C

1. Experiments with capacitors

1.1 Introduction

In many processes in physics, chemistry, and biology, the change in time of a quantity often is proportional to its momentary value. Examples include:

- The number of decays of instable atoms in a radioactive sample within a certain time is proportional to the number of instable atoms present in the sample.
- The change in concentration of a reactant during a chemical reaction is proportional to the concentration.
- The growth rate of a given population is proportional to the number of individuals in that population.
- A body moving through a viscous liquid experiences a force of friction (and thereby an acceleration), which is proportional to its velocity.
- The current during the discharge of a capacitor across an electrical resistance is proportional to the charge of the capacitor.

All these processes are quantitatively described using **exponential functions**. Using the example of charging and discharging of a capacitor, we will introduce the bascis characteristics of exponential functions. Some keywords:

- exponential function,
- electric circuits,
- Kirchhoff's laws,
- charging and discharging of a capacitor, and
- measurement of electrical currents and voltages.

1.2 Theory

1.2.1 Charging and discharging of a capacitor

The generic electrical circuit of these experiments is shown in Fig. 1.1: The capacitor with capacity C is charged across the (Ohmic) resistor R if the switch is in positions 1, and discharged when in position 2.

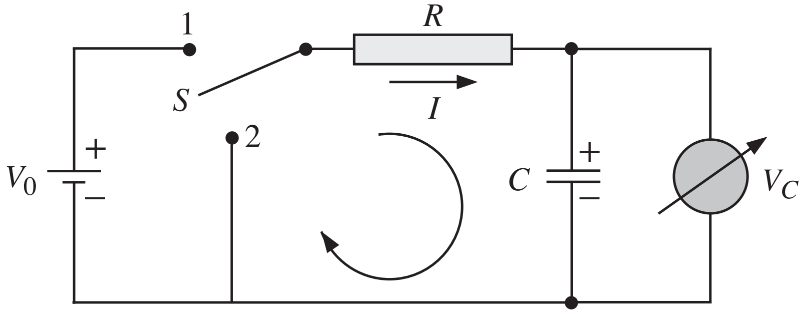


Figure 1.1: Generic circuit.

Charging

A power supply providing the DC voltage V_0 is connected in series to a resistor R and a capacitor C. At time $t = 0$, the switch S is put into position 1; as a consequence, the capacitor is charges via the resistor R. We will derive an expression describing the time dependence of voltage $V(t)$ and current $I(t)$.

Discharging

The capacitor C is charged; the corresponding voltage between the two contacts be V_0. At $t = 0$ the switch S is switched to position 2: the capacitor is discharged across the resistor R. Again, we are interested in voltage $V(t)$ and current $I(t)$ as function of time.

The initial conditions at time $t = 0$ are:

$$Q(0) = 0$$

$$Q(0) = Q_0 = V_0 C$$

Using the first one of Kirchhoff's laws we obtain in both cases:

$$V_0 = I R + \frac{Q}{C}$$

$$0 = I R + \frac{Q}{C}$$

Inserting the relation $I = dQ/dt$ yields:

$$V_0 = \frac{dQ}{dt} R + \frac{Q}{C} \qquad (1.1)$$

$$0 = \frac{dQ}{dt} R + \frac{Q}{C} \qquad (1.2)$$

This means that the change in charge on the capacitor is proportional to the charge $Q(t)$.

The solution of the inhomogeneous differential equation Eq. 1.1 is given by the sum of the solution of the corresponding homogeneous equation Eq. 1.2 and one particular solution of the inhomogeneous equation.

In the limit of very long times $t \to \infty$ the charge approaches the final value $Q_0 = V_0\, C$:

$$Q(t = \infty) \;=\; Q_0 \;=\; V_0\, C.$$

The general solution of Eq. 1.1 becomes:

$$Q(t) \;=\; A'\, e^{-\frac{t}{RC}} + Q_0$$

The solution of the homogeneous equation Eq. 1.2 is found by seperating the variables Q and t:

$$\frac{dQ}{Q} = -\frac{dt}{RC}.$$

Integration leads to:

$$\ln Q = -\frac{t}{RC} + \ln A.$$

Using the constant of integration $\ln A$ we obtain

$$Q(t) = A\, e^{-\frac{t}{RC}}. \qquad (1.3)$$

Determination of the constants from the starting conditions:

$$\begin{aligned} Q(0) &= A' + Q_0 = 0 \;\Rightarrow\; A' = -Q_0 \\ Q(t) &= Q_0\,(1 - e^{-\frac{t}{RC}}) \\ &= V_0\, C\,(1 - e^{-\frac{t}{RC}}) \end{aligned}$$

$$\begin{aligned} Q(0) &= Q_0 = A \\ Q(t) &= Q_0\, e^{-\frac{t}{RC}} = V_0\, C\, e^{-\frac{t}{RC}} \end{aligned}$$

Calculation of the current $I = dQ/dt$:

$$I(t) \;=\; \frac{V_0}{R}\, e^{-\frac{t}{RC}} \qquad (1.4)$$

$$I(t) \;=\; -\frac{V_0}{R}\, e^{-\frac{t}{RC}} \qquad (1.5)$$

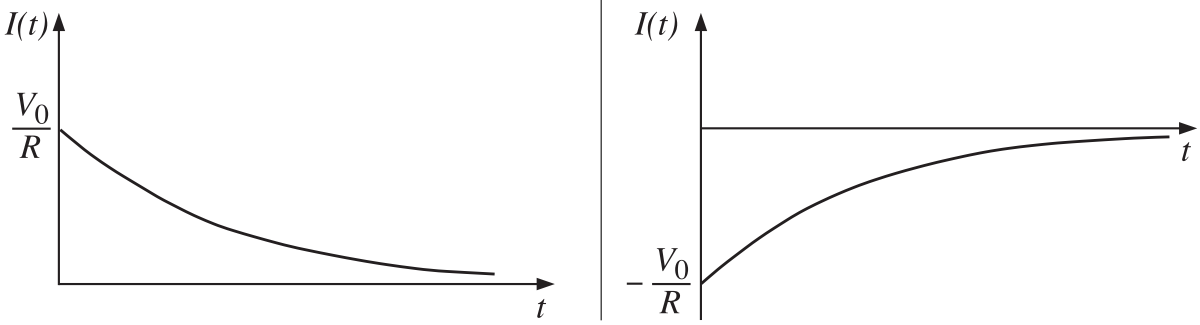


Figure 1.2: Current during charging and discharging of a capacitor.

Calculation of the voltage $V_C = Q/C$:

$$V_C(t) \;=\; V_0\,(1 - e^{-\frac{t}{RC}}) \qquad (1.6)$$

$$V_C(t) \;=\; V_0\, e^{-\frac{t}{RC}} \qquad (1.7)$$

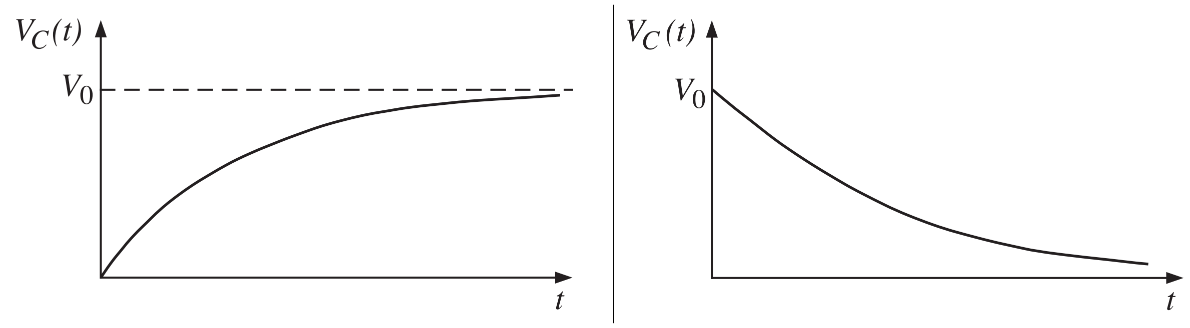
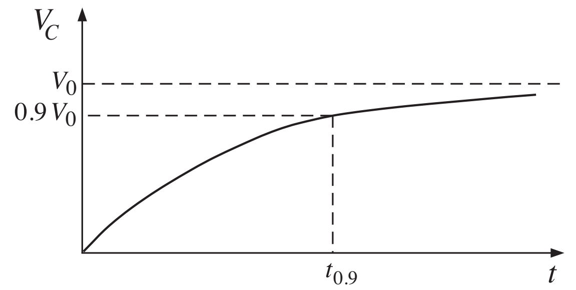


Figure 1.3: Voltage during charging and discharging of a capacitor..

Often, the characteristic timescale $t_{0.9}$ (rise time) is given, which is defined as time needed to reach 90% of the value in the asymptotic limit $V_C = 0.9\,V_0$:



$$V_C(t_{0.9}) \;=\; 0.9\,V_0 \;=\; V_0\,(1 - e^{-\frac{t_{0.9}}{RC}})$$

Figure 1.4: Definition of the rise time $t_{0.9}$.

Problem 1:

- Determine $t_{0.9}$ for given values of R and C.
- How should R and (or) C be changed in order to obtain a shorter rise time $t_{0.9}$?

1.2.2 Properties of the exponential function

The exponential function $f(t) = A\ e^{-\alpha t}$ possesses the following properties:

- At any time t, the derivative df/dt with respect to time is proportional to the current value $f(t)$:
$$\frac{df}{dt} \;=\; -\alpha\, A\, e^{-\alpha t} \;=\; -\alpha f(t).$$
The negative sign indicates that $f(t)$ is monotonically decreasing.

- For equidistant steps Δt, the value of $f(t)$ always changes by the same factor at each step:
$$e^{-\alpha\,\Delta t}.$$
The latter can easily be seen in a graphical representation of an exponential function, like in Fig. 1.5.

The function $f(t) = A\,e^{-\alpha t}$ reaches the value 0 in the asymptotic limit $t \to \infty$. For this reason, processes described by a decreasing exponential function are characterized by a time called *half-life* $T_{1/2}$ or alternatively by the characteristic time constant τ.

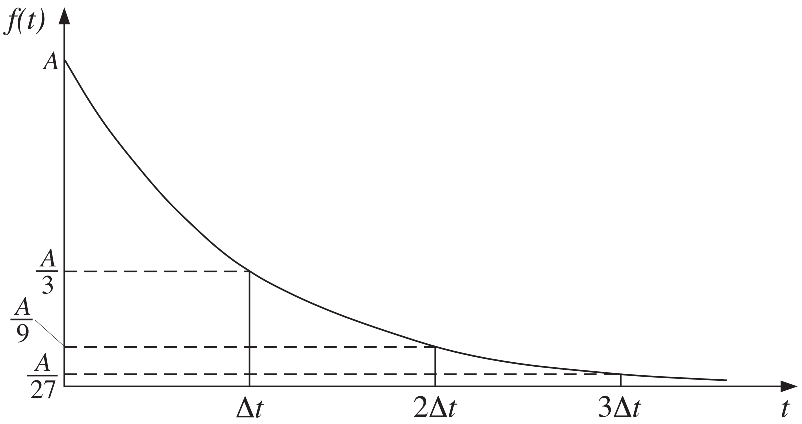


Figure 1.5: Exponential function: decrease by a factor of one third in each interval δt.

Half-life:

After the half-life period $T_{1/2}$ the function $f(t)$ is dropped by half of its initial value:

$$
\begin{aligned}
f(0) &= A \\
f(T_{1/2}) &= \frac{A}{2} = A\,e^{-\alpha T_{1/2}} \\
\ln\frac{1}{2} &= -\alpha\,T_{1/2} \\
T_{1/2} &= \frac{\ln 2}{\alpha}
\end{aligned}
$$

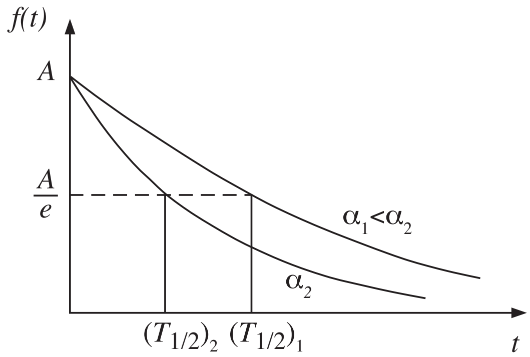


Figure 1.6: Illustration of the half-life.

(Characteristic) Time constant:

After a time corresponding to the time constant τ the function $f(t)$ reaches $1/e$ of its initial value:

$$
\begin{aligned}
f(0) &= A \\
f(\tau) &= \frac{A}{e} = A\,e^{-1} = A\,e^{-\alpha\tau} \\
\tau\alpha &= 1 \qquad \Rightarrow \qquad \tau = \frac{1}{\alpha}
\end{aligned}
$$

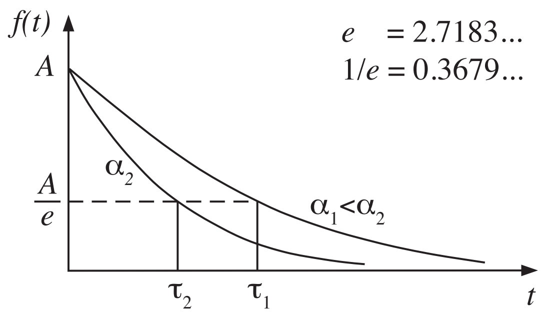


Figure 1.7: Illustration of the time constant.

Ex.: Discharging of a capacitor

According to Eq. 1.6) the voltage behaves as:

$$V_C = V_0 e^{-\frac{t}{RC}}$$
$$\alpha = \frac{1}{RC}$$

$$\text{ergo:} \quad \tau = RC \quad (1.8)$$
$$T_{1/2} = RC \ln 2 \quad (1.9)$$

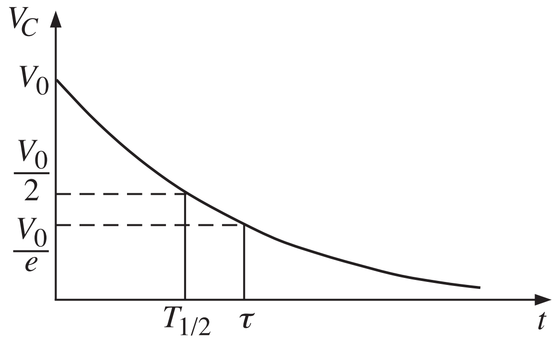


Figure 1.8: Discharging of a capacitor.

1.3 Experimental Part

1.3.1 Goal of this experiment

A capacitor is charged and discharged through a resistor R. Voltage across the capacitor and the current through the resistor are monitored as function of time for two different RC time constants $\tau_1 \simeq 10s$ and $\tau_2 \simeq 15s$.

1.3.2 Setup

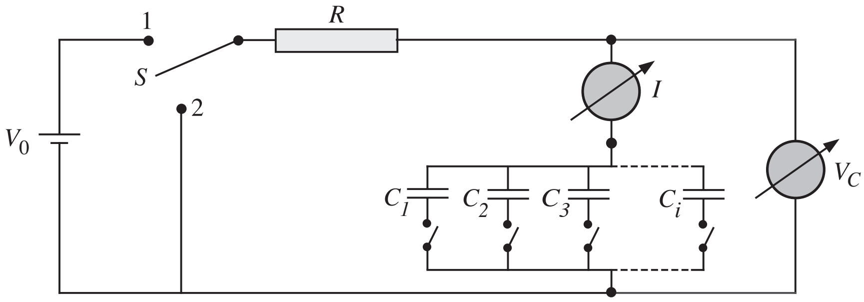


Figure 1.9: Electric circuit.

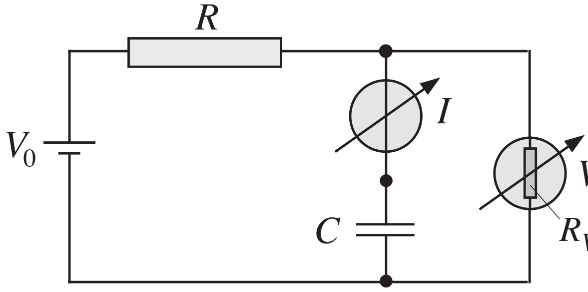
The power supply and volt- and amperemeter are connected to the corresponding jacks. Using the switches On-Off (Ein-Aus) the number of capacitors can be varied. Since the capacitors are connected in parallel, the total capacitance of the circuit is given by the sum of the single capacitances:

$$C_{tot} = \sum C_i$$

The resistor R is to be plugged onto the corresponding jacks. Choose a combination of R and C which leads to a time constant $RC \simeq 10..15$ s according to Eq. 1.8.

Internal resistance of the voltmeter

Every real voltmeter has a finite internal resistance R_V. If we take this internal resistance into account we obtain the following circuit:



The voltmeter used here has an internal resistance of $10^7\Omega$. Therefore $R_V \gg R$, *i.e.* the current passing through the voltmeter can be neglected here.

Figure 1.10: Internal resistance of the voltmeter.

1.3.3 Assembly

- Connect the power supply and the instruments.
- Choose the time constant and the corresponding combination of R and C.
- Plug in the resistor R.
- Activate the capacitors required.
- Turn on the voltage and set the voltage to 15 V.

1.3.4 Measurements

Charging the capacitor (Switch in position 1):

- The voltage across the capacitors is to be measured as function of time after switching to position 1 ($t = 0$): for doing so read and note the voltage V_C every 5 s.
- Discharge the capacitors.
- For the second time, record the current as function of time in the same way than the voltage before. In particular, pay attention to the current at $t = 0$ when switching on.

Problem 2: Why the voltage does not reach V_0?

Discharging the capacitor (Switch in position 2):

- Charge the capacitors up to a voltage V.
- Put the switch in position 2 and record voltage and current as function of time in the same way as during charging.

1.3.5 Report

- Answer the questions in the text.
- Briefly resume the goal and the experiment itself. In particular, mention the following topics:
  - Characteristic properties of the exponential function.
  - Meaning of half-life and time constant τ
- Analysis:
  - Plot the voltage $V_C(t)$ and the current $I(t)$ on linear graph paper.

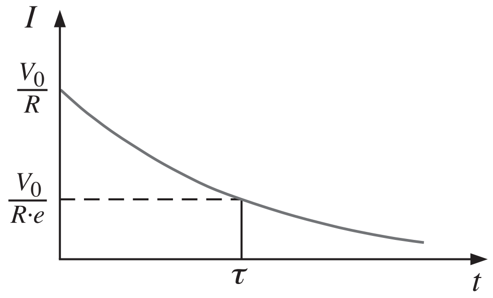
– Moreover, plot the *decreasing* curves (current for charging, voltage for discharging) on semi-logarithmic paper.

– Determine the time constants τ from these plots as outlined below. Compare the values obtained with the values calculated from the resistance and the total capacitance.

Hint: If you use semi-logarithmic graph paper, the slopes are given by the difference of the **logarithms** of the values plotted on the logarithmic axis, as shown in Fig. 1.12. Alternatively, the values $\ln V(t)$ and $\ln I(t)$ may be plotted against a linear scale (normal linear graph paper) if no semi-logarithmic paper is at disposal.

Determination of the time constant from the plots

- From the linear representation of the exponantial function:



As shown in the figure, the characteristic time constant can be read directly from the plot.

Figure 1.11: Determination of the time constant τ from the linear plot.

- From the logarithmic representation of the exponential function:

Logarithmizing

$$I = \frac{V_0}{R}\, e^{-\frac{t}{RC}}$$

leads to

$$\ln \frac{RI}{V_0} = -\frac{t}{RC}.$$

Note: Only values can be logarithmized, not units. Therefore, the units must cancel. Then we can rewrite the equation:

$$\ln I + \ln \frac{R}{V_0} = -\frac{t}{RC}. \qquad (1.10)$$

Using the logarithm at base 10, we obtain the same relation except for a correction factor $\log e$:

$$\log \frac{IR}{V_0} = -\frac{t}{RC} \log e$$

and, after the units having canceled:

$$\log I + \log \frac{R}{V_0} = -\frac{t}{RC} \log e. \qquad (1.11)$$

Equations (1.10) and (1.11) represent straight lines. For the slopes we obtain:

$$\frac{\Delta(\ln I)}{\Delta t} = -\frac{1}{RC} \qquad \frac{\Delta(\log I)}{\Delta t} = -\frac{\log e}{RC}$$

$$RC = \left|\frac{\Delta t}{\Delta(\ln I)}\right| \qquad RC = \log e \left|\frac{\Delta t}{\Delta(\log I)}\right|$$

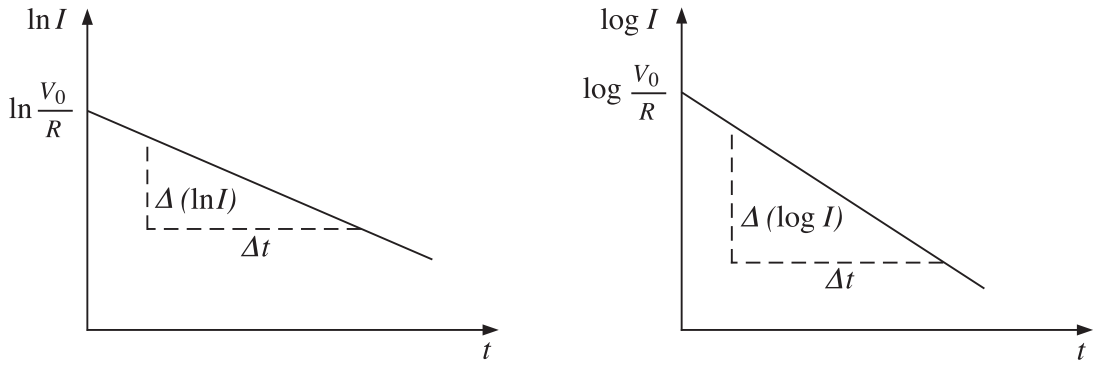
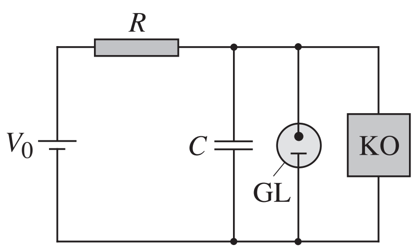


Figure 1.12: Calculations of the slopes using natural (left-hand-side) and common logarithm (right-hand-side).

Demonstration

The assistant will demonstrate sawtooth oscillations (also termed relaxation oscillations) using several different combinations of resistor and capacitor in the electric circuit shown in Fig. 1.3.5. Observe the changes in frequency and ignition time. The circuit is explained in detail in the appendix.



The voltage $V_C(t)$ is measured on the oscilloscope.

Figure 1.13: Electric circuit used for the demonstration. The polarity of the bulb is important!

1.4 Appendix

1.4.1 Generation of sawtooth oscillations using a glow lamp

A glow lamp is a bulb filled with gas and equipped with two electrodes.

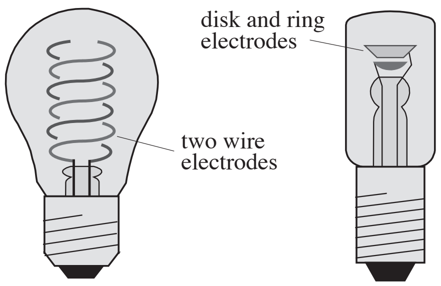


Figure 1.14: Types of glow lamps.

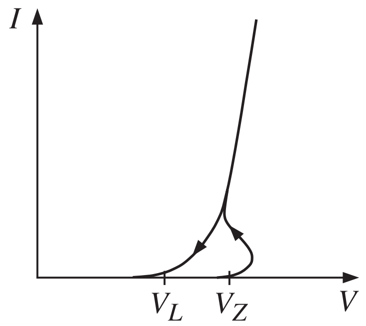


Figure 1.15: $I-V$-curve of a glow lamp. V_Z denotes the voltage at which the lamp ignites, V_L the voltage at which the lamp goes off.

A voltage is applied to the electrodes. Free electrons present in the gas are accelerated by the electric field. If the field is sufficiently strong, the electrons acquire enough energy between two collisions to ionize gas atoms. The number of free charges, thereby, increases generating an avalanche of charge carriers and ionized atoms. The current is large and the inner resistance R_i of the bulb decreases strongly. The critical field or voltage is called **ignition voltage** of the bulb V_Z.

Circuit for generating sawtooth oscillations

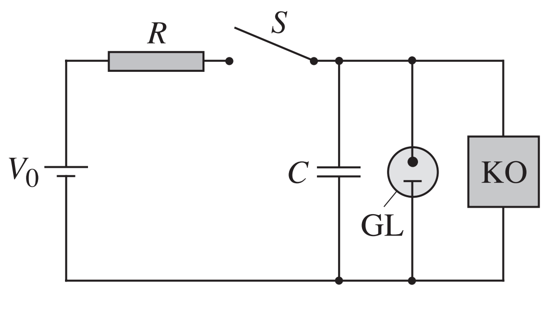
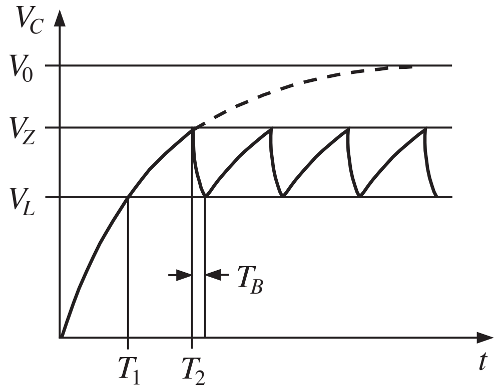


Figure 1.16: Electric circuit.

A capacitor C is connected in parallel to a glow lamp, both are linked to a voltage source through a resistor R. The resistor R shall be large compared to the inner resistance of the ignited bulb R_i. As long as the current through the bulb is small, i.e. $V_C < V_Z$, the capacitor is slowly charged through the resistor R. When the voltage V_C reaches the ignition voltage, the bulb ignites and the capacitor is rapidly discharged through the bulb R_i.

Since $R \gg R_i$, the re-charging of the capacitor through R is slow and V_C decreases until it reaches the voltage V_L, at which the bulb goes off. Then, the capacitor start charging again and the whole cycle repeats. If the voltage V_C is measured using an oscilloscope we will see the following curve: (compare to Fig. 1.3):



$T_B =$ bulb is ignited,
$T_2 - T_1 =$ recharging of the capacitor from V_L to V_Z .

Figure 1.17: Voltage of the capacitor.

Calculation of the frequency

Charging of the capacitor according to Eq. 1.6)):

$$\begin{aligned} V_L &= V_0\,(1 - e^{-\frac{T_1}{RC}}) \Rightarrow \frac{V_0 - V_L}{V_0} &= e^{-\frac{T_1}{RC}} \\ V_Z &= V_0\,(1 - e^{-\frac{T_2}{RC}}) \Rightarrow \frac{V_0 - V_Z}{V_0} &= e^{-\frac{t_2}{RC}} \end{aligned}$$

Division of both equations and logarithmizing leads to

$$\frac{V_0 - V_L}{V_0 - V_Z} = e^{-\frac{1}{RC}\cdot(T_1 - T_2)}$$

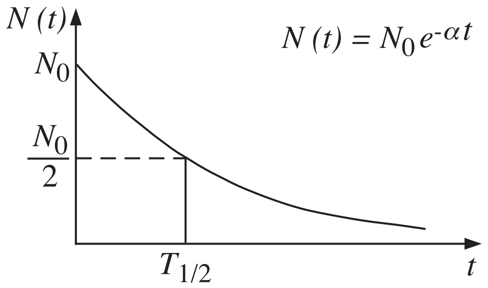
$$\begin{aligned} \ln\frac{V_0 - V_L}{V_0 - V_Z} &= -\frac{1}{RC}\cdot(T_1 - T_2) \\ T_2 - T_1 &= RC\,\ln\frac{V_0 - V_L}{V_0 - V_Z}. \end{aligned}$$

In the case $R \gg R_i$, we may neglect the ignition time T_B with respect to $T_2 - T_1$. Hence we obtain for the frequency ν of the oscillations:

$$\nu \approx \frac{1}{T_1 - T_2} = \frac{1}{RC}\,\frac{1}{\ln\frac{V_0 - V_L}{V_0 - V_Z}} \approx \frac{1}{RC}$$

1.4.2 Some important applications of the exponential function

a) Radioactive decay

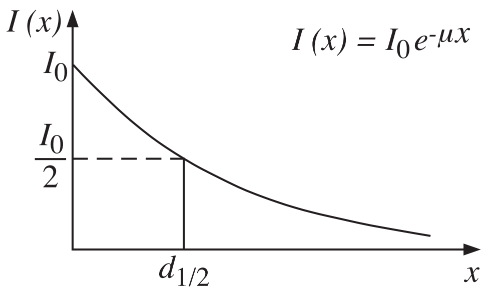


N_0 = number of unstable nuclei at time $t = 0$
$N(t)$ = number of unstable nuclei at time t
At half-time $T_{1/2}$, half of the nuclei did not yet decay.

Figure 1.18: Radioactive decay.

Depending on the stability of the isotopes, typical half-life times range from 10^{-22} seconds up to thousands of years.

b) Absorption of light in matter (electromagnetic radiation including γ-rays)



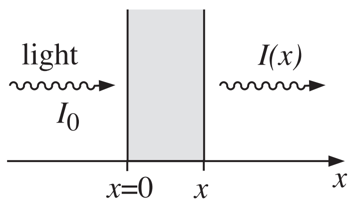


Figure 1.19: Absorption of electromagnetic radiation.

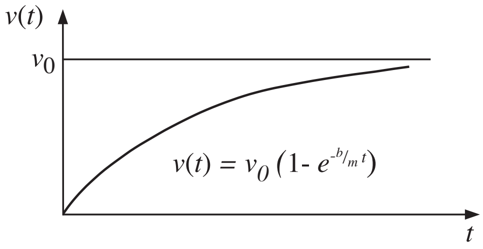
Be I_0 the intensity at $x = 0$. $I(x)$ denotes the intensity of the radiation after having passed a layer of thickness x of the material. $d_{1/2}$ corresponds to the thickness of the layer at which 50% of the radiation is absorbed.

Note: The intensity of $\gamma-$-radiation never reaches zero even for very thick radiation shielding.

c) Motion in viscous liquids

In viscous liquids the friction is proportional to and opposite to the speed. The equation of motion of a body moving through a liquid reads:

$$m\,\frac{d^2x}{dt^2} = m\,\frac{dv_x}{dt} = -\beta\, v_x + F_0$$



F_0 = driving (constant) force, e.g. gravity or electric field
βv_x = friction in a viscous liquid
v_0 = stationary speed

Figure 1.20: Motion through a viscous liquid.

d) In biology

In many biological processes like decomposition, resorption, or excretion, for instance, several processes following exponential functions with different time constants are superimposed in complex ways.