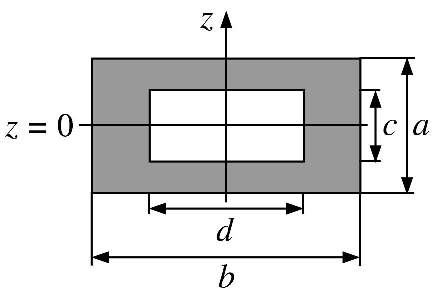
Comparing two rods of equal cross-section but different material distribution, the area moment of inertia increases for increasing distance of the material from the neutral line. We will give a few examples below:

- Area moment of inertia of a plain beam:



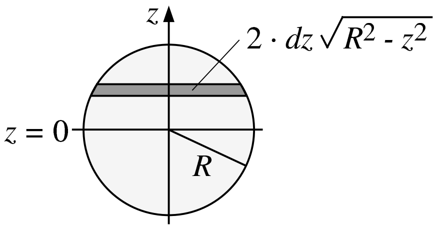
$$I = b \int_{-a/2}^{a/2} z^2 dz = \frac{ba^3}{12} \tag{6.10}$$

- Area moment of inertia of a hollow beam:



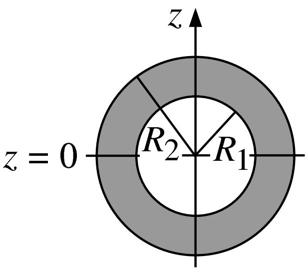
$$I = b \int_{-a/2}^{a/2} z^2 dz - d \int_{-c/2}^{c/2} z^2 dz = \frac{ba^3 - dc^3}{12} \tag{6.11}$$

- Area moment of inertia of a cylindrical rod:



$$I = \int_{-R}^{R} z^2 dz \sqrt{R^2 - z^2} = \frac{\pi R^4}{4} \tag{6.12}$$

- Area moment of inertia of a hollow cylinder:



$$I = I_{R_1} - I_{R_2} = \frac{\pi(R_2^4 - R_1^4)}{4} \tag{6.13}$$

6.3 Experimental

Goal of the experiment

In this laboratory we use the relations 6.6 for an external torque and the corresponding torsion angle, and 6.33 for an external force and the deflection. Our purpose is to determine the elastic constants for the materials used.

a) Torsion of a cylindrical rod:

- Measurement of the shear modulus G of brass, aluminum, and iron;
- verification of the relation $\varphi \propto R^4$;
- comparison of a solid cylinder shape and annular (hollow) cylinder shape.

b) Bending of a beam of rectangular cross-section:

- Measurement of the modulus of elasticity E of steel;
- verification of the relation $h(L) \propto L^3$;
- comparison of different area moments of inertia.

6.3.1 Torsion of a cylindrical rod

Setup

In the first part of the experiments, different cylindrical rods are loaded with external torque and the torsion angle $\phi = \phi(l)$ measured at the end $x = L$ of the rod. The setup is shown in Fig. 6.8: the rods is firmly fixed at one end. The external moment M_0 is produced with the aid of masses m attached to the periphery of a disk with radius R_s, fastened to the other end of the rod. Then:

$$M_0 = mgR_s. \tag{6.14}$$

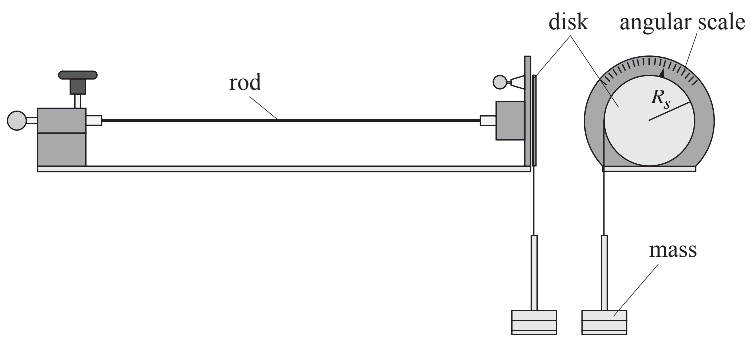


Figure 6.8: Experimental setup for measuring torsion.

The masses are chosen for each rod so that the torsion angle ϕ never increases beyond 15° and the deformation remains linear elastic.

The following rods are at disposal:

Table 6.1: Cylindrical rods

rod #	material	length [mm]	inner radius [mm]	outer radius [mm]
(1)	aluminum	750	-	2.5
(2)	brass	750	-	2.5
(3)	iron	750	-	2.5
(4)	iron	750	-	2.75
(5)	iron	750	1	3.0

Measurement of the shear modulus of iron, brass, and aluminum

Inserting Eq. 6.14 into Eq. (6.6) yields:

$$\varphi = \frac{2LmgR_s}{\pi GR^4} \tag{6.15}$$

If we plot φ as function of the load m we obtain a straight line of slope

$$a = \frac{2LgR_s}{\pi GR^4}. \tag{6.16}$$

From the slope and using the known geometrical parameters, the shear modulus G can be calculated.

Using rods no. (1), (2), and (3) (see table 6.1) measure G for the different materials. The rods all have identical geometry.

- Mount consecutively the rods (1), (2), and (3) allowing for the required pre-stress by putting an initial load of 250 g. Note the starting angle φ_0.
- Load the rod with a torque consecutively using the weights of mass given below and measure the corresponding torsion angle ϕ at the end of the rod:
  - Al: $m = 0.25$, 0.5, and 0.75 kg;
  - brass: $m = 0.25$, 0.5, 0.75, and 1.0 kg;
  - Fe: $m = 0.5$, 1.0, 1.5, 2.0, and 2.5 kg.

  Warning: The load may not exceed the maximum values given here!
- Compile all measurements in a table.

Problem 1: Why do we use a small load to define the experimental zero?

- Draw a diagram of angle ϕ as function of mass m. Draw the best fitting straight line for the points representing the measurements. Note, that the line not necessarily have to pass through the origin (why?).

- Calculate the slope $a = \Delta\varphi/\Delta m$. Note that the angle must be converted into radians before this calculation!

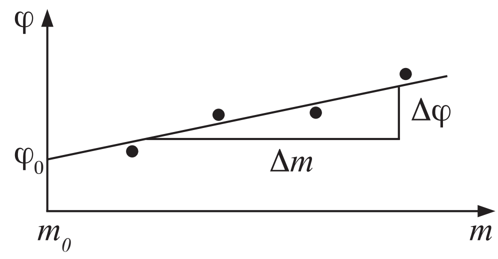


Figure 6.9: Data points of the torsional angle vs. mass.

No error calculus required.

Verification of the R^4-dependence of the torsion angle

Using rods (3) und (4) (see Table 6.1), which are identical except for their radius.

- Mount consecutively rods (3) and (4) ($R_{(3)} = 2.5\,\text{mm}$ and $R_{(4)} = 2.75\,\text{mm}$).
- Put a small load ($m = 250$ g) and determine zero as before: $\varphi_{0,(3)}$ and $\varphi_{0,(4)}$, respectively.
- Measure the torsional angle $\varphi_{(3)}$ and $\varphi_{(4)}$, respectively, using the same torque ($m = 2$ kg).
- Verify whether the torsional angles fulfil the following relation

$$\frac{\Delta\varphi_{(4)}}{\Delta\varphi_{(3)}} = \left(\frac{\varphi_{(4)} - \varphi_{0,(4)}}{\varphi_{(3)} - \varphi_{0,(3)}}\right) = \frac{R^4_{(3)}}{R^4_{(4)}}. \tag{6.17}$$

Comparison of a solid cylinder shape and annular cylinder shape

Using rods (4) and (5) (see Table 6.1) compare the torsion of rods of same material and length, but different shape.

- Mount the rod ($R_{(4)} = 2.75\,\text{mm}$ and $R_{(5),i} = 1\,\text{mm}$, $R_{(5),a} = 3\,\text{mm}$) and put a small load ($m = 250$ g) to determine zero: $\varphi_{0,(4)}$ and $\varphi_{0,(5)}$, respectively.
- Measure the torsional angle, $\varphi_{(4)}$ and $\varphi_{(5)}$, respectively, using the same torque in both cases ($m = 2$ kg).
- Compare the torsion angle of both rods at equal torsional load. Verify that:

$$\frac{\Delta\varphi_{(5)}}{\Delta\varphi_{(4)}} = \left(\frac{\varphi_{(5)} - \varphi_{0,(5)}}{\varphi_{(4)} - \varphi_{0,(4)}}\right) = \frac{R^4_{(4)}}{R^4_{(5),a} - R^4_{(5),i}}. \tag{6.18}$$

Problem 2: Calculate the cross sectional area of both rods from the measurements and discuss the ratio of mass to stiffness. What do you conclude for the ratio between material mass (or volume) to stiffness?

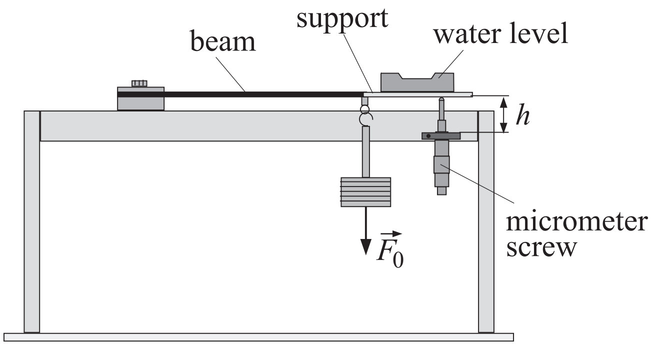


Figure 6.10: Set-up for the measurement of the deflection of a rectangular beam.

6.3.2 Bending of a beam with rectangular cross section

In the second experiment rectangular beams are loaded with known forces and the deflection $h = h(l)$ at the end of the beam measured. For the sake of simplicity, the beam is referred to as (rectangular) rod in the following. The set-up is shown in Fig. 6.10. The deflection of the rod is measured with the help of a second, small reference beam that is connected to the first at one end, and a micrometer at the other. A water level is used with the micrometer to position the reference beam horizontally. The vertical position read from the micrometer represents the position of the rod end, and any change in the reading for a different load is the corresponding deflection of the rod. The external force F_0 is applied by using weights of known masses m that are attached to the end of the rod. The weights are chosen for each rod such that the deflection does not exceed the linear elastic range. Relevant parameters for the rods used in the experiments may be found in Table 6.2.

Table 6.2: Rectangular rods for deflection experiments

rod	material	length [mm]	width [mm]	height [mm]	wall thickness [mm]
1	steel	250	8	6	-
2	steel	350	8	6	-
3	steel	350	6	8	-
4	steel	300	16	4	-
5	steel	300	15	10	1.5

Determination of the elastic modulus of steel

Plotting the deflection h of the rods as function of mass m yields, according to Eq. 6.8 and with $F_0 = mg$ a straight line of slope

$$a = \frac{L^3 g}{3\,EI}. \tag{6.19}$$

From the slope we can deduce the elastic modulus E. For this experiment, use rod (1) from Table 6.2.

- Mount to the rod and determine zero like previously.
- Measure the deflection of the rod for the following loads: $m = 0.25$, 0.5, 1.0, 1.5, and 2.0 kg.
- Plot h as function of m on graph paper and draw the best fitting line by eye, as shown in Fig. 6.11.
- Calculate the slope $a = \Delta h/\Delta m$ of this line and determine the elastic modulus E.

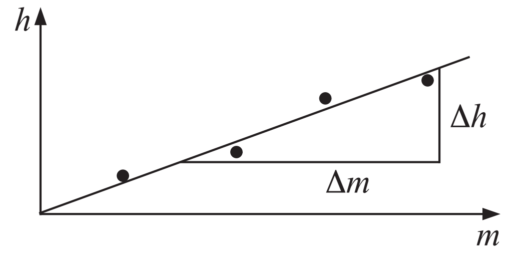


Figure 6.11: Data points of deflection vs. mass.

No error calculus required.

Verification of the l^3-dependence of the deflection

Rods (1) and (2) from Table 6.2 are identical except for their length.

- Mount consecutively rods (1) and (2) ($L_{(1)} = 25$ cm, $L_{(2)} = 35$ cm) and determine zero for small load.
- Load both rods with the same mass ($m = 1$ kg) and measure the deflections $h_{(1)}$ and $h_{(2)}$.
- Verify the relation

$$\frac{h_{(2)}}{h_{(1)}} = \frac{L^3_{(2)}}{L^3_{(1)}}. \tag{6.20}$$

Dependence of the deflection on the area moment of inertia.

According to Eq. (6.8) the deflection is proportional to the inverse of the area moment of inertia. In order to verify this, we compare the deflections of rods (2) and (3) (length 35 cm) and (4) and (59 (length 30 cm). The cross-sections of these rods are depicted in Fig. 6.12.

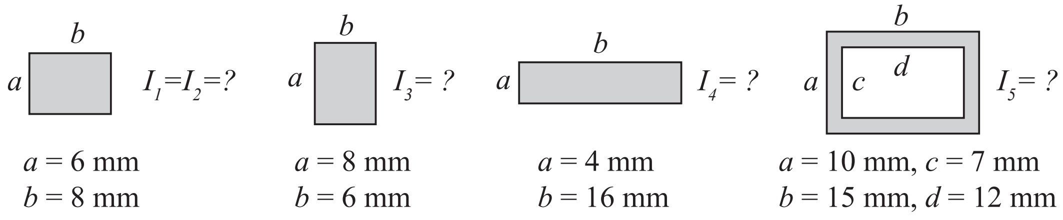


Figure 6.12: Cross-sections of the various rods used in this experiment.

Problem 3: Calculate the area moments of inertia $I_{(2)}$, $I_{(3)}$, $I_{(4)}$, and $I_{(5)}$ for the rods (2), (3), (4), and (5) from Table 6.2.

- Mount rod (2) and determine zero.
- Determine the deformation $h_{(2)}$ for a given load ($m = 1\,\mathrm{kg}$).
- Repeat the measurements for rods (3) through (5) using the following loads: $m = 1$ kg for rods (2) and (3), and $m = 1.5\,\mathrm{kg}$ for rods (4) and (5), determine $h_{(3)}$, $h_{(4)}$, and $h_{(5)}$.
- Verify the following relationships:

$$\frac{h_{(3)}}{h_{(2)}} = \frac{I_{(2)}}{I_{(3)}} \qquad \text{and} \qquad \frac{h_{(5)}}{h_{(4)}} = \frac{I_{(4)}}{I_{(5)}}. \tag{6.21}$$

Problem 4: Calculate and compare the cross-sectional areas of rods (4) and (5).

- Discuss the influence of the cross-sectional shape and the ratio of mass of the rod to the stiffness.
- How can you explain the difference between experimental results and expected values in the case of the hollow rod (5)? (Discuss the stability of the shape when subject to heavy loads.)
- (Further assignment:) Compare qualitatively the sections shown in Fig. 6.12 in upright orientation and rotated by 90 deg ('portrait' and 'landscape').

Appendix: Bending of a rectangular section beam

A rod of length l with a rectangular cross-sectional area A, as in Fig. 6.13, is rigidly fixed at one end, while the other is unsupported. At the free end an external force $\vec{F}_0$ may be applied perpendicular to the symmetry line of the rod. Equilibrium requires again that external forces are balanced by internal reactions.

In Fig. 6.14 a situation is shown when the rod is being acted on by the external force $\vec{F}_0$.

This is called a *bending moment*, by physicists also *torque*. Note that in some textbooks, a torque is created by two equal but opposite forces not in-line, also called a *couple*. There is no resulting translational force in this case, as there is for the rod discussed here. In the rod there will be both tensile stresses and compressive stresses; the upper side the material will be stretched, the lower side will be compressed. This means that in the rod there must be an un-deformed, 'neutral' line (or surface), often called the *neutral axis*, n. A cross section perpendicular to n will remain perpendicular independent of the bending, as long as the material remains in the elastic region.

Failure in bending will occur when the bending moment is sufficient to induce tensile stresses greater than the yield stress of the material, either in extension or compression. Failure because of shear may occur before failure in bending.

A cross-sectional surface perpendicular to the neutral axis is subject to a constant shear stress τ, (whereas the normal stress varies because of the bending).

$$\tau = \frac{F_0}{A} \tag{6.22}$$

The external force F_0 is thus compensated by stress as long as the rod remains in equilibrium. cf Fig. 6.14.

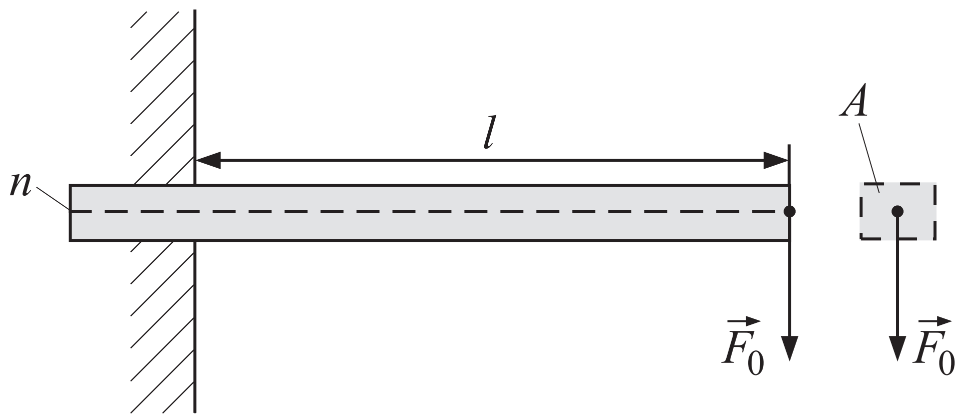


Figure 6.13: Rod with rectangular cross section before force is applied.

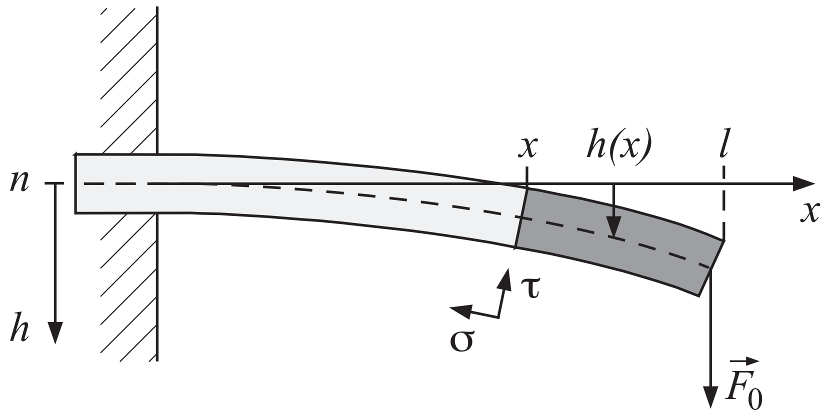


Figure 6.14: Bending of a rod with rectangular cross section under the action of a perpendicular force.

A formula for the deformation

Next our purpose is to develop expressions that connect the measurable deformation with the parameters of stress introduced above. Macroscopic equilibrium in particular means that external forces and moments must be globally balanced. The equation for equilibrium of the moments for the rod would look like this:

$$M_0 = F \cdot L \tag{6.23}$$

where M_0 is the moment at the point of fixture in the wall.

Since we are not dealing with a rigid mechanics problem, this relation is of little use. We therefore apply the same reasoning to the part of the rod shaded grey in Fig. 6.14, and separated from the rod by an imaginary perpendicular cut at co-ordinate x. By doing so, we are able to introduce internal stress into the equilibrium equations.

First, we write the global equilibrium for the piece of the rod cut at x:

$$M(x) = F_0 \cdot (L - x) \tag{6.24}$$

This relation is valid for any cut at a position x. The moment $M(x)$ varies with the co-ordinate x and assumes its maximum at $x = 0$, i.e. at the position where the rod is rigidly fixed.

If the cross section of the rod is constant and equal in form everywhere, we the bent rod assumes the form of a circular shape with constant radius R. We introduce the radial co-ordinate z, with origin at the neutral axis cf Fig. 6.15.

The strain ε, the relative change in length, may be written:

$$\varepsilon(z) = \frac{(R + z) \cdot \phi - R \cdot \phi}{R \cdot \phi} = \frac{z}{R} \tag{6.25}$$

The normal strain $\varepsilon(z)$ is accompanied by a corresponding normal stress $\sigma(z)$

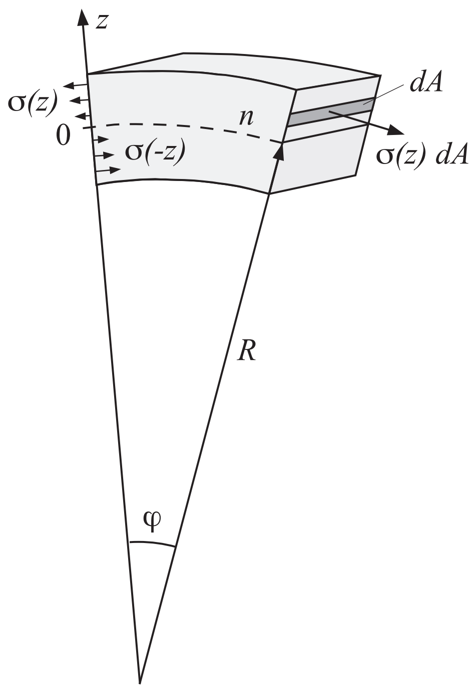


Figure 6.15: Bending a rectangular rod – detail.

In the elastic range Hooke's law is valid, and we write, using that the relative change of length (the normal stress) is proportional to the normal strain, with the elastic modulus E as factor of proportionality:

$$\sigma(z) = E \cdot \varepsilon(z) = E \cdot \frac{z}{R} \tag{6.26}$$

In an infinitesimal area element dA, cf Fig. 6.15, the normal strain is $\sigma(z)$ at distance z from the neutral axis (the origin). Calculating the moment relative to the neutral axis, the moment arm is likewise z, so that the infinitesimal moment contributed by area element dA is $z \cdot \sigma(z)$. Integrating over the cross section A, and inserting $\sigma(z)$ from Eq. 6.26, we get the total moment for the neutral axis:

$$M_0' = \int_A z \cdot \sigma(z)\, dA = \frac{E}{R} \cdot \int_A z^2\, dA = \frac{E}{R} \cdot I_z \tag{6.27}$$

Appearing in this expression is the so called *area moment of inertia* or *second moment of inertia* $I_z = \int_A z^2\, dA$, (sometimes the second moment of area), making a short digression due here.

The resistance of a beam to bending and deflection depends not only on the load but also on the geometry of the beam's cross-section, or more precisely, how area is distributed in the cross section.

A rectangular beam of width b and thickness d in the plane of bending has the area moment:

$$I_z = \frac{b \cdot d^3}{12} \tag{6.28}$$

A beam with I-shaped cross section will have a slightly smaller area moment. We might think of it as cut out of the rectangular beam, its profile fitting inside. The much smaller mass, however, in the end gives a stiffer and lighter structure, which is a consequence of the relatively large area far from the neutral line. For this reason, beams with higher area moments of inertia, such as I-beams, are often preferred in building construction, as opposed to other beams with the same area[2].

Despite the apparent likeness, the second moment of area is not at all the same thing as the moment of inertia, as used, i.e, in Newtons's second law for calculating angular acceleration. As a word of warning, It is not uncommon to refer to the second moment of area as the moment of inertia and use the same symbol I for both. It is usually clear from the context, though, if accelerational or bending moment of inertia is meant, and it becomes obvious from the units: second moment of area has unit length to the fourth power, whereas a moment of inertia has the unit mass times length squared.

In order that the structure element be in equilibrium, we require that

$$M_0'(x) = M_0(x) \quad \text{equivalent to} \quad \frac{E}{R(x)} \cdot I_z = F_0 \cdot (\, l - x \,) \tag{6.29}$$

For large bending radii R, i.e. a small deformation, we approximately have:

$$\frac{1}{R} = \frac{d^2 h}{dx^2} \tag{6.30}$$

with $h(L)$ the deflection from the centre line at the free end, cf Fig. 6.14, and $h(x)$ the deflection at co-ordinate x. Inserting in Eq. 6.14 yields a differential equation

$$\frac{d^2 h}{dx^2} = \frac{F_0}{E \cdot I_z} \cdot (\, L - x \,) \tag{6.31}$$

which is twice integrated to give:

$$h(x) = \frac{F_0}{E \cdot I_z} \cdot \left(\frac{l \cdot x^2}{2} - \frac{x^3}{6} \right) + C_1 \cdot x + C_2 \tag{6.32}$$

The two integration constants may be determined from the boundary conditions at $x = 0$ where the beam is fixed. Here we have $h(0) = 0$ and $\frac{dh}{dx} = 0$, from which follows $C_1 = C_2 = 0$. The maximum deflection at the end ($x = L$) is then

$$h(l) = \frac{F_0}{E \cdot I_z} \cdot \left(\frac{L^3}{2} - \frac{l^3}{6} \right) = \frac{F_0 \cdot L^3}{3\, E \cdot I_z} \tag{6.33}$$

[2]For the case of torsional load, the *polar moment of inertia*, characterises an object's ability to resist torsion.