WS2

12. AC properties of LCR-electric circuits

12.1 Introduction

So-called passive electric components, such as ohmic resistors (R), capacitors (C) and inductors (L), are widely used in various areas of science and electrical engineering. LCR-oscillatory circuits deliver the time base for many electrically generated oscillation processes and RC-elements are utilised as frequency filters for signal forming. The resonance behaviour of LCR-oscillatory circuits features plenty of similarities compared to oscillation-capable systems in other physical areas.

In the first part of this lab course, the behaviour of a RC serial circuit compared to a square-wave voltage shall be examined. In the second part we take a look at the properties of conductors and inductors (coils) compared to harmonic alternating voltages (AC), where in the third part we investigate on the resonance behaviour of a LCR parallel oscillatory circuit.

12.2 Theoretical part

a) Square-wave voltage on the RC serial circuit

A capacitor C and a resistor R are connected to a direct current voltage (DC) source, which creates a constant voltage V_0, via a switch S as it is shown in Fig. 12.1.

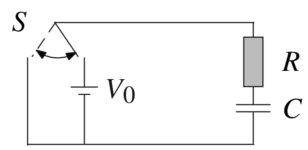


Figure 12.1: Generation of a square-wave voltage with a RC serial circuit.

At time $t = 0$, the capacitor shall be completely discharged and the voltage V_0 is being plugged by turning the switch. After the power-up of the voltage it follows by the 2nd Kirchhoff's circuit law

$$V_0 \;=\; V_R + V_C \;=\; R \cdot I + \frac{Q}{C} \;=\; R \cdot \frac{dQ}{dt} + \frac{Q}{C} \tag{12.1}$$

This is a first-order differential equation for the charge Q which is stored on the capacitor. Considering the initial conditions $V_R = V_C = Q = 0$ for $t = 0$ the solution of the differential equation becomes

$$Q(t) = Q_0 \cdot (1 - e^{-t/RC}) \qquad \text{mit } Q_0 = V_0 \cdot C \tag{12.2}$$

$$I(t) = \frac{dQ}{dt} = I_0 \cdot e^{-t/RC} \qquad \text{mit } I_0 = \frac{Q_0}{R \cdot C} = \frac{V_0}{C} \tag{12.3}$$

$$V_C(t) = \frac{Q(t)}{C} = V_{C_0} \cdot (1 - e^{-t/RC}) \quad \text{mit } V_{C_0} = \frac{Q_0}{C} = V_0 \tag{12.4}$$

The characteristic time constant which defines the behaviour of the circuit is $\tau = RC$.

At time $T \gg \tau$ the capacitor shall be completely charged and the voltage is being broken by turning the switch again. The initial conditions at time T become $Q = Q_0$, $V_C = V_0$ and $I = 0$. Starting with the differential equation for $t > T$

$$0 = V_R + V_C = R \cdot I + \frac{Q}{C} = R \cdot \frac{dQ}{dt} + \frac{Q}{C} \tag{12.5}$$

it follows:

$$Q(t) = Q_0 \cdot e^{-(t-T_0)/RC} \qquad \text{mit } Q_0 = V_0 \cdot C \tag{12.6}$$

$$I(t) = \frac{dQ}{dt} = I_0 \cdot e^{-(t-T_0)/RC} \qquad \text{mit } I_0 = -\frac{Q_0}{R \cdot C} = -\frac{V_0}{C} \tag{12.7}$$

$$V_C(t) = \frac{Q(t)}{C} = V_{C_0} \cdot e^{-(t-T_0)/RC} \quad \text{mit } V_{C_0} = \frac{Q_0}{C} = V_0 \tag{12.8}$$

The temporal progress of the voltage for periodic power-up and shut-off is shown in Fig. 12.2, in particular for the case $T \gg \tau$. If this condition is not fulfilled then the capacitor is not completely charged and the current hasn't dropped down to zero yet when turning the switch. In this case the temporal progress for the voltage is shown in Fig. 12.3.

b) Harmonic alternating voltage on the RC-circuit and the RL-circuit

The properties and behavior of a capacitance or inductance subjected to a harmonic voltage were already introduced in experiment WS1 - *Alternating currents and impedances.* Here we briefly recall the basics and introduce the description of impedances by complex numbers.

When plugging a harmonic alternating current

$$V(t) = V_0 \cdot \cos(\omega t) \tag{12.9}$$

of frequency ω to an ordinary electrical circuit, then a harmonic alternating current of the same frequency flows, which in general is shifted by a phase φ against the voltage $V(t)$:

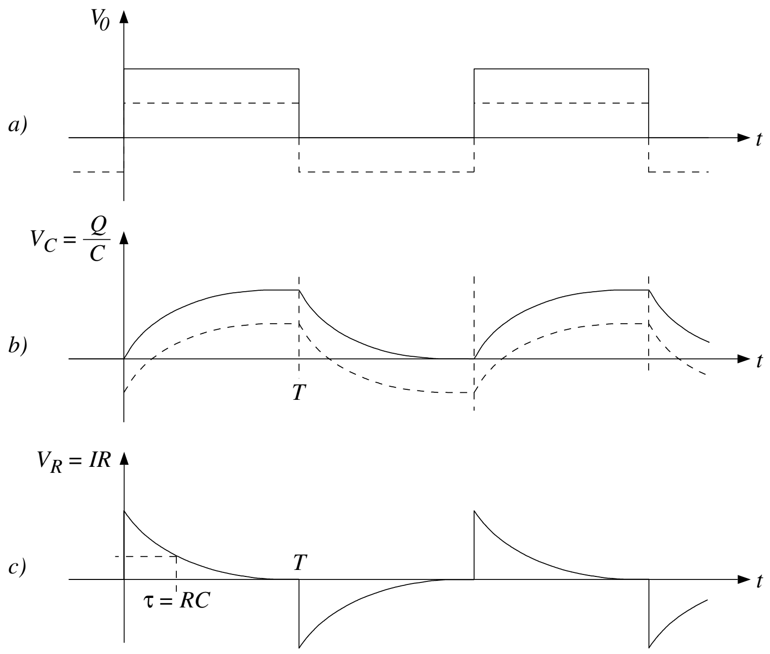


Figure 12.2: Temporal progress of the voltage for periodic power-up and shut-off of the input voltage for $T_0 \gg RC$. The dashed line shows the voltage characteristic for a square-wave voltage with mean zero, as it is being delivered from a square-wave generator.

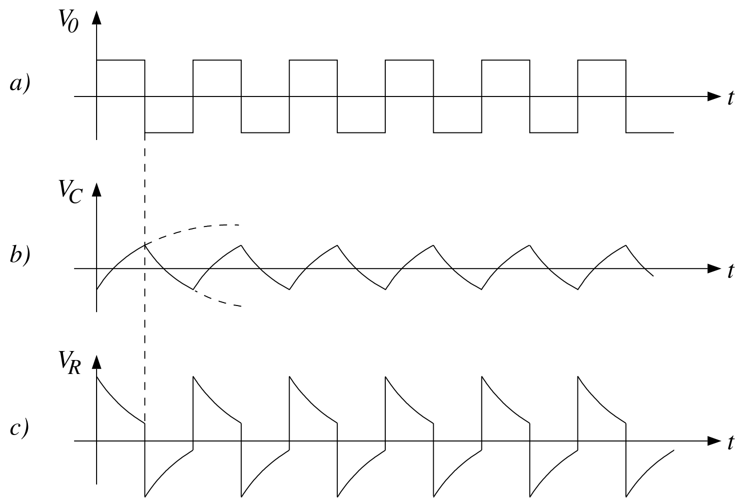


Figure 12.3: Temporal progress of the voltage for periodic power-up and shut-off of the input voltage when the condition $T_0 \gg RC$ is not fulfilled.

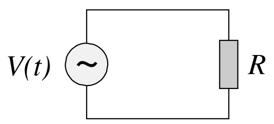
$$I(t) = I_0 \cdot \cos(\omega t - \varphi) \tag{12.10}$$

The behaviour of the circuit is completely described by this phase and the absolute value of the impedance

$$|Z| = \frac{V_0}{I_0} \tag{12.11}$$

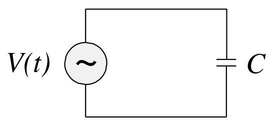
If the circuit contains only one single element, one receives the impedance with the help of Kirchhoff's 2nd law:

- Ohmic resistor R:



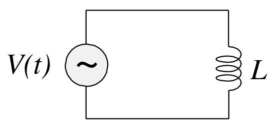
$$\begin{aligned} V(t) &= V_0 \cdot \cos(\omega t) \; = \; V_R(t) \; = \; R \cdot I(t) \\ I(t) &= \frac{V_0}{R} \cdot \cos(\omega t) \\ |Z_R| &= R\,, \quad \varphi_R \; = \; 0 \end{aligned}$$

- Capacitor of capacity C:



$$\begin{aligned} V(t) &= V_0 \cdot \cos(\omega t) \; = \; V_C(t) \; = \; \frac{Q(t)}{C} \\ I(t) &= \frac{dQ}{dt} \; = \; -V_0\,\omega\,C \cdot \sin(\omega t) \; = \; V_0\,\omega\,C \cdot \cos(\omega t + \frac{\pi}{2}) \\ |Z_C| &= \frac{1}{\omega\,C}\,, \quad \varphi_C \; = \; -\frac{\pi}{2} \end{aligned}$$

- Coil of inductance L:



$$\begin{aligned} V(t) &= V_0 \cdot \cos(\omega t) \; = \; V_L(t) = -L \cdot \frac{dI}{dt} \\ I(t) &= \frac{V_0}{\omega\,L} \cdot \sin(\omega t) \; = \; \frac{V_0}{\omega\,L} \cdot \cos(\omega t - \frac{\pi}{2}) \\ |Z_L| &= \omega\,L\,, \quad \varphi_L \; = \; +\frac{\pi}{2} \end{aligned}$$

The phase and the absolute value of the impedance can be displayed as a vector in the complex plane. For the three described special cases this is shown in Fig. 12.4

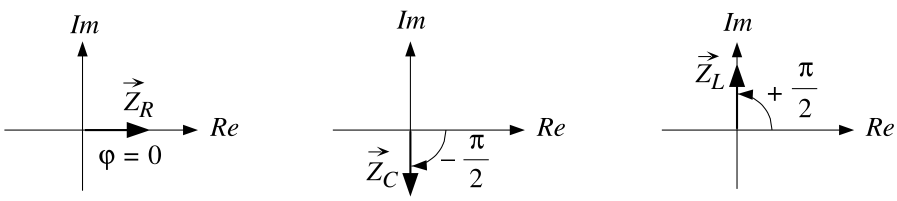
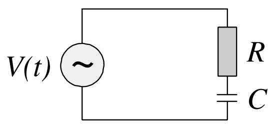
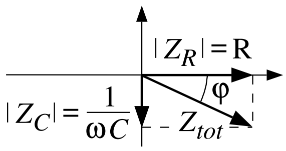


Figure 12.4: Depiction of the impedance in the complex plane.

In case the circuit contains multiple impedances serially, the total impedance is given by vector sum of the single impedances. For the example of the RC serial circuit one receives:



$$V = V_R + V_C = I \cdot (Z_R + Z_C) = I \cdot Z_{tot} \tag{12.12}$$
$$|Z_{tot}| = \sqrt{Z_R^2 + Z_C^2} = \sqrt{R^2 + 1/(\omega^2 C^2)} \tag{12.13}$$
$$\tan\varphi = -\frac{1}{\omega RC} \tag{12.14}$$



and for the amplitude of the current follows:

$$I_0 = \frac{V_0}{|Z_{tot}|} = \frac{V_0}{\sqrt{R^2 + 1/(\omega^2 C^2)}} \tag{12.15}$$

The voltage amplitudes at the capacity and the resistor are given by

$$V_{C_0} = I_0 \cdot |Z_C| = \frac{V_0}{\sqrt{1 + \omega^2 R^2 C^2}} \tag{12.16}$$
$$V_{R_0} = I_0 \cdot |Z_R| = \frac{V_0\, R\, \omega\, C}{\sqrt{1 + \omega^2 R^2 C^2}} \tag{12.17}$$

Their dependence on the frequency ω of the input voltage is shown in Fig. 12.5. At the capacity high frequencies and at the resistor low frequencies of the input signal are being suppressed. Therefore this circuit can be used either as a high-pass or as a low-pass filter. The frequency response of the output voltage V_a can be adjusted in both cases by varying the resistor R. A practical usage of this circuit are tone controls in audio amplifiers.

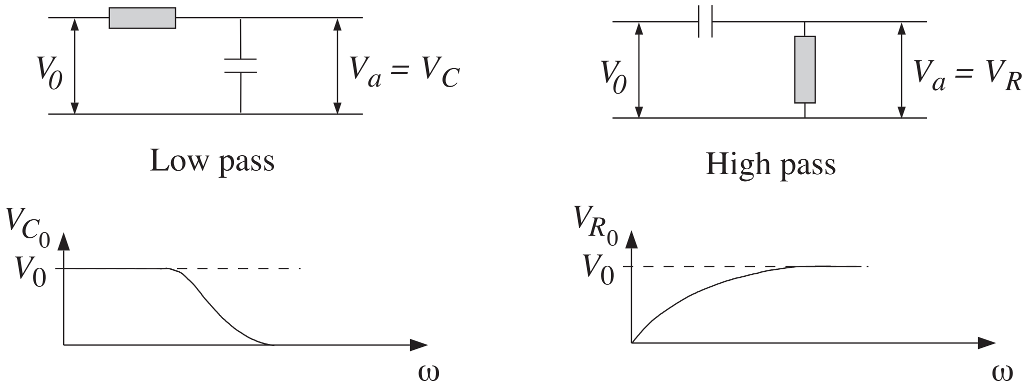
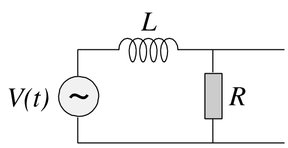
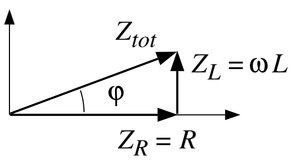


Figure 12.5: RC serial circuit as high-pass or low-pass filter.

In a similar manner one can combine a resistor and an inductance. In this case it holds that:



$$|Z_{tot}| = \sqrt{Z_R^2 + Z_L^2} = \sqrt{w^2 L^2 + R^2} \tag{12.18}$$
$$\tan\varphi = \frac{\omega L}{R} \tag{12.19}$$



and the output amplitude at the resistor is:

$$V_{R_0} = I_0 \cdot R = \frac{V_0}{|Z_{tot}|} \cdot R = \frac{V_0 R}{\sqrt{R^2 + \omega^2 L^2}} \tag{12.20}$$

c) Electrical parallel oscillatory circuit (LCR-circuit)

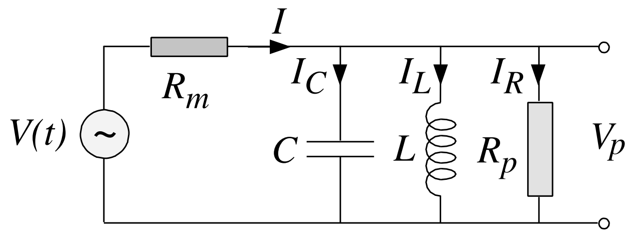


Figure 12.6: parallel oscillatory circuit.

In this experiment we use the circuit which is shown in Fig. 12.6. The oscillatory circuit itself is made up of a coil of inductance L which is parallel-connected with a capacity C and an ohmic resistor R_p. Due to metro-logical reasons a further resistor R_m is connected in series with the oscillatory circuit.

When a harmonic input voltage $V(t) = V_0 \cdot \cos \omega t$ is applied to the circuit with Kirchhoff's second law follows

$$V(t) = R_m \cdot I(t) + V_p(t) \quad \text{and therefore} \tag{12.21}$$

$$I(t) = \frac{1}{R_m} \cdot (V_0 \cdot \cos \omega t - V_p(t)) \tag{12.22}$$

and with Kirchhoff's first law

$$I(t) = I_C(t) + I_R(t) + I_L(t) = C \cdot \frac{dV_p}{dt} + \frac{V_p(t)}{R} + \frac{1}{L} \cdot \int V_p \, dt \tag{12.23}$$

By equalling Equ. 12.22 and 12.23 and differentiating one time with respect to time the characteristic differential equation for this oscillatory circuit follows:

$$-\omega \cdot \frac{V_0}{R_m} \cdot \sin \omega t = C \cdot \frac{d^2V_p}{dt^2} + \left(\frac{1}{R_p} + \frac{1}{R_m}\right) \cdot \frac{dV_p}{dt} + \frac{V_p(t)}{L}. \tag{12.24}$$

It is of the same form as the one for the forced oscillation of the harmonic oscillator in classical mechanics (see lab course R from the winter semester). Its general solution is a superposition of a freely damped oscillation and a forced oscillation with excitation frequency ω.

The free oscillation decays with the time constant $\tau = 2L/R$ and for times $t \gg \tau$ the system oscillates only with the frequency ω of the excitation voltage and is shifted by a phase φ to this excitation.

$$V_p(t) = V_{p_0} \cdot \cos(\omega t + \varphi) \tag{12.25}$$

The amplitude V_{p_0} and the phase φ follow by plugging into the differential equation:

$$V_{p0} = \frac{V_0/R_m}{\sqrt{\left(\frac{1}{R_p} + \frac{1}{R_m}\right)^2 + \left(\frac{1}{\omega L} - \omega C\right)^2}} \tag{12.26}$$

$$\tan\varphi = \left(\frac{1}{\omega L} - \omega C\right) \Big/ \left(\frac{1}{R_p} + \frac{1}{R_m}\right) \tag{12.27}$$

Hence, the amplitude and the phase of the voltage that decays at the oscillatory circuit are dependent on the excitation frequency ω and show the typical resonance behaviour of oscillation-capable systems. The amplitude of the voltage drop is at the maximum for the resonance frequency

$$\omega = \omega_0 = \frac{1}{\sqrt{L\,C}} \tag{12.28}$$

Here, the impedances of the capacity and the coil cancel each other out and the phase is $\varphi = 0$.

The expression on the right side of Equ. 12.26 can be simplified for frequencies in the close region of the resonance frequency as follows. First of all with $\omega_0^2 = \frac{1}{L\,C}$ one has

$$V_{p0}(\omega) = \frac{V_0/R_m}{\sqrt{\left(\frac{1}{R_p} + \frac{1}{R_m}\right)^2 + \frac{C^2}{\omega^2}\cdot(\omega_0^2 - \omega^2)}} \tag{12.29}$$

Near by the resonance frequency we have $\omega \approx \omega_0$ and with $\Delta\omega = \omega - \omega_0$ it is approximately true that

$$\omega_0^2 - \omega^2 = (\omega_0 + \omega)\cdot(\omega_0 - \omega) \approx -2\,\omega\,\Delta\omega \tag{12.30}$$

Plugged into Equ. 12.26 this approximation leads to

$$V_{p0}(\Delta\omega) \approx \frac{V_0/R_m}{\sqrt{\left(\frac{1}{R_p} + \frac{1}{R_m}\right)^2 + \frac{C^2}{\omega^2}\cdot(2\omega\,\Delta\omega)^2}} = \frac{V_0/R_m}{\sqrt{\left(\frac{1}{R_p} + \frac{1}{R_m}\right)^2 + 4C^2\,\Delta\omega^2}} \tag{12.31}$$

The resonance curve is shown in Fig. 12.7. Its width is usually specified by the so-called half-power width $2\,\Delta\omega_{1/2}$, where $\Delta\omega_{1/2}$ denotes the frequency difference for which the square of the amplitude of the voltage drop – i.e. the power in the parallel oscillatory circuit – has fallen off to half of the value at the resonance frequency ω_0. For $\Delta\omega_{1/2}$ the amplitude of the voltage drop itself has fallen off to $1/\sqrt{2}$ times the value at the resonance frequency. From Equ. 12.31 one gets

$$2\,\Delta\omega_{1/2} = \frac{1}{C}\cdot\left(\frac{1}{R_p} + \frac{1}{R_m}\right) \tag{12.32}$$

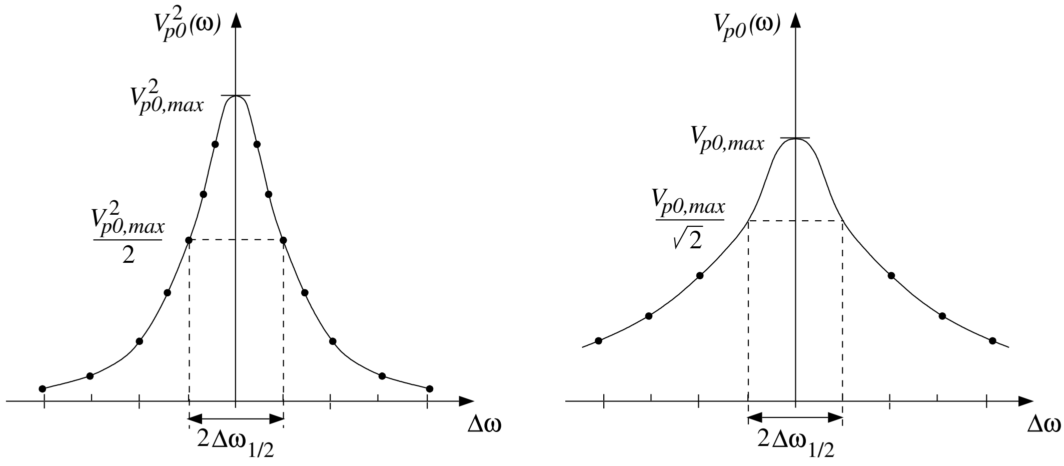


Figure 12.7: Resonance curve.

12.3 Experimental part

In this experiment we observe the measured quantities with a cathode-ray oscillograph. The functional principle of the oscilloscope is explained in the appendix of the lab course manual for EM.

The oscilloscope can only display voltages. Currents can be displayed indirectly via the voltage drop V on a known ohmic test resistor R_t, due to $V = R_t \cdot I$. Therefore the resistor R_t has to be included into the circuit, whereby its value has to be small enough such that it's not affecting the measurement. In this experiment a test resistor of $22\,\Omega$ is being used.

In order to display two measured quantities at the same time (e.g. input voltage and current in the circuit) to determine their phase relation we use a two-channel oscilloscope as in teh preceeding experiment WS1 - *Alternating currents and impedances.*

- **Remember that the outer connectors (shieldings) of the coaxial inputs of the oscilloscope are connected internally with the ground! Therefore, always link the inputs such that both outer connectors lead to the same point in the circuit.**
- **For this experiment *no* error calculation has to be done.**

a) Square-wave voltage n the *RC*-serial circuit

For an applied square-wave voltage the output voltages at the capacitor and the resistor of a *RC* serial circuit shall be observed.

- Assemble the circuit to target the arrangement of Fig. 12.8. Use a resistor of $R = 470\,\Omega$ and a capacitor with a capacity of $C = 1\,\mu$F.
- With the voltage generator set up a square-wave voltage with an amplitude of 5 V and a frequency of 100 Hz. On the oscilloscope, simultaneously study the input voltage $V(t)$ and

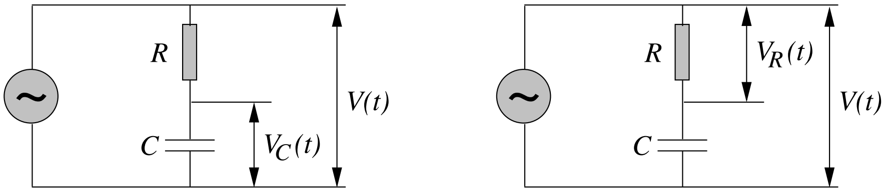


Figure 12.8: Measurements on the RC serial circuit.

the voltage $V_C(t)$ at the capacitor. Plot the voltage characteristic as a function against the time graphically and from the curves determine the periods T and $\tau = RC$. Compare your results with the expected values.

- Repeat the measurement for a square-wave voltage with a frequency of 5000 Hz.
- Reset the square-wave voltage to a frequency of 100 Hz and simultaneously study the input voltage $V(t)$ and the voltage $V_R(t)$ at the resistor. Plot the voltage characteristic as a function against the time graphically and from the curves determine the periods T and $\tau = RC$. Compare your results with the expected values.
- Repeat the measurement for a square-wave voltage with a frequency of 5000 Hz.

b) Resonance curve of the LCR resonance circuit

The resonance curve of a LCR parallel oscillatory circuit shall be measured for different values of the resistor R_p.

- Assemble the circuit after Fig. 12.9. First, use a resistor of $R_p = 19.1\,\mathrm{k\Omega}$.
- With the oscilloscope measure the oscillation amplitude of the oscillatory circuit as a function of the input (excitation) frequency. Initially, raise the input frequency in big steps of 200 kHz to 600 kHz and measure the range around the resonance frequency in more delicate steps. Chose an appropriate step size to do that. Plot the amplitude as a function of the input frequency and determine the half-power width.
- Repeat the measurement for a resistor of $R_p = 57.6\,\mathrm{k\Omega}$.
- If you compare the resonance curve measured here with the one of a mechanical oscillator (cf. experiment R - *Resonance* in the first semester), which role has the resistance R_p in the present case?
- Plot the half-power width against $1/R_p$ and lay a straight line through both measurement points. Determine the capacity C from the slope of the line and the resistor R_m from the abscissa segment. Finally, determine the inductance L from the measured resonance frequency.

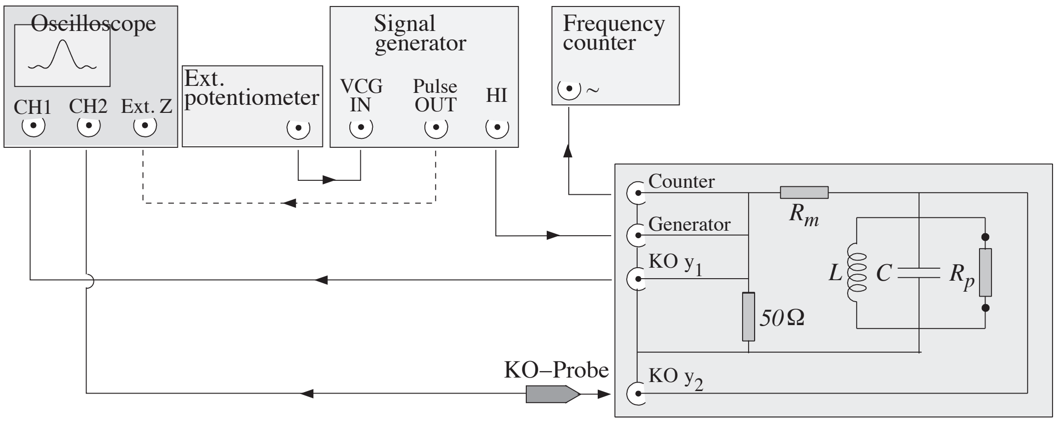


Figure 12.9: Setup for the measurement of the resonance curve of a LCR parallel oscillatory circuit.